International Journal of Mathematical Archive-10(1), 2019, 1-5
Available online through www.ijma.info ISSN 2229 – 5046

# COMMON FIXED POINT THEOREM BY USING COMPATIBLE MAPPINGS IN MENGER SPACES

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*(Received On: 12-08-18; Revised & Accepted On: 05-01-19)*

***ABSTRACT***

*We prove common fixed point theorem by using compatible mappings via in Menger spaces. some others in Menger as well as metric spaces. our result generalize many known results in Menger as well as metric spaces*

***Keywords**: Fixed point, Fixed Point Theorem, Common Property, Finite Family, Compatible Mapping.*

## 1. INTRODUCTION

There have been lots of generalizations of metric space. One such generalization is Menger space in which, used distribution functions instead of nonnegative real numbers as value of metric.

A Menger space is a space in which the concept of distance is considered to be a probabilistic, rather than deterministic. For detail discussion of Menger spaces and their applications we refer to Schweizer and Sklar [15]. The theory of Menger space is fundamental importance in probabilistic functional analysis.

A probabilistic metric space shortly PM-Space, is an ordered pair $(X, F)$ consisting of a non empty set $X$ and a mapping $F$ from $X \times X$ to $L$, where $L$ is the collection of all distribution functions (a distribution function $F$ is non decreasing and left continuous mapping of reals in to $[0,1]$ with properties, $\inf F(x) = 0$ and $\sup F(x) = 1$).

The value of $F$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ are assumed satisfy the following conditions;
1.1(a) $F_{u,v}(x) = 1$, for all $x > 0$, iff $u = v$;
1.1 (b) $F_{u,v}(0) = 0$, if $x = 0$;
1.1 (c) $F_{u,v}(x) = F_{v,u}(x)$;
1.1 (d) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$.

A mapping $t: [0,1] \times [0,1] \to [0,1]$ is a t-norm, if it satisfies the following conditions;
1.1 (e) $t(a, 1) = a$ for every $a \in [0,1]$;
1.1 (f) $t(0, 0) = 0$,
1.1 (g) $t(a, b) = t(b, a)$ for every $a, b \in [0,1]$;
1.1 (h) $t(c, d) \geq t(a, b)$ for $c \geq a$ and $d \geq b$
1.1 (i) $t(t(a, b), c) = t(a, t(b, c))$ where $a, b, c, d \in [0,1]$.

A Menger space is a triplet $(X, F, t)$, where $(X, F)$ is a PM-Space, $X$ is a non-empty set and a $t$-norm satisfying instead of 6.1(i) a stronger requirement.
1.1 (j) $F_{u,w}(x + y) \geq t\left(F_{u,v}(x), F_{v,w}(y)\right)$ for all $x \geq 0, y \geq 0$.

For a given metric space $(X, d)$ with usual metric $d$, one can put $F_{u,v}(x) = H(x - d(u, v))$ for all $x, y \in X$ and $t > 0$. where $H$ is defined as;

$$H(x) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$

and t-norm is defined as $t(a, b) = \min\{a, b\}$.

Our aim of this chapter, we establish a fixed point theorem in the setting of probabilistic metric space using weak compatibility.

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## 2. PRELIMINARIES

To proof of our result we need some known definitions which are follows;

**Definition 2.1:** A probabilistic metric space (PM- space) is an ordered pair $(X, F)$ consisting of a non empty set X and a mapping F from $X \times X$ into the collections of all distribution $F \in R$. For $x, y \in X$ we denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at u in R.

**Definition 2.2:** Self maps A and B of a Menger space $(X, F, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

**Definition 2.3:** Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_m, BAx_n,}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to x$, $Bx_n \to x$ for some x in X as $n \to \infty$.

The concept of neighborhoods in Menger space was introduced by Schweizer and Skalar [91].

**Definition 2.4:** Let $(X, F, t)$ be a Menger space. If $x \in X, \varepsilon > 0$ and $\lambda \in (0, 1)$, then $(\varepsilon, \lambda) -$ neighborhood of x is called $U_x(\varepsilon, \lambda)$, is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X: F_{x,y}(\varepsilon) > (1 - \lambda)\}.$$

an $(\varepsilon, \lambda) -$ topology in X is the topology induced by the family

$$\{U_x(\varepsilon, \lambda): x \in X\, \varepsilon > 0 \textit{ and } \lambda \in (0, 1)\}$$

of neighborhood.

**Remark 1:** If t is continuous, then Menger space $(X, F, t)$ is a Housdorff space in $(\varepsilon, \lambda) -$ topology.

Let $(X, F, t)$ be a complete Menger space and $A \subset X$. Then A is called a bounded set if

$$\lim_{u \to \infty} \inf_{x,y \in A} F_{x,y}(u) = 1.$$

**Definition 2.5:** A sequence $\{x_n\}$ in $(X, F, t)$ is said to be convergent to a point x in X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $x_n \in U_x(\varepsilon, \lambda)$ for all $n \geq N$ or equivalently $F(x_n, x; \varepsilon) > 1 - \lambda$ for all $n \geq N$.

**Definition 2.6:** A sequence $\{x_n\}$ in $(X, F, t)$ is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, $\exists$ an integer $N = N(\varepsilon, \lambda)$ such that $F(x_n, x_m, \varepsilon) > 1 - \lambda$ for all $n, m \geq N$.

**Definition 2.7:** A Menger space $(X, F, t)$ with the continuous $t$ –norm is said to be complete if every cauchy sequence in X converges to a point in X.

**Definition 2.8:** Let $(X, F, t)$ be a Menger space, two mappings $f, g : X \to X$ are said to be weakly compatible if they commute at coincidence point.

**Lemma 2.9:** Let X be a set f, g OWC self maps of X. If f and g have a unique point of coincidence,
$w = fx = gx$, then w is the unique common fixed point of f and g.

**Lemma 2.10:** Let $\{x_n\}$ be a sequence in a Menger space $(X, F, t)$, where t is continuous and $t(p, p) \geq p$ for all $p \in (0,1)$ and $n \in N$

$$F(x_n, x_{n+1}, kp) \geq F(x_{n-1}, x_n, p), \text{ then } \{x_n\} \text{ is Cauchy sequence.}$$

**Lemma2.11:** If $(X, d)$ is a metric space, then the metric d induces a mapping $F: X \times X \to L$ defined by

$$F(p, q) = H(x - d(p, q)), \qquad p, q \in R.$$

Further if $t: [0,1] \times [0,1] \to [0,1]$ is defined by $t(a, b) = \min\{a, b\}$, then $(X, F, t)$ is a Menger space. It is complete if $(X, d)$ is complete.

## 3. MAIN THEOREM

In this section we prove some common fixed point theorems by using seven compatible mappings in complete Menger spaces.

In fact we prove following results.

**Theorem 3.1:** Let $A, B, S, T$ and $P$ be self maps on a complete Menger space $(X, F, *)$ with $t * t \geq t$ for all $t \in [0,1]$, satisfying;
3.1 (a) $P(X) \subseteq AB(X), P(X) \subseteq ST(X)$;
3.1 (b) there exists a constant $k \in (0,1)$ such that

$$M_{Px,Py}(kt) \geq M_{ABx,Px}(t) * M_{Px,STy}(t) * M_{ABx,STy}(t) * \frac{M_{Px,ABx}(t) * M_{Px,STy}(t)}{M_{STy,ABx}(t)} * M_{Px,Py}(t) * M_{Px,Px}(t) * M_{ABx,Py}(n-\alpha)t$$

for all $x, y \in X, \alpha \in (0, n)$ and $t > 0$,
**3.** 1 (c) $PB = BP, PT = TP, AB = BA$ and $ST = TS$,
3.1 (d) $A$ and $B$ are continuous,
3.1 (e) the pair $(P, AB)$ is compatible (if compatible then it is weak compatible). Then $A, B, S, T$ and $P$ have a common fixed point in $X$.

**Proof:** Since $P(X) \subset AB(X)$, for $x_0 \in X$, we can choose a point $x_0 \in X$ such that $Px_0 = ABx_1$. Since $P(X) \subset ST(X)$, for this point $x_1$, we can choose a point $x_2 \in X$ such that $Px_1 = STx_2$. Thus by induction, we can define a sequence $y_n \in X$ as follows;

$$y_{2n} = Px_{2n} = ABx_{2n+1}$$

and $$y_{2n+1} = Px_{2n+1} = STx_{2n+1}$$

for $n = 1,2, ....$from 3.1 (b) ,
for all $t > 0$ and $\alpha = 2 - q$ with $q \in (0,2)$, we have

$$M_{y_{2n+1},y_{2n+2}}(kt) = M_{Px_{2n+1},Px_{2n+2}}(kt) \geq M_{y_{2n+1},y_{2n+1}}(t) * M_{y_{2n},y_{2n+1}}(t)$$
$$M_{y_{2n+1},y_{2n+2}}(t) M_{y_{2n+1},y_{2n+1}}(t) * M_{y_{2n},y_{2n+1}}(t) * \frac{M_{y_{2n+1},y_{2n}}(t) * M_{y_{2n+1},y_{2n+1}}(t)}{M_{y_{2n+1},y_{2n}}(t)} * M_{y_{2n},y_{2n+2}}(1+q)t,$$

$$\begin{aligned} M_{y_{2n+1},y_{2n+2}}(kt) &\geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n},y_{2n+2}}(1+q)t \\ &\geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n+1},y_{2n+2}}(qt) \\ &\geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n+1},y_{2n+2}}(t) \end{aligned}$$

as $q \to 1$. Since $*$is continuous and $M_{x,y}(*)$ is continuous, letting $q \to 1$ in above equation

We get

$$M_{y_{2n+1},y_{2n+2}}(kt) \geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n+1},y_{2n+2}}(t) \dots \dots \quad 3.1\ (i)$$

Similarly, we have

$$M_{y_{2n+2},y_{2n+3}}(kt) \geq M_{y_{2n+1},y_{2n+2}}(t) * M_{y_{2n+2},y_{2n+2}}(t) \dots \dots \quad 3.1\ (ii)$$

Thus from 2.1 (i) and 2.1 (ii), it follows that

$$M_{y_{n+1},y_{n+2}}(kt) \geq M_{y_n,y_{n+1}}(t) * M_{y_{n+1},y_{n+2}}(t)$$

for $n = 1,2, \ldots$ and then for positive integers n and p,

$$M_{y_{n+1},y_{n+2}}(kt) \geq M_{y_n,y_{n+1}}(t) * M_{y_{n+1},y_{n+2}}\left(\frac{t}{k^p}\right).$$

Thus, since $M_{y_{n+1},y_{n+1}}\left(\frac{t}{k^p}\right) \to 1$ as $p \to \infty$ we have

$$M_{y_{n+1},y_{n+2}}(kt) \geq M_{y_n,y_{n+1}}(t).$$

$y_n$ is Cauchy sequence in X and since x is complete, $y_n$ converges to a point $z \in X$. Since $Px_n, ABx_{2n+1}$ and $STx_{2n+2}$ are subsequences of $y_n$, they also converge to the point $z$, since $A, B$ are continuous and pair $\{P, AB\}$ is compatible and also weak compatible, we have

$$\lim_{n\to\infty} PABx_{2n+1} = ABz$$

and $$\lim_{n\to\infty} (AB)^2 x_{2n+1} = ABz,$$

From 2.1 (b) with $\alpha = 2$, we get

$$M_{PABx_{2n+1},\ Px_{2n+2}}(kt) \geq M_{(AB)^2x_{2n+1},\ STx_{2n+2}}(t) * M_{PABx_{2n+1},\ STx_{2n+2}}(t)$$
$$M_{PABx_{2n+1},\ Px_{2n+2}}(t) * M_{PABx_{2n+1}PABx_{2n+1}}(t) *$$
$$* M_{(AB)^2x_{2n+1},\ STx_{2n+2}}(t) * \frac{M_{PABx_{2n+1},(AB)^2 x_{2n+1}}(t) * M_{PABx_{2n+1},\ STx_{2n+2}}(t)}{M_{STx_{2n+2},\ (AB)^2 x_{2n+1}}(t)}$$
$$* M_{(AB)^2x_{2n+1},\ Px_{2n+2}}(t)$$

which implies that

$$\begin{aligned} M_{ABz,z}(kt) &= \lim_{n\to\infty} M_{PABx_{2n+2}}(kt) \\ &\geq 1 * M_{ABz,z}(t) M_{ABz,z}(t) M_{ABz,Bz}(kt) * M_{ABz,z}(t) * \frac{1 * M_{ABz,z}(t)}{M_{z,ABz}(t)} * M_{ABz,z,z}(t) \end{aligned}$$

We have
$ABz = z$, since $M_{z,STz}(t) \geq M_{z,ABz}(t) = 1$ for all $t > 0$,

We get $STz = z$. Again by 3.1 (b) with $\alpha = 2$,

We have

$$M_{PABx_{2n+1},\,Pz}(kt) \geq M_{(AB)^2x_{2n+1},\ PABx_{2n+1}}(t) * M_{PABx_{2n+1},\,STz}(t) * M_{PABx_{2n+1},\,Pz}\Big(t) * M_{PABx_{2n+1},\,PABx_{2n+1},}(kt)\Big)$$
$$* M_{(AB)^2x_{2n+1},\,STz}(t) * \frac{M_{PABx_{2n+1},(AB)^2x_{2n+1}}(t) * M_{PABx_{2n+1},\,STz}(t)}{M_{STz,(AB)^2x_{2n+1}}(t)} * M_{(AB)^2x_{2n+1}\,Pz}(t)$$

which implies that

$$M_{ABz,Pz,Pz}(kt) = \lim_{n\to\infty} M_{PABx_{2n+1},\,Pz}(kt)$$
$$\geq 1 * 1 * 1 * 1 * 1 * 1M_{ABz,Pz}(t)$$
$$\geq M_{ABz,Pz}(t).$$

We have $ABz = Pz$. Now, we show that $Bz = z$.Infact, from 2.1(b) with $\alpha = n$, and 3.1 (c) we get,

$$M_{Bz,z}(kt) = M_{BPz,Pz}(kt)$$
$$= M_{PBz,Pz}(kt)$$
$$M_{PBz,Pz}(kt) \geq M_{PBz,STz}(t) * M_{ABBz,STz}(t) * M_{PBz,Pz}(t) M_{PBz,PBz}(t)$$
$$* \frac{M_{PBz,\ ABBz}(t) * M_{PBz,z,z}(t)}{M_{z,PBz}(t)} * M_{PBz,z}(t)$$
$$= 1 * M_{Bz,z}(t) * M_{Bz,z}(t) * 1 * 1 * 1 * M_{Bz,z}(t)$$
$$= M_{Bz,z}(t).$$

which implies that $Bz = z$. Since $ABz = z$,

We have $Az = z$. Next, we show that $Tz = z$. Indeed from 6.3.1 (b) with $\alpha = 2$, and 2.1 (c) we get

$$M_{Tz,z}(kt) = M_{TPz,Pz}(kt) = M_{Pz,Pz}(kt)$$
$$\geq 1 * M_{z,Tz}(t) * M_{Pz,Pz}(t) * M_{Pz,Pz}(t) * M_{z,Tz}(t) * 1 * M_{z,Tz}(t)$$
$$\geq M_{Tz,z}(t).$$

which implies that $Tz = z$. Since $STz = z$, we have $Sz = STz = z$. Therefore, by combining the above results we obtain,

$$Az = Bz = Sz = Tz = Pz.$$

Therefore $z$ is the common fixed point of $A, B, S, T$ and $P$.

Finally, the uniqueness of the fixed point of $A, B, S, T$ and $P$.

## REFERENCES

1. Y. J. Cho, P. P. Murthy and M. Stojakovic, Compatible mappings of type and common fixed points in Menger spaces, *Comm. Korean Math. Soc.*, 7 (2) (1992), 325-339.
2. Lj. Ciric, On fixed points of generalized contractions on probabilistic metric spaces, *Publ. Inst. Math.*, 18 (32)(1975), 71-78.1206 *H. K. Pathak and R. K. Verma*
3. R. Dedic and N. Sarapa, On common fixed point theorems for commuting mappings in Menger spaces, *Radovi Mathematicki*, 4 (1988), 269-278.
4. J. Jachymski, Common fixed point theorems for some families of maps, *Indian J. Pure Appl. Math.*, 25 (9) (1994), 925-937; MR1294061 (95g: 54038).
5. G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.*, 9 (1986), 771-779.
6. G. Jungck, Common fixed points for non-continuous, non-self mappings on a non-numeric spaces, *Fareast. J. Math. Sci.*, 4 (2) (1996), 199-212.
7. G. Jungck, P. P. Murthy and Y. J. Cho, Compatible mappings of type (A) and common fixed points, *Jour. Math. Japon.*, 38 (1993), 381-390.
8. H. K. Pathak, Y. J. Cho, S. M. Kang and B. S. Lee, Fixed point theorems for compatible mappings of type (P) and application to dynamic programming, *L´e Mathematiche*, 50 (1995) 15-33.
9. H. K. Pathak, S. M. Kang and J. S. Baek, Weak compatible mappings of type (A) and common fixed points in Menger space, *Comm. Korean. Math. Soc.*, 10 (1995), 63-67.
10. S. N. Mishra, Common fixed points of compatible mappings in PM-spaces, *Math. Japon.*, 36 (2)(1991), 283-289; MR1095742.
11. B. Singh and S. Jain, A fixed point theorem in Menger space through weak-compatibility, *J. Math. Anal. Appl.*, 301 (2005), 439-448; MR2105684.

12. V. M. Sehgal and A. T. Bharucha-Reid, Fixed point of contraction mappings on probabilistic metric spaces, *Math. Syst. Theory*, 6 (1972), 97-102.
13. S. Sessa, On a weak commutative condition in fixed point consideration, *Publ. Inst. Math. (Beograd)*, 32 (1982), 146-153.
14. B. K. Sharma and D. R. Sahu, 2-Compatibility and its application, *Kyungpook J. Math.*, 36 (1) (1996), 171-181.
15. B. Schweizer and A. Sklar, Probabilistic metric spaces, *North Holland series in Probability and Applied Mathematics*, 1983; MR0790314 (86g: 54045).

**Source of support: Nil, Conflict of interest: None Declared.**