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GEOMETRIC DIFFERENCE SEQUENCE SPACES IN NUMERICAL ANALYSIS

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ABSTRACT

Grossman and Katz [4] introduced the non-Newtonian calculus consisting of the branches of geometric, anageometric and bigeometric calculus. Cengin Türkmen and Feyzi Başar [1] have some basic results on the sets of sequences with geometric calculus. The main purpose of this paper is to introduce the geometric difference sequence space $c^G(\Delta_G)$ and prove that $c^G(\Delta_G)$ is a Banach space with respect to norm $\|.\|^G_{\Delta_G}$. Finally we obtain the Geometric Newton-Gregory interpolation formulae.

Key words: *Difference sequence spaces, Geometric Calculus, interpolation formula.*

AMS subject classification (2000): *26A06, 11U10, 46A45, 08A05.*

INTRODUCTION

In 1967 Robert Katz and Michael Grossman created the first system of non-Newtonian calculus, which we call the geometric calculus. In 1970 they had created an infinite family of non-Newtonian calculi, each of which differs markedly from the classical calculus of Newton and Leibniz. Among other things, each non-Newtonian calculus possesses four operators: a gradient (i.e. an average rate of change), a derivative, an average and an integral. For each non-Newtonian calculus there is a characteristic class of functions having a constant derivative.

We should know that all concepts in classical arithmetic have natural counterparts in $\alpha -$ arithmetic. Consider any generator α with range $A \subseteq \mathbb{C}$. By $\alpha -$ arithmetic, we mean the arithmetic whose domain is A and operations are defined as follows. For $x, y \in A$ and any generatorα,

$\alpha - addition$ $\quad x \dot{+} y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)]$
$\alpha - subtraction$ $\quad x \dot{-} y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)]$
$\alpha - multiplication$ $\quad x \;\; y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)]$
$\alpha - division$ $\quad x \dot{/} y = \alpha[\alpha^{-1}(x) / \alpha^{-1}(y)]$
$\alpha - order$ $\quad x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y).$

If we choose exp as an $\alpha - generator$ defined by $\alpha(z) = e^z$ for $z \in \mathbb{C}$ then $\alpha^{-1}(z) = \ln z$ and $\alpha - arithmetic$ turns out to Geometric arithmetic.

$\alpha - addition$ $\quad x \oplus y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] = e^{(\ln x + \ln y)} = x.y$ $geometric\ addition$
$\alpha - subtraction$
$$x \ominus y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] = e^{(\ln x - \ln y)} = x \div y, y \neq 0 \ geometric\ subtraction$$
$\alpha - multiplication$ $\quad x \odot y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] = e^{(\ln x \times \ln y)} = x^{\ln y}$ $geometric\ multiplication$
$\alpha - division$ $\quad x \oslash y = \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)] = e^{(\ln x \div \ln y)} = x^{\frac{1}{\ln y}}, y \neq 1$ $geometric\ division.$

In [11] defined the geometric complex numbers $\mathbb{C}(G)$ as follows:
$$\mathbb{C}(G) := \{e^z : z \in \mathbb{C}\} = \mathbb{C}\ \{0\}.$$
Then $(\mathbb{C}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e.

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Then for all $x, y \in \mathbb{C}(G)$

- $x \oplus y = xy$
- $x \ominus y = x/y$
- $x \odot y = x^{\ln y} = y^{\ln x}$
- $x \oslash y$ or $\frac{x}{y}G = x^{\frac{1}{\ln y}}, y \neq 1$
- $x^{2_G} = x \odot x = x^{\ln x}$
- $x^{p_G} = x^{\ln^{p-1} x}$
- $\sqrt{x}^G = e^{(\ln x)^{\frac{1}{2}}}$
- $x^{-1_G} = e^{\frac{1}{\log x}}$
- $x \odot e = x$ and $x \oplus 1 = x$
- $e^n \odot x = x^n = x \oplus x \oplus \ldots\ldots$ (upto n number of x)
- $\mid x \mid_G = \begin{cases} x, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \\ \frac{1}{x}, & \text{if } x < 1 \end{cases}$

  Thus $\mid x \mid_G \geq 1$.
- $\sqrt{x^{2_G}}^G = \mid x \mid_G$
- $\mid e^y \mid_G = e^{\mid y \mid}$
- $\mid x \odot y \mid_G = \mid x \mid_G \odot \mid y \mid_G$
- $\mid x \oplus y \mid_G \leq \mid x \mid_G \oplus \mid y \mid_G$
- $\mid x \oslash y \mid_G = \mid x \mid_G \oslash \mid y \mid_G$
- $\mid x \ominus y \mid_G \geq \mid x \mid_G \ominus \mid y \mid_G$
- $0_G \ominus 1_G \odot (x \ominus y) = y \ominus x, i.e.$ in short $\ominus (x \ominus y) = y \ominus x$.

Let l_∞, c and c_0 be the linear spaces of complex bounded, convergent and null sequences, respectively, normed by

$$\|x\|_\infty = \sup_k \mid x_k \mid.$$

Türkmen and Başar [11] have proved that

$$\omega(G) = \{(x_k): x_k \in \mathbb{C}(G) \text{ for all } k \in \mathbb{N}\}$$

is a vector space over $\mathbb{C}(G)$ with respect to the algebraic operations $\oplus$ addition and $\odot$ multiplilcation

$$\oplus: \omega(G) \times \omega(G) \to \omega(G)$$
$$(x, y) \to x \oplus y = (x_k) \oplus (x_y) = (x_k y_k)$$
$$\odot: \mathbb{C}(G) \times \omega(G) \to \omega(G)$$
$$(\alpha, y) \to \alpha \odot y = \alpha \odot (y_k) = (\alpha^{\ln y k}),$$

where $x = (x_k), y = (y_k) \in \omega(G)$ and $\alpha \in \mathbb{C}(G)$. Then

$l_\infty(G) = \{x = (x_k) \in \omega(G): sup_{k \in \mathbb{N}} \mid x_k \mid_G < \infty\}$

$c(G) = \{x = (x_k) \in \omega(G): G \lim_{k \to \infty} \mid x_k \ominus l \mid_G = 1\}$

$c_0(G) = \{x = (x_k) \in \omega(G): G \lim_{k \to \infty} x_k = 1\}$, where ${}_G \lim$ is the geometric limit

$l_p(G) = \{x = (x_k) \in \omega(G): {}_G\sum_{k=0}^{\infty} (\mid x_k \mid_G)^{pG} < \infty\}$, where ${}_G\sum$ is the geometric sum,

are classical sequence spaces over the field $\mathbb{C}(G)$. Also it is shown that $l_\infty(G), c(G)$ and $c_0(G)$ are Banach spaces with the norm

$$\|x\|_G = \sup_k \mid x_k \mid_G, x = (x_1, x_2, x_3 \ldots) \in \lambda(G), \lambda \in \{l_\infty, c, c_0\}.$$

For the convenience, in this paper we denote $l_\infty(G), c(G), c_0(G)$, respectively as l_{G_∞}, c^G, c_0^G

In 1981, Kizmaz [6] introduced the notion of difference sequence spaces using forward difference operator Δ and studied the classical difference sequence spaces $l_\infty(\Delta), c(\Delta), c_0(\Delta)$. In this section we define the following new geometric sequence space

$$l_{G_\infty}(\Delta_G) = \{x = (x_k) \in \omega(G): \Delta_G x \in l_{G_\infty}\}, \text{ where } \Delta_G x = x_k \ominus x_{k+1}.$$

Theorem: *The space $l_{G_\infty}(\Delta_G)$ is a normed linear space $w.r.t.$ the norm*

$$\|x\|_{\Delta_G}^G = \mid x_1 \mid_G \oplus \|\Delta_G x\|_{G_\infty}.$$

Theorem: *The space $c^G(\Delta_G)$ is a Banach space w.r.t. the norm $\|.\|^G_{\Delta_G}$.*

Proof: Let (x_n) be a Cauchy sequence in $c^G(\Delta_G)$, where $x_n = (x_k^{(n)}) = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots\dots)\forall n \in \mathbb{N}, x_k^{(n)}$ is the k^{th} coordinate of x_n. Then

$$\begin{aligned}\|x_n \ominus x_m\|^G_{\Delta_G} &= |x_1^{(n)} \ominus x_1^{(m)}|^G \oplus \|\Delta_G x_n \ominus \Delta_G x_m\|^G_\infty \to 1 \text{ as } m,n \to \infty\\ &= |x_1^{(n)} \ominus x_1^{(m)}|^G \oplus \|(x_k^{(n)} \ominus x_{k+1}^{(n)}) \ominus (x_k^{(m)} \ominus x_{k+1}^{(m)})\|^G_\infty \to 1\\ &= |x_1^{(n)} \ominus x_1^{(m)}|^G \oplus \|x_{k+1}^{(n)} \ominus x_{k+1}^{(m)} \ominus (x_k^{(n)} \ominus x_{k+1}^{(m)})\|^G_\infty \to 1\\ &= |x_1^{(n)} \ominus x_1^{(m)}|^G \oplus \sup_k |(x_k^{(n)} \ominus x_k^{(m)}) \ominus (x_{k+1}^{(n)} \ominus x_{k+1}^{(m)})\|^G_\infty \to 1 \text{ as } m,n \to \infty.\end{aligned}$$

This implies that $\left|x_1^{(n)} \ominus x_1^{(m)}\right|^G \to 1$ as $n,m \to \infty\ \forall\ k \in \mathbb{N}$, since $\left|x_1^{(n)} \ominus x_1^{(m)}\right|^G \geq 1$.

Therefore for fixed k, k^{th} co-ordinates of all sequences form a Cauchy sequence in $\mathbb{C}(G)$ i.e. $x_k^{(n)} = (x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, x_k^{(4)}, \dots\dots\dots)$ is a Cauchy sequence. Then by the completeness of $\mathbb{C}(G), (x_k^{(n)})$ converges to x_k (say) as follows:

$$\begin{array}{llllllll}
x_1 & = (& x_1^{(1)}, & x_2^{(1)}, & x_3^{(1)}, & \dots, & x_k^{(1)}, & \dots)\\
x_2 & = (& x_1^{(2)}, & x_2^{(2)}, & x_3^{(2)}, & \dots, & x_k^{(2)}, & \dots)\\
x_3 & = (& x_1^{(3)}, & x_2^{(3)}, & x_3^{(3)}, & \dots, & x_k^{(3)}, & \dots)\\
. & & . & . & . & & . & \\
. & & . & . & . & & . & \\
. & & . & . & . & & . & \\
x_m & = (& x_1^{(m)}, & x_2^{(m)}, & x_3^{(m)}, & \dots, & x_k^{(m)}, & \dots)\\
. & & . & . & . & & . & \\
. & & . & . & . & & . & \\
. & & . & . & . & & . & \\
x_n & = (& x_1^{(n)}, & x_2^{(n)}, & x_3^{(n)}, & \dots, & x_k^{(n)}, & \dots)\\
. & & . & . & . & & . & \\
. & & . & . & . & & . & \\
. & & . & . & . & & . & \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & \\
x & = (& x_1, & x_2, & x_3, & \dots, & x_k, & \dots)
\end{array}$$

i.e.

$$G\lim_{n\to\infty} x_k^{(n)} = x_k\ \forall k \in \mathbb{N}.$$

Further for each $\varepsilon > 1, \exists N = N(\varepsilon)$ s.t. $\forall n,m \geq N$ we have

$$\left|x_1^{(n)} \ominus x_1^{(m)}\right|^G < \varepsilon \left|x_{k+1}^{(n)} \ominus x_{k+1}^{(m)} \ominus (x_k^{(n)} \ominus x_k^{(m)})\right|^G < \varepsilon$$

and

$$G\lim_{m\to\infty} \left|x_1^{(n)} \ominus x_1^{(m)}\right|^G = \left|x_1^{(n)} \ominus x_1\right|^G < \varepsilon.$$

This implies

$$G\lim_{n\to\infty} |(x_{k+1}^{(n)} \ominus x_{k+1}^{(m)}) \ominus (x_k^{(n)} \ominus x_k^{(m)})|^G = |(x_{k+1}^{(n)} \ominus x_{k+1}) \ominus (|x_k^{(n)} \ominus x_k)|^G < \varepsilon\ \forall\ n \geq N.$$

Since ε is independent of k,

$$\begin{aligned}&\sup_k |(x_{k+1}^{(n)} \ominus x_{k+1}) \ominus (x_k^{(n)} \ominus x_k)|^G < \varepsilon\\ \Longrightarrow &\sup_k |(x_{k+1}^{(n)} \ominus x_k^{(n)}) \ominus (x_{k+1} \ominus x_k)|^G = \|\Delta_G x_n \ominus \Delta_G x\|^G_\infty < \varepsilon.\end{aligned}$$

Consequently we have $\|x_n \ominus x\|^G_{\Delta_G} = |x_1^{(n)} \ominus x_1|^G \oplus \|\Delta_G x_n \ominus \Delta_G x\|^G_\infty < \varepsilon^2\ \forall\ n \geq N$.

Hence we obtain $x_n \to x$ as $n \to \infty$.

Now we must show that $x \in c^G(\Delta_G)$. We have

$$\begin{aligned}|x_k \ominus x_{k+1}|^G &= |x_k \ominus x_k^N \ominus x_{k+1}^N \oplus x_{k+1}^N \ominus x_{k+1}|^G\\ &\leq |x_k^N \ominus x_{k+1}^N|^G \oplus \|x^N \ominus x\|^G_{\Delta_G} = O(e).\end{aligned}$$

This implies $x = (x_k) \in c^G(\Delta_G)$.

Now we define $s: l_{G_\infty}(\Delta_G) \to l_\infty(\Delta_G), x \to sx = y = (1, x_2, x_3, \ldots.)$. It is clear that s is a bounded linear operator on $l_\infty(\Delta_G)$ and $\|s\|_{G\infty} = e$.
Also

$$s[l_{G_\infty}(\Delta_G)] = sl_{G_\infty}(\Delta_G) = \{x = (x_k) : x \in l_{G_\infty}(\Delta_G), x_1 = 1\} \subset l_{G_\infty}(\Delta_G)$$

is a subspace of $l_\infty^G(\Delta_G)$ and as $|x_1|^G = 1$ for $x_1 = 1$ we have

$$\|x\|_{\Delta_G}^G = \|\Delta_G x\|_\infty^G \text{ in } sl_{G_\infty}(\Delta_G).$$

On the other hand we can show that

$$\Delta_G: sl_{G_\infty}(\Delta_G) \to l_{G_\infty}$$
$$x = (x_k) \to y = (y_k) = (x_k \ominus x_{k+1})$$

is a linear homomorphism. So $sl_{G_\infty}(\Delta_G)$ and l_{G_∞} are equivalent as topological space. Δ_G and Δ_G^{-1} are norm preserving and $\|\Delta_G\|_{G\infty} = \|\ \|_{G\infty} = e$.

Let $[sl_{G_\infty}(\Delta_G)]^*$ and $[l_{G\infty}]^*$ denote the continuous duals of $sl_{G_\infty}(\Delta_G)$ and $l_{G\infty}$, respectively. We can prove that

$$T := [sl_{G_\infty}(\Delta_G)]^* \to [l_{G_\infty}]^*, f_{\Delta_G} \to f = f_{\Delta_G} o \Delta_G^{-1}$$

is a linear isometry. Thus $[sl_{G_\infty}(\Delta_G)]^*$ is equivalent to $[l_{G_\infty}]^*$. In the same way we can show that $sc^G(\Delta_G)$ and $c^G, sc_0^G(\Delta_G)$ and c_0^G are equivalent as topological spaces and $[sc^G(\Delta_G)]^* = [sc_0^G(\Delta_G)]^* = l_{G_1}$ (l_{G_1}, the space of geometric absolutely convergent series).

2. GEOMETRIC FORM OF ABEL'S PARTIAL SUMMATION FORMULA

Abel's partial summation formula states that if $(a_k)$and (b_k) are sequences then

$$\sum_{k=1}^n a_k b_k = \sum_{k=1}^n S_k (b_k - b_{k+1}) + S_n b_{n+1},$$

where $S_k = \sum_{i=1}^k a_i$. Then

$$\sum_{k=1}^\infty a_k b_k = \sum_{k=1}^\infty S_k (b_k - b_{k+1}) + lim_{n\to\infty} S_n b_{n+1}$$
$$\sum_{k=1}^\infty a_k b_k = \sum_{k=1}^\infty S_k (b_k - b_{k+1}), \text{ if } (b_k) \text{ monotonically decreases to zero.}$$

Similarly as $\odot$ is distributive over $\oplus$ we have

$$G\sum_{k=1}^\infty a_k \odot b_k = G\sum_{k=1}^\infty S_k \odot (b_k \ominus b_{k+1}), \text{ where } S_k = G\sum_{i=1}^k a_i.$$

In particular, if $(b_k) = (e^{-k})$, then (b_k) monotonically decreases to zero. Then $G\sum_{k=1}^\infty a_k \odot e^{-k} = G\sum_{k=1}^\infty S_k \odot (e^{-k} \ominus e^{-(k+1)}) = G\sum_{k=1}^\infty S_k \odot e = G\sum_{k=1}^\infty S_k$.

Let (p_n) be a sequence of geometric positive numbers monotonically increasing to infinity. Then $(\frac{e}{p_n}G)$ is a sequence monotonically decreasing to zero (i.e. to 1).

Lemma 2.1: $If\ sup_n |G\sum_{v=1}^n c_v|^G \le \infty\ then\ sup_n \left(p_n \odot |G\sum_{k=1}^\infty \frac{c_{n+k-1}}{p_{n+k}} G|^G\right) < \infty.$

Lemma 2.2: If the series $\sum_{k=1}^\infty c_k$ is convergent then

$$lim_n \left(p_n \odot G\sum_{k=1}^\infty \frac{c_{n+k-1}}{p_{n+k}} G\right) = 1.$$

Corollary 2.3: Let (p_n) be monotonically increasing. If

$$sup_n |G\sum_{v=1}^n p_v \odot a_v|^G < \infty\ then\ sup_n |p_n \odot G\sum_{k=n+1}^\infty a_k|^G < \infty.$$

Proof: We put $p_{k+1} \odot a_{k+1}$ instead of c_k in Lemma 2.1 we get

$$\begin{aligned} p_n \odot G\sum_{k=1}^\infty \frac{c_{n+k-1}}{p_{n+k}} G &= p_n \odot G\sum_{k=1}^\infty \frac{p_{n+k} \odot a_{n+k}}{p_{n+k}} G \\ &= p_n \odot G\sum_{k=1}^\infty a_{n+k} \\ &= p_n \odot G\sum_{k=n+1}^\infty a_k = O(e). \end{aligned}$$

Corollary 2.4: If $G\sum_{k=1}^\infty p_k \odot a_k$ is convergent then

$$lim_n\, p_n \odot G\sum_{k=n+1}^\infty a_k = 1$$

Corollary 2.5: $G\sum_{k=1}^\infty e^k \odot a_k$ is convergent iff $G\sum_{k=1}^\infty R_k$ is convergent with $e^n \odot R_n = O(e)$, where $R_n = G\sum_{k=n+1}^\infty a_k$.

3. SOME APPLICATIONS OF GEOMETRIC DIFFERENCE

In this section we find the Geometric Newton-Gregory interpolation formulae.

Geometric Factorial: Let us define geometric factorial notation $!_G$as

$$n!_G = e^n \odot e^{n-1} \odot e^{n-2} \odot \ldots . \odot e^2 \odot e = e^{n!}.$$

For example,

$$0!_G = e^{0!} = e^0 = 1$$
$$1!_G = e^{1!} = e = 2.71828$$
$$2!_G = e^{2!} = e^2 = 7.38906$$
$$3!_G = e^{3!} = e^6 = 4.03429 \times 10^2$$
$$4!_G = e^{4!} = e^{24} = 2.64891 \times 10^{10}$$
$$5!_G = e^{5!} = e^{120} = 1.30418 \times 10^{52} \qquad \text{etc.}$$

Generalized Geometric Forward Difference Operator: Let

$$\Delta_G f(a) = f(a \oplus h) \ominus f(a).$$
$$\begin{aligned}\Delta_G^2 f(a) &= \Delta_G f(a \oplus h) \ominus \Delta_G f(a)\\ &= \{f(a \oplus e^2 \odot h) \ominus f(a \oplus h)\} \ominus \{f(a \oplus h) \ominus f(a)\}\\ &= f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \oplus f(a).\end{aligned}$$
$$\begin{aligned}\Delta_G^3 f(a) &= \Delta_G^2 f(a \oplus h) \ominus \Delta_G^2 f(a)\\ &= \{f(a \oplus e^3 \odot h) \ominus e^2 \odot f(a \oplus e^2 \odot h) \oplus f(a \oplus h)\}\\ &\quad \ominus \{f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \oplus f(a)\}\\ &= f(a \oplus e^3 \odot h) \ominus e^3 \odot f(a \oplus e^2 \odot h) \oplus e^3 \odot f(a \oplus h) \ominus f(a).\end{aligned}$$

Thus, n^{th} geometric forward difference is

$$\Delta_G^n f(a) =_G \sum_{k=0}^{n} (\ominus\, e)^{k_G} \odot e^{\binom{n}{k}} \odot f(a \oplus e^{n-k} \odot h), \text{ with } (\ominus\, e)^{0_G} = e.$$

Generalized Geometric Backward Difference Operator: Let

$$\nabla_G f(a) = f(a) \ominus f(a \ominus h).$$
$$\begin{aligned}\nabla_G^2 f(a) &= \nabla_G f(a) \ominus \nabla_G f(a \ominus h)\\ &= \{f(a) \ominus f(a \ominus h)\} \ominus \{f(a \ominus h) \ominus f(a \ominus e^2 \odot h)\}\\ &= f(a) \ominus e^2 \odot f(a \ominus h) \oplus f(a \ominus e^2 \odot h)\}.\end{aligned}$$

$$\begin{aligned}\nabla_G^3 f(a) &= \nabla_G^2 f(a) \ominus \nabla_G^2 f(a \ominus h)\\ &= \{f(a) \ominus e^2 \odot f(a \ominus h) \oplus f(a \ominus e^2 \odot h)\}\\ &\quad \ominus \{f(a \ominus h) \ominus e^2 \odot f(a \ominus e^2 \odot h) \oplus f(a \ominus e^3 \odot h)\}\\ &= f(a) \ominus e^3 \odot f(a \ominus h) \oplus e^3 \odot f(a \ominus e^2 \odot h) \ominus f(a \ominus e^3 \odot h).\end{aligned}$$

Thus, n^{th} geometric backward difference is

$$\Delta_G^n f(a) =_G \sum_{k=0}^{n} (\ominus\, e)^{k_G} \odot e^{\binom{n}{k}} \odot f(a \ominus e^k \odot h).$$

Factorial Function: The product of n consecutive factors each at a constant geometric difference, h, the first factor being x is called a factorial function of degree n and is denoted by $x^{(nG)}$. Thus
$x^{(nG)} = x \odot (x \ominus e \odot h) \odot (x \ominus e^2 \odot h) \odot (x \ominus e^3 \odot h) \odot \ldots \odot (x \ominus e^{n-1} \odot h).$

In particular, for $h = e$,

$$x^{(nG)} = x \odot (x \ominus e) \odot (x \ominus e^2) \odot (x \ominus e^3) \odot \ldots \odot (x \ominus e^{n-1}).$$

Geometric Newton-Gregory Forward Interpolation Formula
Let $y = f(x)$ be a function which takes the values
$f(a), f(a \oplus h), f(a \oplus e^2 \odot h), f(a \oplus e^3 \odot h), \ldots\ldots, f(a \oplus e^n \odot h)$ for the $n+1$ geometrically equidistant values (which form a Geometric Progression in ordinary sense) $a, a \oplus h, a \oplus e^2 \odot h, a \oplus e^3 \odot h, \ldots\ldots, a \oplus e^n \odot h$ of the independent variable x and let $P_n(x)$ be a geometric polynomial in x of degree n defined as:

$$\begin{aligned}P_n(x) = A_0 \oplus A_1 \odot (x \ominus a) \oplus A_2 \odot (x \ominus a) \odot (x \ominus a \ominus h) \oplus A_3 \odot (x \ominus a) \odot (x \ominus a \ominus h)\\ \odot (x \ominus a \ominus e^2 \odot h) \oplus \ldots \oplus A_n \odot (x \ominus a) \odot (x \ominus a \ominus h) \odot \ldots\\ \odot (x \ominus a \ominus e^{n-1} \odot h). \qquad (i)\end{aligned}$$

We choose the coefficients $A_0, A_1, A_2, \ldots., A_n$ such that
$P_n(a) = f(a), P_n(a \oplus h) = f(a \oplus h), P_n(a \oplus e^2 \odot h) = f(a \oplus e^3 \odot h), \ldots., p_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h).$

Putting $x = a, a \oplus h, a \oplus e^2 \odot h, a \oplus e^3 \odot h, \dots\dots, a \oplus e^n \odot h$ in (i) and then also putting the values of $P_n(a), P_n(a \oplus h), \dots\dots, P_n(a \oplus e^n \odot h)$, we get

$$f(a) = A_0 \Rightarrow A_0 = f(a).$$

$$f(a \oplus h) = A_0 \oplus A_1 \odot h \Rightarrow A_1 = \frac{f(a \oplus h) \ominus f(a)}{h} G = \frac{\Delta_G f(a)}{h} G.$$

$$f(a \oplus e^2 \odot h) = A_0 \oplus e^2 \odot h \odot A_1 \oplus e^2 \odot h \odot h \odot A_2$$

$$\Rightarrow A_2 = \frac{f(a \oplus e^2 \odot h) \ominus e^2 \odot [f(a \oplus h) \ominus f(a)] \ominus f(a)}{e^2 \odot h^{2_G}} G$$

$$= \frac{f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \ominus f(a)}{2!_G \, e^2 \odot h^{2_G}} G$$

$$= \frac{\Delta_G^2 f(a)}{2!_G \odot h^{2_G}} G$$

Similarly $A_3 = \frac{\Delta_G^3 f(a)}{3!_G \odot h^{3_G}} G$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$A_n = \frac{\Delta_G^n f(a)}{n!_G \odot h^{n_G}} G.$$

Putting the values of $A_0, A_1, A_2, \dots, A_n$ found above in (i), we get

$$P_n(x) = f(a) \oplus \frac{\Delta_G f(a)}{h} G \odot (x \ominus a) \oplus \frac{\Delta_G^2 f(a)}{2!_G \odot h^{2_G}} G \odot (x \ominus a) \odot (x \ominus a \ominus h) \oplus \frac{\Delta_G^3 f(a)}{3!_G \odot h^{3_G}} G \odot (x \ominus a) \odot (x \ominus a \ominus h) \odot (x \ominus a \ominus e^2 \odot h) \oplus \dots \oplus \frac{\Delta_G^n f(a)}{n!_G \odot h^{n_G}} G \odot (x \ominus a) \odot (x \ominus a \ominus h) \odot \dots \odot (x \ominus a \ominus e^{n-1} \odot h).$$

This is the Geometric Newton-Gregory forward interpolation formula. Putting $\frac{x \ominus a}{h} G = u$ or $x = a \oplus h \odot u$, formula takes the form

$$P_n(x) = f(a) \oplus u \odot \Delta_G f(a) \oplus \frac{u \odot (u \ominus e)}{2!_G} G \odot \Delta_G^2 f(a) \oplus \frac{u \odot (u \ominus e) \odot (u \ominus e^2)}{3!_G} G \odot \Delta_G^3 f(a) \oplus \dots \oplus \frac{u \odot (u \ominus e) \odot (u \ominus e^2) \odot \dots \odot (u \ominus e^{n-1})}{n!_G} G \odot \Delta_G^n f(a). \qquad (ii)$$

The result (ii) can be written as

$$P_n(x) = P_n(a \oplus h \odot u) = f(a) \oplus u^{(1_G)} \odot \Delta_G f(a) \oplus \frac{u^{(2_G)}}{2!_G} G \odot \Delta_G^2 f(a) \oplus \frac{u^{(3_G)}}{3!_G} G \odot \Delta_G^3 f(a) \oplus \dots\dots \oplus \frac{u^{(n_G)}}{n!_G} G \odot \Delta_G^n f(a).$$

Where $u^{(n_G)} = u \odot (u \ominus e) \odot (u \ominus e^2) \odot \dots \odot (u \ominus e^{n-1})$.

Geometric Newton-Gregory Backward Interpolation Formula

Let $y = f(x)$ be a function which takes the values $f(a \oplus e^n \odot h), f(a \oplus e^{n-1} \odot h), f(a \oplus e^{n-2} \odot h), f(a \oplus e^{n-3} \odot h), \dots\dots, f(a)$ for the $n+1$ geometrically equidistant values $a \oplus e^n \odot h, a \oplus e^{n-1} \odot h, a \oplus e^{n-2} \odot h, a \oplus e^{n-3} \odot h, \dots\dots, a$ of the independent variable x and let $P_n(x)$ be a geometric polynomial in x of degree n defined as:

$$P_n(x) = A_0 \oplus A_1 \odot (x \ominus a \ominus e^n \odot h) \oplus A_2 \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \oplus A_3 \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \odot (x \ominus a \ominus e^{n-2} \odot h) \oplus \dots \oplus A_n \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \odot \dots \odot (x \ominus a \ominus h). \qquad (iii)$$

where $A_0, A_1, A_2, \dots, A_n$ are constants which are to be determined so as to make

$P_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h), P_n(a \oplus e^{n-1} \odot h) = f(a \oplus e^{n-1} \odot h), \dots, P_n(a) = f(a)$

Putting $x = a \oplus e^n \odot h, a \oplus e^{n-1} \odot h, \dots\dots$ in (iii) and also putting $P_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h), \dots\dots$, we get

$$A_0 = f(a \oplus e^n \odot h)$$

$$A_1 = \frac{\nabla_G f(a \oplus e^n \odot h)}{h} G$$

$$A_2 = \frac{\nabla_G^2 f(a \oplus e^n \odot h)}{2!_G \odot h^{2_G}} G$$

$$A_3 = \frac{\nabla_G^3 f(a \oplus e^n \odot h)}{3!_G \odot h^{3_G}} G \dots \qquad \dots\dots\dots\dots\dots\dots\dots$$

$$A_n = \frac{\nabla_G^n f(a \oplus e^n \odot h)}{n!_G \odot h^{n_G}} G$$

Substituting the values of $A_0, A_1, A_2, \ldots$ in (iii), we get

$$\begin{aligned}P_n(x) = f(a \oplus e^n \odot h) \oplus \frac{\nabla_G f(a \oplus e^n \odot h)}{h} G \odot (x \ominus a \ominus e^n \odot h) \oplus \frac{\nabla_G^2 f(a \oplus e^n \odot h)}{2!_G \odot h^{2_G}} G \\ \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \oplus \frac{\nabla_G^3 f(a \oplus e^n \odot h)}{3!_G \odot h^{3_G}} G \odot (x \ominus a \ominus e^n \odot h) \\ \odot (x \ominus a \ominus e^{n-1} \odot h) \odot (x \ominus a \ominus e^{n-2} \odot h) \oplus \ldots \oplus \frac{\nabla_G^n f(a \oplus e^n \odot h)}{n!_G \odot h^{n_G}} G \\ \odot (x \ominus a \ominus e^n \odot h) \odot (x \ominus a \ominus e^{n-1} \odot h) \odot \ldots \odot (x \ominus a \ominus h). \qquad (iv)\end{aligned}$$

This is the Geometric Newton-Gregory backward interpolation formula. Putting $u = \frac{x \ominus (a \oplus e^n \odot h)}{h} G$ or $x = a \oplus e^n \odot h \oplus u \odot h$, we get

$$\begin{aligned}P_n(x) &== P_n(a \oplus e^n \odot h \oplus u \odot h) \\ &= f(a \oplus e^n \odot h) \oplus u \odot \nabla_G f(a \oplus e^n \odot h) \oplus \frac{u \odot (u \oplus e)}{2!_G} G \odot \nabla_G^2 f(a \oplus e^n \odot h) \\ &\oplus \frac{u \odot (u \oplus e) \odot (u \oplus e^2)}{3!_G} G \odot \nabla_G^3 f(a \oplus e^n \odot h) \oplus \ldots \\ &\oplus \frac{u \odot (u \oplus e) \odot (u \oplus e^2) \odot \ldots \odot (u \oplus e^{n-1})}{n!_G} G \odot \nabla_G^n f(a \oplus e^n \odot h).\end{aligned}$$

Advantages of Geometric Interpolation Formulae over Ordinary Interpolation Formulae

All the ordinary interpolation formulae are based upon the fundamental assumption that the data is expressible or can be expressed as a polynomial function with fair degree of accuracy. But geometric interpolation formulae have no such restriction. Because geometric interpolation formulae are based on geometric polynomials which are not polynomials in ordinary sense. So geometric interpolation formulae can be used to generate transcendental functions, mainly to compute exponential and logarithmic functions. Also geometric forward and backward interpolation formulae are based on the values of the argument that are geometrically equidistant but need not be equidistant like classical interpolation formulae.

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