

ON THE LAPLACIAN ENERGY AND COLOR SIGNLESS LAPLACIAN ENERGY
OF BOOK GRAPH B_n

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ABSTRACT

A book graph B_n is a graph $S_n \times P_2$ where S_n is a star graph with n edges. The energy $E(G)$ of a simple graph G is the sum of the absolute values of the eigenvalues of the adjacency matrix of G . In this paper, the Energy, Laplacian Energy, and Color Signless Laplacian Energy of Book Graph B_n were determined.

Keywords: Book Graph, Laplacian Energy, Color Signless Laplacian Energy.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. Let A be the adjacency matrix of G , and let the eigenvalues of A be

$$\lambda_1 \geq \dots \geq \lambda_n.$$

In 1978, Gutman [3] defined the energy of a graph G , $E(G)$, as

$$E(G) = \sum_{i=1}^n |\lambda_i|$$

This concept originated from chemistry to estimate the total π -electron energy of a molecule. The conjugated hydrocarbons can be represented by a graph called molecular graph where carbon atoms are the vertices while carbon-carbon bonds are the edges and hydrogen atoms are ignored. The eigenvalues of the molecular graph are the energy level of the electron in the molecule. [4]

Recently, graph energy has become the focus of interest among mathematicians and different energies were observed such as Laplacian energy, color energy and color signless Laplacian energy.

In this paper, we calculate the Energy, Laplacian Energy, and Color Signless Laplacian Energy of a Book Graph B_n which is a Cartesian product of a star graph S_n and path graph P_2 .

1.1 Laplacian Energy

Let $\mu_1 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of the Laplacian matrix $L = (L_{ij})$ of G where

$$L_{ij} = \begin{cases} \delta_i, & \text{if } i = j \\ -1, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are adjacent} \\ 0, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are not adjacent} \end{cases}$$

where $\delta_i = \text{degree of vertex } v_i$.

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The Laplacian energy of a graph G is then defined as

$$LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|$$

This Laplacian energy of a graph G has been recently defined by Gutman and Zhou. [5]

1.2 Color Signless Laplacian Energy

A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring a graph G is called chromatic number and denoted by $\chi(G)$.

Adiga, Sampathkumar, Sriraj, Shrikanth [1] have studied color energy which is the sum of the absolute values of the color eigenvalues of G .

The color adjacency matrix $A_c(G) = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j) \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) \neq c(v_j) \\ 0, & \text{if } v_i = v_j \text{ or } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j) \end{cases}$$

, $c(v_i)$ is the color of vertex v_i .

The color energy of a graph denoted by $E_c(G)$ is defined as

$$E_c(G) = \sum_{i=1}^n |\lambda_i|$$

where λ_i of $A_c(G)$ are called the color eigenvalues of G .

Given the Signless Laplacian matrix of the graph G , of order n and size m , is denoted by $LE^+(G) = D(G) + A(G)$, where $D(G)$ is the degree matrix and $A(G)$ is the adjacency matrix, the Signless Laplacian energy of the graph G is

$$LE^+(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|$$

where μ_i are the Signless Laplacian eigenvalues of the graph G .

Bhat and D'Souza [2] defined color Laplacian energy as

$$LE_c(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|$$

where μ_i are eigenvalues of color Laplacian matrix $L_c(G) = D(G) - A_c(G)$.

Then, the color Signless Laplacian energy of G , denoted by $LE_c^+(G)$ is defined as

$$LE_c^+(G) = \sum_{i=1}^n \left| \mu_i^+ - \frac{2m}{n} \right|$$

where μ_i^+ are eigenvalues of color Signless Laplacian matrix $L_c^+(G) = D(G) + A_c(G)$.

1.3 Book Graph B_n

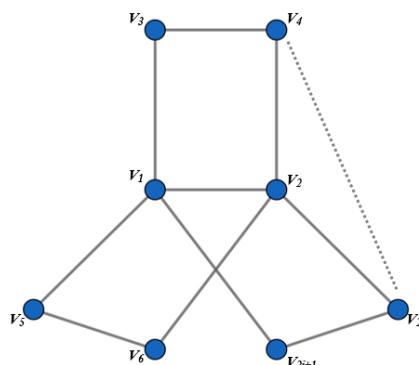


Figure-1: The book graph B_n

The book graph B_n is a graph $S_n \times P_2$ where S_n is a star graph with n edges. The order of B_n is $N = 2n + 2$, and its size is $M = 3n + 1$. The principle of labelling vertices of the book graph B_n is that the center vertex of the first star graph is labelled by V_1 (which is adjacent to V_2) other vertices are labelled by $V_{2i+1}, 1 \leq i \leq n$, and center of the second star graph is labelled by V_2 (which is adjacent to V_1) and the other vertices are labelled by $V_{2i}, 2 \leq i \leq n + 1$.

1.4 Colored Book Graph B_n

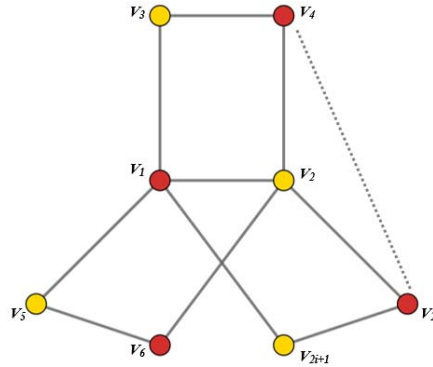


Figure-2: Colored book graph B_n

Colored book graph B_n , is a book graph with chromatic number $\chi(B_n) = 2$, with vertex color of red (R) and yellow (Y). The principle of coloring, the vertices of the book graph B_n , is that $c(V_1) = R$ and $c(V_{2i+1}) = Y, 1 \leq i \leq n$; and $c(V_2) = Y$ and $c(V_{2i}) = R, 2 \leq i \leq n + 1$.

2. MAIN RESULTS

This section presents the energy, Laplacian energy and Color Signless Laplacian energy of book graph B_n and its proofs.

Theorem 1: Let B_n be a book graph with order $N = 2n + 2$ and size $M = 3n + 1$. Then, the energy of B_n is

$$E(B_n) = 2(2\sqrt{n} + n - 1)$$

Proof: The adjacency matrix of book graph B_n is

$$A(B_n) = \begin{pmatrix} 0 & 1 & 1 & 0 & \dots & \dots & \dots & \dots & 1 & 0 \\ 1 & 0 & 0 & 1 & \dots & \dots & \dots & \dots & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the matrix is given by

$$P(B_n, \lambda) = (\lambda + 1)^{n-1}(\lambda - 1)^{n-1}[\lambda - (\sqrt{n} - 1)][\lambda - (\sqrt{n} + 1)][\lambda - (-\sqrt{n} - 1)][\lambda - (\sqrt{n} + 1)]$$

Hence, the adjacency spectrum of book graph B_n , is

$$E_{spec}(B_n) = \left\{ \begin{matrix} -1 & 1 & \sqrt{n} - 1 & \sqrt{n} + 1 & -\sqrt{n} - 1 & -\sqrt{n} + 1 \\ n - 1 & n - 1 & 1 & 1 & 1 & 1 \end{matrix} \right\}$$

Therefore, the energy is given by

$$\begin{aligned} E(B_n) &= \sum_{i=1}^N |\lambda_i| \\ &= (n - 1)|-1| + (n - 1)|1| + |\sqrt{n} - 1| + |\sqrt{n} + 1| + |-\sqrt{n} - 1| + |-\sqrt{n} + 1| \\ &= 2(2\sqrt{n} + n - 1) \end{aligned}$$

Theorem 2: Let B_n be a book graph with order $N = 2n + 2$ and size $M = 3n + 1$. Then, the Laplacian energy of B_n is given by

$$LE(B_n) = 4n$$

Proof: The Laplacian matrix of book graph B_n is

$$L(B_n) = \begin{pmatrix} n+1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & -1 & 0 \\ -1 & n+1 & 0 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

The characteristic polynomial of the Laplacian matrix is given by

$$P(B_n, \mu) = \mu(\mu - 1)^{n-1}(\mu - 2)(\mu - 3)^{n-1}[\mu - (n + 1)][\mu - (n + 3)]$$

Hence, the Laplacian spectrum of book graph B_n is

$$LE_{spec}(B_n) = \left\{ \begin{matrix} 0 & 1 & 2 & 3 & n+1 & n+3 \\ 1 & n-1 & 1 & n-1 & 1 & 1 \end{matrix} \right\}$$

Therefore, the Laplacian energy is given by

$$\begin{aligned} LE(B_n) &= \sum_{i=1}^N \left| \mu_i - \frac{2M}{N} \right| \\ &= \left| 0 - \frac{2(3n+1)}{2n+2} \right| + (n-1) \left| 1 - \frac{2(3n+1)}{2n+2} \right| + \left| 2 - \frac{2(3n+1)}{2n+2} \right| + \\ &\quad (n-1) \left| 3 - \frac{2(3n+1)}{2n+2} \right| + (n+1) \left| n+1 - \frac{2(3n+1)}{2n+2} \right| + \left| (n+3) - \frac{2(3n+1)}{2n+2} \right| \\ &= 4n \end{aligned}$$

Theorem 3: Let B_n be a book graph with order $N = 2n + 2$ and size $M = 3n + 1$. Then, the Color Signless Laplacian energy of B_n if $n > 1$ is given by

$$LE_c^+(B_n) = 2(3n - 1)$$

Proof: The Color Signless Laplacian matrix of book graph B_n , is

$$LE_c^+(B_n) = \begin{pmatrix} n+1 & 1 & 1 & -1 & \cdots & \cdots & \cdots & \cdots & 1 & -1 \\ 1 & n+1 & -1 & 1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 \\ 1 & -1 & 2 & 1 & -1 & 0 & \cdots & \cdots & -1 & 0 \\ -1 & 1 & 1 & 2 & 0 & -1 & \cdots & \cdots & 0 & -1 \\ \vdots & \vdots & -1 & 0 & \ddots & \ddots & -1 & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & -1 & \ddots & \ddots & 0 & -1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & -1 & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & -1 & \ddots & \ddots & 0 & -1 \\ 1 & -1 & -1 & 0 & \cdots & \cdots & -1 & 0 & 2 & 1 \\ -1 & 1 & 0 & -1 & \cdots & \cdots & 0 & -1 & 1 & 2 \end{pmatrix}$$

The characteristic polynomial of Color Signless Laplacian matrix is given by

$$P(B_n, \mu^+) = (\mu^+ - 2)^{n-1}(\mu^+ - 4)^{n-1}[\mu^+ - (n + 2)]^2(\mu^+ + n)[\mu^+ - (4 - n)]$$

Hence, the Color Signless Laplacian spectrum of book graph B_n , is

$$LE_{c\ spec}^+(B_n) = \left\{ \begin{matrix} 2 & 4 & n+2 & -n & 4-n \\ n-1 & n-1 & 2 & 1 & 1 \end{matrix} \right\}$$

Therefore, the Color Signless Laplacian energy if $n > 1$ is given by

$$\begin{aligned} LE_c^+(B_n) &= \sum_{i=1}^N \left| \mu_i - \frac{2M}{N} \right| \\ &= (n-1) \left| 2 - \frac{2(3n+1)}{2n+2} \right| + (n-1) \left| 4 - \frac{2(3n+1)}{2n+2} \right| + 2 \left| (n+2) - \frac{2(3n+1)}{2n+2} \right| + \left| -n - \frac{2(3n+1)}{2n+2} \right| \\ &\quad + \left| (4-n) - \frac{2(3n+1)}{2n+2} \right| \\ &= 2(3n-1) \end{aligned}$$

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