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Fixed point theorems in dislocated quasi b-metric spaces

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ABSTRACT

In this paper we have given some fixed point theorems in dislocated quasi b-metric Spaces which are generalizations of dislocated quasi metric, partial b-metric spaces.

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Key words: *Metric Space, Dislocated Quasi b- Metric Space, Contraction, Fixed Point.*

1. INTRODUCTION

In 1922, Banach [3] proved a fixed point theorem for contraction mappings in metric spaces. Since then a number of fixed point theorems have been proved by different authors, and many more generalizations of this theorem have been established. The study of common fixed point of mappings satisfying certain contractive conditions have been at the center of vigorous research activity.

Many generalizations of Banach's contraction theorem in different types of generalizations if metric spaces are made is clear from the literature. Some problems particularly the problems of convergence of measurable functions with respect to measure leads to the generalization of metric space. Czerwik [5] introduced the concept of b-metric space and proved Banach contraction theorem in so called b-metric spaces.

Alaghamdi *et al.* [2] introduced the notion of a b-metric like space which generalized the notion of a b-metric space, where they proved some exciting new fixed point results in b-metric like spaces. Recently Shukla [11] introduced the concept of partial b-metric space and gave some fixed point results and examples.

The notion of dislocated metric spaces was introduced by Hitzler [6] as a part of the study of logic programming semantics. Zeyada [12] initiated the concept of dislocated quasi metric space and generalized the result of Hitzler [6]. Recently Rehman and Sarwar [9] introduced the concept of dislocated quasi b-metric space and proved the Banach's contraction principle, Kannan [7] and Chatterjea [4] type fixed point results for self mappings in such spaces.

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Banach's contraction principle has been generalized by various authors by putting different types of contractive conditions either on mappings or on the space. A comprehensive literature and generalization of the same can be found in Rhodes [10].

In this paper we generalize the results of Rehman and Sarwar [9] and introduce the definition of limit of a sequence in a dislocated quasi b- metric space in a suitable way. Based on this, we establish some fixed point results.

The following definitions and results are needed in the sequel.

Definition 1.1 ([2], [12]): Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function, called a distance function, which satisfies one or more of the conditions
d_1) $d(x,x) = 0$.
d_2) $d(x,y) = d(y,x) = 0$ then $x = y$,
d_3) $d(x,y) = d(y,x)$
d_4) $d(x,y) \leq d(x,z) + d(z,y)$), for all $x, y, z \in X$.

If d satisfies the conditions d_1, d_2, d_3 and d_4 then d is called a metric on X.

If d satisfies the conditions d_1, d_2 and d_4 then d is called a quasi metric on X.

If d satisfies the conditions d_2, d_3 and d_4 then d is called a dislocated metric on X.

If d satisfies the conditions d_2 and d_4 then d is called a dislocated quasi metric on X.

A non empty set X with dislocated quasi metric d, i.e. (X, d) is called a dislocated quasi metric space.

Definition 1.2: Let X be a non empty set and $k \geq 1$ be real number. Then a mapping $d : X \times X \to [0, \infty)$ is called b-metric if
(1.2.1) $d(x,x) = 0$.
(1.2.2) $d(x,y) = d(y,x) = 0$ then $x = y$,
(1.2.3) $d(x,y) = d(y,x)$
(1.2.4) $d(x,y) \leq k(d(x,z) + d(z,y))$ for all $x, y, z \in X$.
Then the pair (X, d) is called b-metric space.

We observe that, b-metric is more general than usual metric.

Definition 1.3 [9]: Let X be a non empty set and $k \geq 1$ be real number. Then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi metric b-metric with index k if
(1.3.1) $d(x,y) = d(y,x) = 0$ then $x = y$,
(1.3.2) $d(x,y) \leq k(d(x,z) + d(z,y))$ for all $x, y, z \in X$.
Then the pair (X, d) is called a dislocated quasi b-metric space with index k.
(simply d q b-metric space)

Preposition 1.4 [9]: Suppose X is a non empty set and d^* is d q- metric and d^{**} is a d q b-metric with k ≥ 1 on X. Then the function $d : X \times X \to [0, \infty)$ defined by
$$d(x,y) = d^*(x,y) + d^{**}(x,y) \text{ for all } x, y \in X \text{ is d q- metric on } X.$$

Definition 1.5 [9]: Suppose (X, d) is a d q b-metric space. A sequence $\{x_n\}$ in X is called d q b-convergent to x in X if for $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x) < \boldsymbol{\varepsilon}$ for all n$\geq n_0$. In this case x is called d q b-limit of the sequence $\{x_n\}$.

Note: We observe that $x_n \to x$ if $d(x_n, x) \to 0$.

Definition 1.6 [9]: A sequence $\{x_n\}$ in a d q b-metric space X is called a Cauchy sequence if for > 0, there exists $n_0 \in N$ such that for all $m, n \geq n_0, d(x_m, x_n) < \boldsymbol{\varepsilon}$.

Definition 1.7 [9]: A d q b-metric space (X, d) is said to be complete if every Cauchy sequence in it is d q b-convergent.

Lemma 1.8 [9]: Limit of convergent sequence in a d q b-metric space is unique.

Definition 1.9: Let $: X \to X$, then T is called contraction if $d(Tx, Ty) \leq \alpha\, d(x, y)$ for all $x, y \in X$, where $0 \leq \alpha k < 1$

Lemma 1.10 [9]: Let (X, d) be a d q b-metric space with index k and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)$ for $n = 1,2,3, \dots \dots ....$ and $0 \leq \alpha k < 1$,$\alpha \in [0,1)$, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.11: Let (X, d) be a d q b-metric space with index k and $T: X \to X$ be a function. Then T is said to be continuous if $Tx_n \to Tx$ whenever $x \in X$ and $\{x_n\}$ is a sequence in X such that $x_n \to x$.

Theorem 1.12 [9]: Let (X, d) be a complete d q b-metric space with index k and $T: X \to X$ be a continuous contraction with $\alpha \in [0,1)$ and $0 \leq \alpha k < 1$, where $k \geq 1$ then T has a unique fixed point in X.

Theorem 1.13 [9]: Let (X, d) be a complete d q b-metric space with index k and $T: X \to X$ be a continuous contraction with $\alpha \in \left[0, \frac{1}{2}\right)$ satisfying the following condition:

$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \quad \forall\, x, y \in X$. Then T has a unique fixed point in X.

Theorem 1.14 [9]: Let (X, d) be a complete d q b-metric space with index k and $T: X \to X$ be a continuous contraction with $\alpha \in \left[0, \frac{1}{4}\right)$ (1.14.1)
and satisfying the following condition: $d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)]$ $\forall x, y \in X$. Then T has a unique fixed point in X.

Note: In Theorems 1.12, 1.13 and 1.14, no dependence on k is given, as a consequence, the results may not hold.

2. MAIN RESULTS

In this section we introduce the definitions of convergence, which is symmetric and establish some fixed point Theorems which in the light of the new definitions, bear dependence on k.

Now, we define limit of a sequence, Cauchy sequence, Convergent sequence and Completeness in the context of dislocated quasi b-metric space with index k.

Definition 2.1: Suppose (X, d) is a d q b-metric space with index k. A sequence $\{x_n\}$ in X is called d q b-convergent to x in X if for $n \geq N$ we have

$$d(x_n, x) + d(x, x_n) \to 0 \text{ as } n \to \infty.$$

Also x is called d q b- limit of the sequence $\{x_n\}$.

Note: In this case we write $x_n \to x$ as $n \to \infty$.

Definition 2.2: A sequence $\{x_n\}$ in a d q b-metric is called a Cauchy sequence if $d(x_m, x_n) \to 0$ as $m.n \to \infty$.That is, given $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(x_m, x_n) < \varepsilon$ for $m, n \geq n_0$.

Definition 2.3: A d q b-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Definition 2.4: A self map $T: X \to X$ is said to be continuous if $x_n \to x \Rightarrow Tx_n \to Tx$.

Now, we begin with the following lemmas which are useful.

Lemma 2.5: Limit of convergent sequence in a d q b-metric space with index k is unique.

Proof: Let $\{x_n\}$ be a convergent sequence in a d q b-metric space with index k.
Let x and y be two limits of $\{x_n\}$.
Then $d(x, x_n) + d(x_n, x) \to 0$
and $d(y, x_n) + d(x_n, y) \to 0$
so that, $d(x, x_n) \to 0, d(x_n, x) \to 0$
$d(y, x_n) \to 0, d(x_n, y) \to 0$
Now,

$$d(x, y) \leq k\, d(x, x_n) + kd(x_n, y) \to 0.$$

$$\therefore\ d(x, y) = 0.$$

Similarly $d(y,x) = 0$.

$\therefore\ x = y$.

Hence, limit of a convergent sequence in a d q b-metric space with index k is unique.

Lemma 2.6: Let (X,d) be a d q b-metric space with index k, $0 \leq \alpha k < 1$ and $\{x_n\}$ be a sequence in X, such that

$$d(x_n, x_{n+1}) \leq \alpha\, d(x_{n-1}, x_n) \text{ and } \tag{2.6.1}$$
$$d(x_{n+1}, x_n) \leq \alpha\, d(x_n, x_{n-1}) \tag{2.6.2}$$

for $n = 1,2,3, \dots \dots$ .Then $\{x_n\}$ is a Cauchy sequence.

Proof: From (2.6.1) and (2.6.2),

$$d(x_n, x_{n+1}) \leq \alpha^n\, d(x_0, x_1) \tag{2.6.3}$$
$$d(x_{n+1}, x_n) \leq \alpha^n\, d(x_1, x_0) \tag{2.6.4}$$

Write $d^*(x,y) = d(x,y) + d(y,x),\quad x,y \in X$

Hence
$$\begin{aligned} d^*(x_n, x_{n+1}) &= d(x_n, x_{n+1}) + d(x_{n+1}, x_n) \\ &= \alpha^n\big(d(x_0, x_1) + d(x_1, x_0)\big) \\ &= \alpha^n d^*(x_0, x_1) \leq k\alpha^n d^*(x_0, x_1) \end{aligned} \tag{2.6.5}$$

Now
$$\begin{aligned} d^*(x_n, x_{n+2}) &= d(x_n, x_{n+2}) + d(x_{n+2}, x_n) \\ &\leq kd(x_n, x_{n+1}) + kd(x_{n+1}, x_{n+2}) + kd(x_{n+2}, x_{n+1}) + kd(x_{n+1}, x_n) \\ &\leq k\alpha^n\, d(x_0, x_1) + k\alpha^{n+1}\, d(x_0, x_1) + k\alpha^{n+1} d(x_1, x_0) + k\alpha^n d(x_1, x_0) \\ &= k\alpha^n(1+\alpha)d^*(x_0, x_1) \\ &\leq k\alpha^n(1+k\alpha)d^*(x_0, x_1) \end{aligned}$$

Now, we claim that

$$d^*(x_n, x_{n+l}) \leq k\alpha^n\, (1 + (k\alpha) + (k\alpha)^2 + \cdots \dots \dots \dots . + (k\alpha)^{l-1})d^*(x_0, x_1) \text{ for } l = 1,2 \dots$$

The result is true for $l = 1$ by (2.6.5) and will prove that it is true for $l + 1$.

$$\begin{aligned} d^*(x_n, x_{n+l+1}) &\leq k[d^*(x_n, x_{n+1}) + d^*(x_{n+1}, x_{n+l+1})] \\ &\leq k\, d^*(x_n, x_{n+1}) + k^2\alpha^{n+1}(1 + (k\alpha) + (k\alpha)^2 + \cdots \dots \dots . + (k\alpha)^{l-1})d^*(x_0, x_1) \\ &\leq k\alpha^n d^*(x_0, x_1) + k\alpha^n\, k\alpha\, (1 + (k\alpha) + (k\alpha)^2 + \cdots \dots \dots \dots . + (k\alpha)^{l-1})d^*(x_0, x_1) \\ &= k\alpha^n d^*(x_0, x_1)(1 + k\alpha\, (1 + (k\alpha) + (k\alpha)^2 + \cdots \dots \dots \dots . + (k\alpha)^{l-1}) \\ &= k\alpha^n d^*(x_0, x_1)\, (1 + (k\alpha) + (k\alpha)^2 + \cdots \dots \dots . + (k\alpha)^{l}) \end{aligned}$$

$\therefore$ It is true for $l + 1$. By induction

Hence
$$\begin{aligned} d^*(x_n, x_{n+l}) &\leq k\alpha^n\, (1 + (k\alpha) + (k\alpha)^2 + \cdots \dots \dots \dots . + (k\alpha)^{l-1})d^*(x_0, x_1) \\ &\leq \frac{k\alpha^n}{1-k\alpha}\, d^*(x_0, x_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

$\therefore d^*(x_n, x_{n+l}) \to 0.$ as $n \to \infty$.

$\therefore d(x_n, x_{n+l}) + d(x_{n+l}, x_n) \to 0.$ as $n \to \infty$.

$\therefore d(x_n, x_{n+l}) \to 0$ and $d(x_{n+l}, x_n) \to 0$ as $n \to \infty$.

Hence, $\{x_n\}$ is a Cauchy sequence.

Definition 2.7: Let $T{:}\, X \to X$ be such that $d(Tx,Ty) \leq \alpha\, d(x,y)\ \forall\, x,y \in X$ where $0 \leq \alpha k < 1$ Then T is called a contraction.

Now we will prove Banach contraction theorem in the context of d q b-metric space with index k.

In proving this Theorem in [9], continuity of T is assumed.

Theorem 2.8: Let (X,d) be a complete d q b-metric space with index k and $T{:}\, X \to X$ be a contraction with $\alpha \in [0,1)$ and $0 \leq \alpha k < 1$. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$.

We define a sequence $\{x_n\}$ in X such that, $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$,….and in general $x_n = T^n x_0$ for $n = 1,2,3, \dots \dots \dots$.

Now,

$$d(x_n, x_{n+1}) = d(T^n x_0, T^{n+1}x_0) = d(T(T^{n-1}x_0), T(T^n x_0))$$
$$\leq \alpha\, d(T^{n-1}x_0, T^n x_0)$$
$$= \alpha\, d(x_{n-1}, x_n)$$
$$\leq \alpha^2\, d(x_{n-1}, x_n) \leq \cdots \ldots \ldots \ldots \ldots \ldots \leq \alpha^n\, d(x_0, x_1).$$
$$\therefore d(x_n, x_{n+1}) \leq \alpha^n\, d(x_0, x_1).$$

Similarly, $d(x_{n+1}, x_n) \leq \alpha^n\, d(x_1, x_0)$.

and $$d(x_n, x_{n+l}) \leq k\, d(x_n, x_{n+1}) + k\, d(x_{n+1}, x_{n+l})$$
$$\leq k\, d(x_n, x_{n+1}) + k(k\, d(x_{n+1}, x_{n+2}) + kd(x_{n+2}, x_{n+l}))$$
$$= k\, d(x_n, x_{n+1}) + k^2\, d(x_{n+1}, x_{n+2}) + k^3\, d(x_{n+2}, x_{n+3}) + \cdots \ldots \ldots + k^n d(x_{n+l-1}, x_{n+l})$$
$$\leq k\, \alpha^n\, d(x_0, x_1) + k^2\, \alpha^{n+1}\, d(x_0, x_1) + \cdots \ldots \ldots + k^n\, \alpha^{n+l} d(x_0, x_1)$$
$$= \frac{k\alpha^n}{1-k\alpha}\, d(x_0, x_1). \qquad (\because k\alpha < 1)$$
$$\leq \frac{k\alpha^n}{1-k\alpha}\,(d(x_0, x_1) + d(x_1, x_0))$$
$$\therefore d(x_n, x_{n+l}) \leq \frac{k\alpha^n}{1-k\alpha}\big(d(x_0, x_1) + d(x_1, x_0)\big) \to 0 \text{ as } n \to \infty.$$

Similarly $d(x_{n+l}, x_n) \leq \frac{k\alpha^n}{1-k\alpha}(d(x_1, x_0) + d(x_1, x_0)) \to 0$ as $n \to \infty$.
$\therefore d(x_n, x_m) \to 0$ as $m, n \to \infty$.
$\therefore$ $\{x_n\}$ is a Cauchy sequence in complete d q b-metric space X.

So there exists $x \in X$, such that
$d(x_{n+1}, x) \to 0$
But $d(Tx_n, Tx) \leq \alpha\, d(x_n, x) \to 0$.

Also $d(Tx, x_{n+1}) \to 0$.
$\therefore x_{n+1} \to Tx$
$\therefore x = Tx$.
Hence x is a fixed point of T.

Uniqueness: Let x, y be two fixed points of T.
Then $d(x, y) = d(Tx, Ty) \leq \alpha d(x, y)$
$\therefore d(x, y) = 0$.

Similarly $d(y, x) = 0$. Hence $x = y$.
$\therefore T$ has unique fixed point.

The above theorem improves the result of [9] in complete d q b-metric space, since continuity of T is not assumed here.

Theorem 2.9: Let (X, d) be a complete d q b-metric space with index k and $0 \leq \alpha < \frac{1}{k+1}$ with $k \geq 1$ and $T: X \to X$ be a self mapping satisfying the condition:

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$.

We define a sequence $\{x_n\}$ in X such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$,….
And in general $x_n = T^n x_0$ for $n = 1,2,3, \ldots \ldots$….
Now,

$$d(x_n, x_{n+1}) = d(T^n x_0, T^{n+1}x_0)$$
$$= d(T(T^{n-1}x_0), T(T^n x_0))$$
$$\leq \alpha\, d(T^{n-1}x_0, T^n x_0) + \alpha\, d(T^n x_0, T^{n+1}x_0)$$
$$= \alpha(d(x_{n-1}, x_n) + \alpha\, d(x_n, x_{n+1})$$

$$\therefore (1-\alpha) d(x_n, x_{n+1}) \leq \alpha\, d(x_{n-1}, x_n)$$

$$\Rightarrow d(x_n, x_{n+1}) \leq \frac{\alpha}{1-\alpha}\, d(x_{n-1}, x_n)$$
$$= (x_{n-1}, x_n) \qquad \text{where } \beta = \frac{\alpha}{1-\alpha}$$

so that $k\beta = \frac{k\alpha}{1-\alpha} < 1$ $(\because\ \alpha < \frac{1}{k+1})$

Similarly, $d(x_{n+1}, x_n) \le \frac{\alpha}{1-\alpha} d(x_n, x_{n-1}) = \beta\, d(x_n, x_{n-1})$
$\therefore$ by Lemma 2.6 , $\{x_n\}$ is a Cauchy sequence in complete d q b-metric space X.

Therefore, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.

Now, we show that x is a fixed point of T.

$$\begin{aligned} d(x_{n+1}, Tx) &= d(Tx_n, Tx) \\ &\le \alpha\, (d(x_n, Tx_n) + d(x, Tx)) \\ &= \alpha\, (d(x_n, x_{n+1}) + d(x, Tx)) \\ &\le \alpha\, d(x_n, x_{n+1}) + \alpha k\, d(x, x_{n+1}) + \alpha k d(x_{n+1}, Tx) \end{aligned}$$

$$\therefore d(x_{n+1}, Tx) \le \alpha\, d(x_n, x_{n+1}) + \alpha k\, d(x, x_{n+1}) + \alpha k d(x_{n+1}, Tx)$$

$$\Rightarrow (1\text{-}\alpha k)\, d(x_{n+1}, Tx) \le \alpha\, d(x_n, x_{n+1}) + \alpha k\, d(x, x_{n+1})$$
$$\therefore \quad d(x_{n+1}, Tx) \to 0$$

Similarly, $d(Tx, x_{n+1}) \to 0$

$\therefore$ $x_{n+1} \to Tx$

$\therefore$ $Tx = x$.

$\therefore$ x is a fixed point of T.

Uniqueness: Let x, y be two fixed points of T,

Now $d(x,x) = d(Tx, Tx) \le \alpha\, (d(x, Tx) + d(x, Tx)) = 2\alpha\, d(x, Tx) = 2\alpha\, d(x, x)$

$\therefore$ $d(x,x) = 0$. $\quad (\because\ 2\alpha < \frac{2}{1+k} \le 1.)$

Similarly, $d(y,y) = 0$.

Now, $d(x,y) = d(Tx, Ty) \le \alpha\, (d(x, Tx) + d(y, Ty)) = \alpha\left(d(x,x) + d(y,y)\right) = 0$.

$\therefore$ $d(x,y) = 0$.

Similarly, $d(y,x) = 0$.

$\therefore x = y$.

Hence T has unique fixed point.

Theorem 2.10: Let (X, d) be a complete d q b-metric space with index k, and $T: X \to X$ be a self mapping with $\alpha k(3k+1) < 1$ (2.10.1)
satisfying the following condition: $d(Tx, Ty) \le \alpha[d(x, Ty) + d(y, Tx)]$ $\forall x, y \in X$.
Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$

We define a sequence $\{x_n\}$ in X such that, $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$,….and in general, $x_n = T^n x_0$ for $n = 1,2,3, \dots \dots ..$
Now,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T^n x_0, T^{n+1} x_0) \\ &= d(T(T^{n-1}x_0), T(T^n x_0)) \\ &\le \alpha\, d(T^{n-1}x_0, T^{n+1}x_0) + \alpha\, d(T^n x_0, T^n x_0)) \\ &= \alpha(d(x_{n-1}, x_{n+1}) + \alpha\, d(x_n, x_n) \\ &\le \alpha\, (k\, d(x_{n-1}, x_n) + k\alpha\, d(x_n, x_{n+1})) + \alpha\, d(x_n, x_n) \end{aligned}$$
$$\therefore (1 - \alpha k) d(x_n, x_{n+1}) \le \alpha k d(x_{n-1}, x_n) + \alpha\, d(x_n, x_n)$$

But $d(x_n, x_n) \le k\, d(x_{n-1}, x_n) + k d(x_{n-1}, x_n)$

$$\begin{aligned} \therefore (1 - \alpha k) d(x_n, x_{n+1}) &\le \alpha k\, d(x_{n-1}, x_n) + \alpha k\, d(x_{n-1}, x_n) + \alpha k\, d(x_n, x_{n-1}) \\ &= 2\alpha k\, d(x_{n-1}, x_n) + \alpha k\, d(x_n, x_{n-1}) \end{aligned}$$

Similarly $(1 - \alpha k)\, d(x_{n+1}, x_n) \le 2\alpha k\, d(x_n, x_{n-1}) + \alpha k\, d(x_{n-1}, x_n)$

Now,

$$(1-\alpha k)\,(\,d(x_n,x_{n+1})+\,d(x_{n+1},x_n)\,)\le 3\alpha k\,(d(x_{n-1},x_n)+\,d(x_n,x_{n+1})\,)$$
$$(1-\alpha k)\,d^*(\,x_n,x_{n+1})\,\le 3\alpha k\,d^*(\,x_{n-1},x_n)$$
$$\therefore d^*(\,x_n,x_{n+1})\le \tfrac{3\alpha k}{1-\alpha k}\,d^*(\,x_{n-1},x_n)$$
$$d^*(\,x_n,x_{n+1})\le \beta\,d^*(\,x_{n-1},x_n) \tag{2.10.2}$$
(where $\beta=\frac{3\alpha k}{1-\alpha k}<1$ by (2.10.1))

Similarly $d^*(\,x_{n+1},x_n)\,\le\,\beta\,d^*(\,x_n,x_{n-1})$ (2.10.3)

Now,

$$\begin{aligned}d^*(\,x_n,x_{n+2})&\le k\,d^*(\,x_n,x_{n+1})+k\,\ \,d^*(\,x_{n+1},x_{n+2})\\&\le k(\,\beta\,d^*(\,x_{n-1},x_n)+\,\beta\,d^*(\,x_n,x_{n+1}))\\&=\,k\beta\,(d^*(\,x_{n-1},x_n)+\,\ d^*(\,x_n,x_{n+1}))\\&\le k\beta\,(\,\beta^{n-1}\,d^*(\,x_o,x_1)+\,\beta^n\,d^*(\,x_o,x_1))\qquad\text{(From (2.10.2) and (2.10.3))}\\&=(\,k\beta^n\,+k\beta^{n+1})d^*(\,x_o,x_1)\\&\le\beta^{n-1}\,(\,k\beta+(k\beta)^2)d^*(\,x_o,x_1)\qquad(\,\because\beta^2\le k\beta^2)\end{aligned}$$

Now,

$$\begin{aligned}d^*(\,x_n,x_{n+3})\,&\le k(\,d^*(\,x_n,x_{n+2})+\,\ d^*(\,x_{n+2},x_{n+3}))\\&\le k(\,\beta^{n-1}\,(\,k\beta+(k\beta)^2+(k\beta)^3)d^*(\,x_o,x_1)\end{aligned}$$

$$d^*(\,x_n,x_{n+l})\,\le k(\,\beta^{n-1}\,(\,k\beta+(k\beta)^2+(k\beta)^3+\cdots\ldots\ldots..+(k\beta)^l)d^*(\,x_o,x_1) \tag{2.10.4}$$

Now,

$$\begin{aligned}d^*(\,x_n,x_{n+l+1})&\le k(\,d^*(\,x_n,x_{n+l})+\,\ d^*(\,x_{n+l},x_{n+l+1}))\\&\le k(\beta^{n-1}\,(\,k\beta+(k\beta)^2+(k\beta)^3+\cdots\ldots\ldots..+(k\beta)^l)\,d^*(\,x_o,x_1)+\beta^{n+l}\,d^*(\,x_o,x_1)\\&=k\,(\beta^{n-1}\,(\,k\beta+(k\beta)^2+(k\beta)^3+\cdots\ldots\ldots..+(k\beta)^l+(k\beta)^{l+1})\,d^*(\,x_o,x_1)\end{aligned}$$

Hence, by mathematical induction (2.10.4) is true for all $n,l=1,2,3\ldots\ldots....$

$$\begin{aligned}\therefore\ d^*(\,x_n,x_{n+l})\,&\le k(\,\beta^{n-1}\,(\,k\beta+(k\beta)^2+(k\beta)^3+\cdots\ldots\ldots..+(k\beta)^l)d^*(\,x_o,x_1)\\&\le\tfrac{k\beta^{n-1}}{1-k\beta}\,d^*(\,x_o,x_1)\,\to 0\quad\text{as }n\,\to\,\infty.\end{aligned}$$
$$\therefore\ d^*(\,x_n,x_{n+l})\,\to 0\quad\text{as }n\,\to\,\infty.$$

Similarly, $d^*(\,x_{n+l},x_n)\,\to 0$ as $n\,\to\,\infty$.
$\therefore\{x_n\}$ is a Cauchy sequence in complete d q b-metric space X.

Therefore, there exists $p\in X$ such that $x_n\to p$ as $n\,\to\,\infty$.

Now, we show that p is a fixed point of T.

$$\begin{aligned}d(x_{n+1},Tp)&=d(Tx_n,Tp)\\&\le\alpha\,(d(\,x_n,Tp)+d(p,Tx_n))\\&\le\alpha\,(kd(\,x_n,x_{n+1})+k\,d(x_{n+1},Tp)+\alpha d(p,x_{n+1}))\end{aligned}$$

$$\Rightarrow\ (1\text{-}\alpha k)\,d(x_{n+1},Tp)\,\le\alpha k\,d(\,x_n,x_{n+1})+\,\alpha d(p,x_{n+1})\,\to\,0\quad\text{as }n\,\to\,\infty.\ \ (\because\ \alpha k<1)$$
$$\therefore\quad d(x_{n+1},Tp)\to\,0$$
$$\therefore\ \ x_{n+1}\to Tp\qquad\therefore\ Tp=p.$$
$\therefore p$ is a fixed point of T.

Uniqueness: Let p,q be two fixed points of T.Then

$$d(p,q)=\,d(Tp,Tq)\le\,\alpha\big(d(p,Tq)+d(q,Tp)\big)=\,\alpha(d(p,q)+\alpha d(q,p)$$
$$\therefore(1-\alpha)d(p,q)\le\,\alpha\,d(q,p)$$
$$\therefore d(p,q)\le\,\frac{\alpha}{1-\alpha}\,d(q,p)\le\,\frac{\alpha}{1-\alpha}\,.\frac{\alpha}{1-\alpha}\,d(p,q)$$
$$\therefore d(p,q)=0.$$

Similarly, $d(q,p)=0$. Hence $p=q$.

Hence T has unique fixed point.

Note: The above Theorem 2.10 improves the result of [9] in the context of d q b-metric space, since continuity of T is not assumed. Also condition (2.10.1) depends on k and coincides with (2.14.1). when $k=1$.

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