WEAKLY CONVEX DOUBLY CONNECTED DOMINATION IN GRAPHS
UNDER SOME BINARY OPERATIONS

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ABSTRACT

Let \( G \) be a connected simple graph. A weakly convex dominating set \( S \) of \( G \) is a weakly convex doubly connected dominating set if \( S \) is a doubly connected dominating set of \( G \). The weakly convex doubly connected domination number of \( G \), denoted by \( \gamma_{ccw}(G) \), is the smallest cardinality of a convex doubly connected dominating set \( S \) of \( G \). In this paper, we characterized the weakly convex doubly connected dominating sets of the composition and Cartesian product of graphs.

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1. INTRODUCTION

Let \( G \) be a connected simple graph. A subset \( S \) of \( V(G) \) is a dominating set of \( G \) if for every \( v \in (V(G)\setminus S) \), there exists \( x \in S \) such that \( xv \in E(G) \). The domination number \( \gamma(G) \) of \( G \) is the smallest cardinality of a dominating set of \( G \). A graph \( G \) is connected if there is at least one path that connects every two vertices \( x, y \in V(G) \), otherwise, \( G \) is disconnected. A component of a graph is a maximal connected subgraph. Clearly, if a graph has only one component, then it is connected, otherwise it is disconnected. A dominating set \( S \subseteq V(G) \) is called a connected dominating set of \( G \) if the subgraph \( \langle S \rangle \) induced by \( S \) is connected. The connected domination number of \( G \), denoted by \( \gamma_c(G) \), is the smallest cardinality of a connected dominating set of \( G \). A connected dominating set of cardinality \( \gamma_c(G) \) is called a \( \gamma_c \)-set of \( G \). A set \( S \subseteq V(G) \) is a doubly connected dominating set if it is dominating and both \( \langle S \rangle \) and \( \langle V(G) \setminus S \rangle \) are connected. The doubly connected domination number of \( G \), denoted by \( \gamma_{cc}(G) \), is the smallest cardinality of a doubly connected dominating set \( S \) of \( G \). A doubly connected dominating set of cardinality \( \gamma_{cc}(G) \) is called a \( \gamma_{cc} \)-set of \( G \). Studies on doubly connected domination in graphs are found in [1, 2, 3, 4, 5].

For any two vertices \( u \) and \( v \) in a connected graph, the distance \( d_G(u,v) \) between \( u \) and \( v \) is the length of a shortest path in \( G \). A \( u-v \) path of length \( d_G(u,v) \) is also referred to as \( u-v \) geodesic. A subset \( C \) of \( V(G) \) is called a convex set of \( G \) if for every two vertices \( u, v \in C \), the vertex-set of every \( u-v \) geodesic is contained in \( C \). A subset \( C \) of \( V(G) \) is called a weakly convex set of \( G \) if for every two vertices \( u, v \in C \), there exists a \( u-v \) geodesic whose vertices bolong to \( C \). Convexity in graphs was studied in [6,7,8,9]. Some variants of convex domination in graphs are found in [10, 11, 12, 13, 14, 15, 16, 17].

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A dominating set of $G$ which is weakly convex is called a weakly convex dominating set. The weakly convex domination number of $G$, denoted by $\gamma_{wcon}(G)$, is the smallest cardinality of a weakly convex dominating set of $G$. A dominating set $S$ which is also convex is called a convex dominating set of $G$. The convex domination number $\gamma_{con}(G)$ of $G$ is the smallest cardinality of a convex dominating set of $G$. A convex dominating set of cardinality $\gamma_{con}(G)$ is called a $\gamma_{con}$-set of $G$. A weakly convex dominating set $S$ of $G$ is a weakly convex doubly connected dominating set if $S$ is a doubly connected dominating set of $G$. The weakly convex doubly connected dominating number of $G$, denoted by $\gamma_{wccc}(G)$, is the smallest cardinality of a weakly convex doubly connected dominating set $S$ of $G$. A weakly convex doubly connected dominating set of cardinality $\gamma_{wccc}(G)$ is called a $\gamma_{wccc}$-set of $G$. For general concepts we refer the reader to [19].

2. RESULTS

The following remarks are immediate from the definitions.

**Remark 2.1:** Let $G$ be a connected graph. If $C \subseteq V(G)$ is a convex dominating set, then $C$ is a weakly convex dominating set of $G$.

**Remark 2.2:** Let $G$ be a non-trivial connected graph of order $n$. Then

(i) $\gamma(G) \leq \gamma_{wcon}(G) \leq \gamma_{wccc}(G) \leq \gamma_{ccc}(G)$, and

(ii) $1 \leq \gamma_{wccc}(G) \leq n$.

The composition of two graphs $G$ and $H$ is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

A subset $S$ of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} \{x\} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. We shall be using this form to denote any subset $C$ of $V(G[H])$.

The following results are needed for the characterization of the weakly convex doubly connected dominating sets of the composition to two of graphs.

**Theorem 2.3[7]:** Let $G$ be connected graph of order $m \geq 2$ and $H$ any graph. A subset $C = \bigcup_{x \in S} \{x\} \times T_x$ is a weakly convex dominating set of $G[H]$ if and only if $S$ is a weakly convex dominating set of $G$ and $T_x$ is a dominating set of $H$ with $\text{diam}_H(T_x) \leq 2$ if $|S| = 1$.

**Remark 2.4:** Let $G$ and $H$ be non-complete connected graphs. If $S$ is a weakly convex dominating set of $G$ with $|S| \geq 2$, then a subset $C = \bigcup_{x \in S} \{x\} \times T_x$ is a weakly convex dominating set of $G[H]$.

The following result is the characterization of the weakly convex doubly connected dominating sets of the composition to two of graphs.

**Theorem 2.5:** Let $G$ and $H$ be non-complete connected graphs. A subset $C = \bigcup_{x \in S} \{x\} \times T_x$ is a weakly convex doubly connected dominating set of $G[H]$ if and only if $S$ is a weakly convex dominating set of $G$ and $T_x$ is a weakly convex set of $H$ and one of the following holds:

i) $S = \{x\}$ and $T_x$ is a dominating set of $H$ with $\text{diam}_H(T_x) \leq 2$, where $T_x \neq V(H)$ whenever $|V(G) \setminus S|$ is not connected.

ii) $S = V(G) \setminus \{z\}$.

iii) $S = S_1 \cup \{z\} = V(G)$ and $(V(H) \setminus T_x)$ is connected.

iv) $T_x \neq V(H)$ for all $x \in S$ whenever $S \neq V(G) \setminus \{z\}$.

**Proof:** Suppose that a subset $C = \bigcup_{x \in S} \{x\} \times T_x$ is a weakly convex doubly connected dominating set of $G[H]$. Then $C$ is a weakly convex dominating set of $G[H]$. Thus $S$ is a weakly convex dominating set of $G$ by Theorem 2.3. Suppose that $T_x$ is not a weakly convex set of $H$. Let $S = \{x\}$ and $T_x = \{a, b\}$ such that $ab \notin E(H)S$. Then $C = \{(x, a), (x, b)\}$ and $(x, a)(x, b) \notin E(G[H])$ contrary to the assumption that $C$ is a weakly convex doubly connected dominating set of $G[H]$. Thus $T_x$ must be a weakly convex set of $H$. Further, $S = \{x\}$ and $T_x$ is a dominating set of $H$ with $\text{diam}_H(T_x) \leq 2$ holds by Theorem 2.3. Suppose that $(V(G) \setminus S)$ is not connected. If $T_x = V(H)$, then

$$V(G[H]) \setminus C = (V(G) \times V(H)) \setminus (S \times V(H)) = (V(G) \setminus S) \times V(H).$$

Since $(V(G) \setminus S)$ is not connected, it follows that $(V(G[H]) \setminus C)$ is not connected contrary to our assumption that $C$ is doubly connected dominating set of $G[H]$. Thus, $T_x \neq V(H)$. This proves statement i).

Next, suppose that $|S| \neq 1$. If $S = V(G) \setminus \{z\}$ then we are done with statement ii). Suppose that $S \neq V(G) \setminus \{z\}$. Consider the following cases:

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Case-2: Suppose that $S \subseteq V(G)$ such that $wz \notin E(G)$. If $T_x = V(H)$ for all $x \in S$, then $V(G[H]) \setminus C = \bigcup_{x \in S, uz \notin E(G)} \{(u, z)\}$. Since $wz \notin E(G)$, it follows that $(w, u)(z, u) \notin E(G[H])$ for all $(w, u), (z, u) \in V(G[H]) \setminus C$. Thus, $C$ is not a doubly connected dominating set of $G[H]$ contrary to our assumption. Hence $T_x \neq V(H)$. This proves statement iii).

For the converse, suppose that $S$ is a weakly convex dominating set of $G$ and $T_x$ is a weakly convex set of $H$ and one of the following statements (i), (ii), or (iv) holds.

First, suppose that statement i) holds. Then $C = \bigcup_{x \in S}((x) \times T_x)$ is a weakly convex dominating set of $G[H]$ by Theorem 2.3. Clearly, if $(V(G) \setminus S)$ is connected, then $C$ is a weakly convex doubly connected dominating set of $G[H]$. Suppose that $(V(G) \setminus S)$ is not connected. Let $T_x \neq V(H)$. If $T_x = \{a\}$ then $C = \{(x, a)\}$. Since $H$ is non-complete connected graph, let $b, c \in V(H) \setminus T_x$. If $bc \in E(H)$, then $(x, b)(x, c) \in E(G[H])$ for all $z \in N_G(x)$, that is, $(V(G[H]) \setminus C)$ is connected. Suppose that $bc \notin E(H)$. Since $H$ is connected, then exists a path $\{b = u_1, u_2, \ldots, u_i = c\}$ such that $\{(x, b), (z, u_1), \ldots, (z, c)\}$ is a path in $(V(G[H]) \setminus C)$ for all $z \in N_G(x)$, that is, $(V(G[H]) \setminus C)$ is connected. Thus, $C$ is a doubly connected dominating set of $G[H]$. Similarly, if $|T_x| \geq 2$, then $C$ is a doubly connected dominating set of $G[H]$.

Next, suppose that ii) holds. If $T_x = V(H)$ for each $x \in S$, then $T_x$ is a dominating set of $H$. Since $S$ is a weakly convex dominating set of $G$, it follows that $C = \bigcup_{x \in S}((x) \times T_x)$ is a weakly convex dominating set of $G[H]$ by Theorem 2.3. Since $G$ is a non-complete connected graphs, $|V(G)| \geq 3$ and $|S| \neq 1$. Since $H$ is non-complete connected graph, let $a, b \in V(H)$. If $ab \in E(H)$, then $(x, a)(x, b) \in E(G[H])$ for all $(x, a), (x, b) \in V(G[H]) \setminus C$ and for all $z \in V(G[H]) \setminus S$. Thus, $(V(G[H]) \setminus C)$ is connected, and hence $C$ is a doubly connected dominating set of $G[H]$.

Since $C$ is connected dominating set of $G[H]$. Now, suppose that $T_x \neq V(H)$ for some $x \in S$. Consider $T_x$ is a dominating set of $H$ for each $x \in S$. Then $C = \bigcup_{x \in S}((x) \times T_x)$ is a weakly convex dominating set of $G[H]$ by Theorem 2.3. This further implies that $C$ is connected dominating set in $G[H]$. Since $S$ is a dominating set in $G$, there exists $x \in S$ such that $xz \in E(G)$ for all $z \in V(G) \setminus S$. Let $a \in V(H) \setminus T_x$ for each $x \in S$. Then $(x, a)(z, u) \in E(G[H])$ for all $u \in V(H)$ and $(x, a)(y, a) \in E(G[H])$ for all $y \in N_G(x) (y \neq z)$. Thus, $(V(G[H]) \setminus C)$ is connected. Hence $C$ is a doubly connected dominating set of $G[H]$. Consider $T_x$ is not a dominating set of $H$ for each $x \in S$. Since $S$ is a weakly convex dominating set of $G$ and $|S| \geq 2$, then $C$ is a weakly convex dominating set of $G[H]$ by Remark 2.4. Let $a \in V(H) \setminus T_x$. By following similar arguments used earlier, $C$ is a doubly connected dominating set of $G[H]$.

Suppose that statement iii) holds. Consider the following cases.

Case-1: Suppose that $T_x = V(H)$ for all $x \in S_x$.

Since $S$ is a weakly convex dominating set of $G$ and $T_x$ is a dominating set of $H$, it follows that $C$ is a weakly convex dominating set of $G[H]$ by Theorem 2.3. If $T_x = V(H)$, then

$C = \bigcup_{x \in S}((x) \times T_x)$

$= \bigcup_{x \in S, z \in \{1\}}((x) \times T_x)$

$= \bigcup_{x \in S}((x) \times T_x) \cup \{(z) \times T_x\}$

$= \bigcup_{x \in S}((x) \times V(H)) \cup \{(z) \times V(H)\}$

$= \bigcup_{x \in S}((x) \times V(H))$

$= S \times V(H) = V(G[H]) \times V(H) = E(G[H])$.

This implies that $V(G[H]) \setminus C = \emptyset$ and hence $(V(G[H]) \setminus C)$ is connected.

If $T_x \neq V(H)$, then let $[a] \subseteq T_x \subset V(H)$. Consider $T_x = \{a\}$. Since $H$ is a non-complete connected graph, $|V(H)| \geq 3$. Let $b, c \in V(H) \setminus T_x$. If $bc \in E(H)$, then $(b, z)(c, z) \notin E(G[H])$ for all $(b, z), (c, z) \in V(G[H]) \setminus C$ and hence $(V(G[H]) \setminus C)$ is connected. Suppose that $bc \notin E(H)$. Since $(V(H) \setminus T_x)$ is connected, there exists a path $[b = v_1, v_2, \ldots, v_i = c]$ in $(V(H) \setminus T_x)$ such that $(b, v_1)(v_2, z) \ldots, (z, c)$ is also a path in $(V(G[H]) \setminus C)$. Thus, $(V(G[H]) \setminus C)$ is connected. Similarly, if $[a] \subset T_x$, then $(V(G[H]) \setminus C)$ is connected. Consider $T_x = V(H) \setminus \{a\}$. Then $V(G[H]) \setminus C = \{(z, a)\}$ and hence $(V(G[H]) \setminus C)$ is connected. Thus, $C$ is a doubly connected dominating set of $G[H]$.
Accordingly, Corollary 2.6: As a consequence of Theorem 2.5, we obtain the following result.

Case-2: Suppose that \( T_x \neq V(H) \) for all \( x \in S \).

Since \( G \) is a non-complete connected graph and \( S = V(G) \), it follows that \( |S| \geq 3 \). Since \( S \) is a weakly convex dominating set of \( G \) with \( |S| \geq 3 \), \( C = \cup_{x \in S} \left\{ (x) \times T_x \right\} \) is a weakly convex dominating set of \( G[H] \) by Remark 2.4. This implies that \( \{C\} \) is connected. Let \( a \in V(H) \setminus T_x \) and \( x, y \in S \). If \( xy \in E(G) \), then \( (x, a)(y, a) \in E(G[H]) \) for all \( (x, a), (y, a) \in V(G[H]) \). This implies that \( \{V(G[H])\} \setminus C \) is connected. Suppose that \( xy \notin E(G) \). Since \( G \) is connected, there exists a path \( [x = x_1, x_2, \ldots, x_r = y] \) in \( G \) such that \( [(x, a), (x_2, a), \ldots, (y, a)] \) is also a path in \( \{V(G[H])\} \setminus C \). Thus, \( \{V(G[H])\} \setminus C \) is connected, that is, \( C \) is a doubly connected dominating set of \( G[H] \).

Finally, suppose that statement (iv) holds. If \( |S| = 1 \), then \( C \) is a weakly convex doubly connected dominating set of \( G[H] \) by statement (i). Suppose that \( |S| \geq 2 \). If \( S = V(G) \), then \( C \) is a weakly convex doubly connected dominating set of \( G[H] \) by statement (iii). If \( S = V(G) \setminus \{z\} \), then \( C \) is a weakly convex doubly connected dominating set of \( G[H] \) by statement (ii). Since \( S \) is a weakly convex dominating set of \( G \) with \( |S| \geq 2 \), \( C = \cup_{x \in S} \left\{ (x) \times T_x \right\} \) is a weakly convex dominating set of \( G[H] \) by Remark 2.4. Let \( w \in V(G[H]) \setminus \{z\} \) and \( a \in V(H) \setminus T_x \).

Case-1: If \( wz \notin E(G) \), then \( (w, a)(z, a) \in E(G[H]) \) for all \( (w, a), (z, a) \in V(G[H]) \setminus \{C\} \), that is, \( \{V(G[H])\} \setminus C \) is connected.

Case-2: Suppose that \( wz \notin E(G) \). Since \( G \) is connected, there exists a path \( \{w = x_1, x_2, \ldots, x_r = z\} \) in \( G \) such that \( [(w, a), (x_2, a), \ldots, (z, a)] \) is a path in \( \{V(G[H])\} \setminus C \). Thus, \( \{V(G[H])\} \setminus C \) is connected, that is, \( C \) is a doubly connected dominating set of \( G[H] \).

Accordingly, \( C \) is a weakly convex doubly connected dominating set of \( G[H] \). ■

As a consequence of Theorem 2.5, we obtain the following result.

**Corollary 2.6:** Let \( G \) and \( H \) be non-complete connected graphs. Then

\[
y_{\text{wconv}}(G[H]) = \begin{cases} 
1 & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) = 1 \\
\begin{cases} 
\gamma(G) & \text{if } \gamma(G) = k \\
1 & \text{if } \gamma(G) = 1 
\end{cases} & \text{if } \gamma(G) = k \text{ where } k \geq 2 
\end{cases}
\]

**Proof:** Suppose that \( \gamma(G) = 1 \) and \( \gamma(H) = 1 \). Let \( S = \{x\} \) be a \( y \)-set in \( G \) and \( T_x = \{a\} \) be a \( y \)-set in \( H \). Then \( S \) is a weakly convex dominating set of \( G \) and \( T_x = V(H) \) is a weakly convex dominating set of \( H \) with diam\(_p\)(\( T_x \)) < 2. Thus \( C = \cup_{x \in S} \left\{ (x, a) \right\} \) is a weakly convex doubly connected dominating set of \( G[H] \) by Theorem 2.5. Hence, \( y_{\text{wconv}}(G[H]) = |C| = 1 \).

Suppose that \( y_{\text{wconv}}(G) = k \) where \( k \geq 2 \). Let \( S = \{x_1, x_2, \ldots, x_k\} \) be a \( y_{\text{wconv}} \)-set in \( G \). Since \( S \) is a weakly convex dominating set of \( G \) with \( |S| \geq 2 \), a subset \( C = \cup_{x \in S} \left\{ (x) \times T_x \right\} \) is a weakly convex dominating set of \( G[H] \) by Remark 2.4. Let \( T_x = \{a\} \) for all \( x \in S \). Then \( C = \{(x_1, a), (x_2, a), \ldots, (x_k, a)\} \), that is, \( |C| = k \). Let \( x, y \in S \) and let \( b \in V(H) \setminus T_x \) for all \( v \in S \). If \( xy \in E(G) \), then by similar arguments used to prove Theorem 2.5, \( C \) is a weakly convex doubly connected dominating set of \( G[H] \). Similarly, if \( xy \notin E(G) \), then \( C \) is a weakly convex doubly connected dominating set of \( G[H] \). Thus, \( y_{\text{wconv}}(G[H]) \leq |C| = k \). Since \( k = y_{\text{wconv}}(G[H]) \leq y_{\text{wconv}}(G[H]) \) by Remark 2.2, it follows that \( y_{\text{wconv}}(G[H]) = k \). ■

The Cartesian product of two graphs \( G \) and \( H \) is the graph \( G \square H \) with vertex-set \( V(G \square H) = V(G) \times V(H) \) and edge-set \( E(G \square H) \) satisfying the following conditions: \( (x, a)(y, b) \in E(G \square H) \) if and only if either \( xy \in E(G) \) and \( a = b \) or \( x = y \) and \( ab \in E(H) \).

The next result is needed for the characterization of the weakly convex doubly connected dominating sets of the Cartesian product of two of graphs.

**Lemma 2.7:** Let \( G \) and \( H \) be non-trivial connected graphs. Then \( C = \cup_{x \in S} \left\{ (x) \times T_x \right\} \) is a weakly convex dominating set of \( G \square H \) if \( S \) is a weakly convex dominating set of \( G \) and \( T_x = V(H) \) for all \( x \in S \), or \( S = V(G) \) and \( T_x = V(H) \) is a weakly convex dominating set of \( H \) for all \( x \in S \).

**Proof:** Suppose that \( C = \cup_{x \in S} \left\{ (x) \times T_x \right\} \) is not a weakly convex dominating set of \( G \square H \). Let \( (x, a) \in C \). If there exists \( (y, a) \in C \) whose vertices in any \((x, a)-(y, a)\) geodesic are not all in \( C \), then for each \( x \in S \), there exists \( y \in S \) whose vertices in any \( x-y \) geodesic are not all in \( S \), that is, \( S \) is not a weakly convex dominating set of \( G \). If there exists \((x, b) \in C \) whose vertices in any \((x, a)-(x, b)\) geodesic are not all in \( C \), then for each \( a \in T_x \), there exists \( b \in T_x \) for all \( x \in S \) whose vertices in any \( a-b \) geodesic are not all in \( T_x \), that is, \( T_x \) is not a weakly convex dominating set of \( H \) for all \( x \in S \). ■

The following result is the characterization of the weakly convex doubly connected dominating sets of the Cartesian product of two of graphs.
Theorem 2.8: Let G and H be non-trivial connected graphs. Then \( C = \bigcup_{x \in S} \{x \times T_x\} \) is a weakly convex doubly connected dominating set of \( G \square H \) if and only if \( S \) is a weakly convex dominating set of \( G \) and \( H \) is a weakly convex dominating set of \( H \) and one of the following statements holds:

i) \( S \neq \emptyset \) and \( T_x = V(H) \) for all \( x \in S \) where \( (V(H) \setminus S_1) \) is connected.

ii) \( S = \emptyset \) and \( T_x \neq V(H) \) for all \( x \in S \) where \( (V(H) \setminus S_1) \) is connected.

iii) \( S = S_1 \cup S_2 \) where \( S_1 = \{x \in V(G) : T_x = V(H)\}, S_2 = \{x \in V(G) : T_x \neq V(H)\}, (S_1) \) is connected, \( (S_2) \) is connected, and \( (V(H) \setminus S_1) \) is connected for all \( z \in S_2 \).

iv) \( T_x = T_x^r \cup T_x^r \), where \( T_x^r = \{a \in V(H) : S = V(G)\}, T_x^r = \{a \in V(H) : S \neq V(G)\}, (T_x^r) \) is connected, \( (T_x^r) \) is connected, and \( (V(G) \setminus S') \) is connected where \( S' = \{x \in V(G) : a \in T_x^r\} \).

Proof: Suppose that \( C = \bigcup_{x \in S} \{x \times T_x\} \) is a weakly convex doubly connected dominating set of \( G \square H \). Suppose that \( S \) is not a weakly convex dominating set of \( G \). Let \( x \in S \). If \( S \) is not a dominating set of \( G \), then there exists \( y \in V(G) \setminus S \) such that \( xy \notin E(G) \). Let \( a \in T_x \) for all \( x \in S \). Then there exists \( y, a \in V(G (H) \setminus C \) such that \( (x, a) \notin E(G) \) for all \( (z, a) \in C \). Hence \( C \) is not a dominating set of \( G \square H \) contrary to our assumption. If \( S \) is not a weakly convex set in \( G \), then \(|S| \geq 2 \). Let \( x, y \in S \) such that \( xy \notin E(G) \). For each \( v \in S \), let \( a \in T_v \). If \(|S| = 2 \), then \((x, a) \notin E(G) \) for all \((x, a) \in C \). Hence \( C \) is not a dominating set of \( G \square H \) contrary to our assumption. If \(|S| = 3 \), then there exists \( z \in V(G) \setminus S \) such that for every \( x \), \( y \) geodesic in \( (S) \), \( z \in I_{G}(x, y) \). Thus, for every \((x, y, a) \) geodesic in \( (C) \), \( G \) is connected for all \( z \in V(G \square H) \). This is contrary to our assumption that \( C \) is a weakly convex dominating set of \( G \square H \). Thus, \(|S| \geq 4 \).

Next, suppose that \( S = S_1 \cup S_2 \) where \( S_1 = \{x \in V(G) : T_x = V(H)\}, S_2 = \{x \in V(G) : T_x \neq V(H)\} \). Suppose that \(|S_1| = 2 \). If \(|V(H)| = 2 \), then \(|S_1| = 1 \) and \(|S_2| = 1 \). Hence \((S_1) \) is connected and \((S_2) \) is connected. Clearly \((V(H) \setminus T_x) \) is connected for all \( z \in S_2 \). Similarly, if \(|V(H)| \geq 3 \), then \((S_1) \) is connected, \((S_2) \) is connected. Suppose that \((V(H) \setminus T_x) \) is not connected for some \( z \in S_2 \). Then there exists \( a, b \in T_x \) such that \( a, b \) geodesic is not a path in \( (T_x) \) for all \( z \in S_2 \). Thus, there exists \((z, a), (z, b) \in (C) \) such that \( a, b \) geodesic is not a path in \( (C) \). This contradicts our assumption that \( C \) is a weakly convex set of \( G \square H \). Thus, \((V(H) \setminus T_x) \) must be connected for all \( z \in S_2 \). This proves statement i).

Similarly, statement iv) holds.

For the converse, suppose that \( S \) is a weakly convex dominating set of \( G \) and \( H \) is a weakly convex dominating set of \( H \) and one of the statements i), ii), iii), or iv) holds. Then \( C = \bigcup_{x \in S} \{x \times T_x\} \) is a weakly convex dominating set of \( G \square H \) by Lemma 2.7. Suppose first that statement i) holds. Let \( z \in V(G) \setminus S \). Consider \( |V(G) \setminus S| = 1 \). Since \( H \) is connected, there exists an \( a-b \) path in \( H \) such that \((z, a)-(z, b) \) is a path in \( V(G \square H) \). This implies that \( C \) is a doubly connected dominating set of \( G \square H \). Hence \( C \) is a weakly convex doubly connected dominating set of \( G \square H \).

Next, suppose that \( iii \) holds. Let \( a \in V(H) \setminus T_x \) for all \( a \in S_2 \). Consider that \(|S_2| = 1 \). Then \((z, a) \in V(G \square H) \). If \(|V(H) \setminus T_x| = 1 \), then \((V(G \square H) \setminus C) = \{(z, a)\} \). This implies that \( V(G \square H) \) is connected and hence \( C \) is weakly convex doubly connected dominating set of \( G \square H \). Suppose that \(|V(H) \setminus T_x| \geq 2 \). Then there exists \( b \in V(H) \setminus T_x \) such that \( a-b \) is a path in \( V(H) \setminus T_x \) for all \( a \in S_2 \). Thus, for each \((z, a) \in V(G \square H) \), there exists \((z, b) \in V(G \square H) \) such that \((z, a)-(z, b) \) is a path in \( V(G \square H) \). This implies that \((V(G \square H) \setminus C) \) is connected and hence \( C \) is a weakly convex doubly connected dominating set of \( G \square H \).

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The next result is the consequence of Theorem 2.8.

**Corollary 2.9:** Let $G$ and $H$ be non-trivial connected graphs. Then

$$
\gamma_{wcc}^w(G \boxplus H) = (\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|) - 1)
$$

if $S$ is a weakly convex dominating set of $G$ and $T_x$ is a weakly convex dominating set of $H$ for all $x \in S$ and one of the following statements holds:

i) $S = V(G) \setminus \{x\}$ and $T_x = V(H)$ for all $x \in S$ and $|V(G)| \leq |V(H)|$.

ii) $S = V(G)$ and $T_x = V(H) \setminus \{a\}$ for all $x \in S$ and $|V(G)| \geq |V(H)|$.

**Proof:** Suppose that $S$ is a weakly convex dominating set of $G$ and $T_x$ is a weakly convex dominating set of $H$ for all $x \in S$ and one of the statements i) or ii) holds. Then $C = \cup_{x \in S} \{x\} \cup T_x$ is a weakly convex doubly connected dominating set of $G \boxplus H$ by Theorem 2.8. Further, $C = S \times V(H)$ if $C = (G \boxplus H)$ for all $x \in S$.

Let $|C| = \min(|S \times V(H)|, |V(G) \times T_x|)$ for all $x \in S$.

$$
\gamma_{wcc}^w(G \boxplus H) \leq |C| = \min(|S \times V(H)|, |V(G) \times T_x|) = \min(|S||V(H)|, |V(G)||T_x|).
$$

If i) holds, then $|C| = |S \times V(H)| = |S||V(H)|$

$$
= (\min(|S|, |V(H)|))(\max(|V(G)|, |V(H)|))
$$

$$
= (\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|))
$$

$$
(\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|) - 1)
$$

$$
(\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|) - 1).
$$

If ii) holds, then $|C| = |V(G) \times T_x| = |V(G)||T_x|$

$$
= (\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|))
$$

$$
(\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|) - 1)
$$

$$
(\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|) - 1).
$$

Thus, $\gamma_{wcc}^w(G \boxplus H) \leq (\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|) - 1)$.

Since $C$ is also a weakly convex dominating set of $G \boxplus H$, it follows that $\gamma_{wcon}(G \boxplus H) \leq |C|$. Let $(x, a) \in C$ and $C' = C \setminus \{(x, a)\}$. Then $(x, a),(z, b) \in E(G \boxplus H)$ for all $x \in N_G(x)$ and $(z, a),(z, b) \in E(G \boxplus H)$ for all $b \in N_H(z)$. If $z \in V(G) \setminus S$, then $(z, a) \in V(G \boxplus H) \setminus C$ is not dominated by any element of $C$ since $(x, a),(z, b) \notin C'$. This implies that $C'$ is not a weakly convex dominating set of $G \boxplus H$ and hence $C$ is a minimum weakly convex dominating set of $G \boxplus H$. Thus, $|C| = \gamma_{wcon}(G \boxplus H) \leq \gamma_{wcc}^w(G \boxplus H)$ by Remark 2.2.

Therefore $\gamma_{wcc}^w(G \boxplus H) = |C| = (\max(|V(G)|, |V(H)|))(\min(|V(G)|, |V(H)|) - 1)$.

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