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COMMON FIXED POINT THEOREM IN COMPLEX VALUED B-METRIC SPACES

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ABSTRACT

In this paper, we prove a common fixed point theorem in complex valued b-metric space satisfying rational inequality using compatible and weakly compatible mappings. Our result extend and generalize some well known results from the existing literature.

***Key Words**: Weakly Compatible Mapping, Complex Valued b-Metric Space, common fixed point.*

***2010 AMS Subject Classification:** 47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

In 2011, Azam et al.[1] introduced the concept of complex valued metric space and proved some fixed point theorem for mappings satisfying a rational inequality. After then, many authors have worked in this direction see in [8, 9, 12 and 13].

Recently, Rao *et al.* [11] introduced the concept of complex valued b-metric space which is more general than the notion of well known complex valued metric space and proved some common fixed point results. Further, several authors [2, 3, 4, 5, 6, 7, 10] continue the study of common fixed point in complex valued b-metric space.

In this paper, we establish common fixed point theorem for rational type inequality in the framework of complex valued b-metric spaces.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) or (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied, Notice that

(C1) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$,
(C2) $z_1 \preccurlyeq z_{2,} z_2 \prec z_3 \Rightarrow z_1 \prec z_3$,
(C3) if $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}$.

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The following definition is recently introduced by Rao *et al.* [11].

Definition 1.1: [11] Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{C}$ is called a complex valued b-metric if the following conditions are satisfied:

(1) $0 \precsim d(x,y)$ and $d(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,y) \precsim s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a complex valued b-metric space.

Example 1.2: [11] Let $X = [0,1]$. Define the mapping $d: X \times X \to \mathbb{C}$ by $d(x,y) = |x-y|^2 + i|x-y|^2$ for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 1.3: Let (X, d) be a complex valued b-metric space.

(1) A point $x \in X$ is called an interior point of a subset $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X: d(x,y) \prec r\} \subseteq A$.
(2) A point $x \in X$ is called a limit of A whenever for every $0 \prec r \in \mathbb{C}$ such that $B(x,r) \cap (A - \{x\}) \neq \emptyset$.
(3) The set A is called open whenever each element of A is an interior point of A. A subbset B is called closed whenever each limit point of B belongs to B.
(4) A Sub-basis for a Hausdorff topology τ on X is a family $\mathcal{F} \coloneqq \{B(x,r): x \in X, 0 \prec r\}$.

Definition 1.4: [11] Let (X, d) be a complex valued b-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is a called convergent, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \prec c$. Also, $\{x_n\}$ converges to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) and x is the limit of $\{x_n\}$.
(ii) $\{x_n\}$ is called a Cauchy sequence in X, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x_{n+m}) \prec c$. If for every Cauchy sequence converges in X, then X is called a complete complex valued b-metric space.

Lemma 1.5: [11] Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} |d(x_n, x)| = 0$.

Lemma 1.6: [11] Let (X, d) be a complex valued b- metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.

Definition 1.7: If f and g are mappings from a metric space (X, d) into itself, are called commuting on X, if $d(fgx, gfx) = 0$ for all $x \in X$.

Definition 1.8: If f and g are mappings from a metric space (X, d) into itself, are called weakly commuting on X, if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Definition 1.9: If f and g are mappings from a metric space (X, d) into itself are called compatible on X, if $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = x$, for some point $x \in X$.

Definition 1.10: Let f and g be two self-maps defined on a set X, then f and g are said to be weakly compatible if they commute at coincidence point.

Lemma 1.11: Let f and g be compatible mappings from a metric space (X, d) into itself. Suppose that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = x$, for some point $x \in X$. Then $\lim\limits_{n\to\infty} gfx_n = fx$, if f is continuous.

2. MAIN RESULTS

Theorem 2.1: Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$. Suppose that the mappings f, g, S and $T: X \to X$ satisfying

(i) $S \subset g, T \subset f$;
(ii) $d(Sx, Ty) \precsim \propto d(fx, gy) + \beta \left[\frac{d(fx,Sx)d(gy,Ty)}{d(fx,Ty)+d(gy,Sx)+d(fx,gy)}\right]$ for all $x, y \in X$ such that $x \neq y, d(fx, Ty) + d(gy, Sx) + d(fx, gy) \neq 0$ where $\propto, \beta$ are nonnegative reals with $\propto + s\beta < 1$.
(iii) Suppose that one of S or f is continuous, pair (S, f) is compatible and (T, g) is weak compatible.
(iv) One of T or g is continuous, pair (S, f) is weak compatible and (T, g) is compatible. Then f, g, S and T have a unique common fixed point in X.

Proof: Suppose $x_0 \in X$ be an arbitrary point. We define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = gx_{2n+1}$$
$$y_{2n+1} = Tx_{2n+1} = fx_{2n+2},\ n = 0,1,2, \dots \dots \dots$$

Then,

$$\begin{aligned} d(y_{2n,}y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \propto d(fx_{2n,}, gx_{2n+1}) + \beta\left[\frac{d(fx_{2n}, Sx_{2n})d(gx_{2n+1}, Tx_{2n+1})}{d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n}) + d(fx_{2n}, gx_{2n+1})}\right] \\ &= \propto d(y_{2n-1}, y_{2n}) + \beta\left[\frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n})}\right] \\ &= \propto d(y_{2n-1}, y_{2n}) + \beta\left[\frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{d(y_{2n-1}, y_{2n+1}) + d(y_{2n-1}, y_{2n})}\right] \\ &= \propto d(y_{2n-1}, y_{2n}) + s\beta\left[\frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{d(y_{2n}, y_{2n+1})}\right] \end{aligned}$$

$$d(y_{2n}, y_{2n+1}) \precsim (\propto + s\beta)d(y_{2n}, y_{2n-1}).$$

Similarly, we can show that

$$d(y_{2n+1}, y_{2n+2}) \precsim (\propto + s\beta)d(y_{2n}, y_{2n+1}).$$

If $(\propto + s\beta) = \delta < 1$, then

$$|d(y_{2n+1}, y_{2n+2})| \leq \delta|d(y_{2n}, y_{2n+1})| \leq -- \leq \delta^{2n+1}|d(y_0, y_1)|.$$

Let $m, n \geq 1$ *and* $m > n$, we have

$$\begin{aligned} |d(y_{2n}, y_{2m})| &\leq s|d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2m})| \\ &= s|d(y_{2n}, y_{2n+1})| + s|d(y_{2n+1}, y_{2m})| \\ &\leq s|d(y_{2n}, y_{2n+1})| + s^2|d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, y_{2m})| \\ &= s|d(y_{2n}, y_{2n+1})| + s^2|d(y_{2n+1}, y_{2n+2})| + s^2|\, d(y_{2n+2}, y_{2m})\,| \\ &\leq s|d(y_{2n}, y_{2n+1})| + s^2|d(y_{2n+1}, y_{2n+2})| + s^3|d(y_{2n+2}, y_{2n+3})| \\ &\quad + ---- + s^{2n+2m-1}|d(x_{2n+2m-1}, x_{2m})| \\ &\leq [s\delta^{2n} + s^2\delta^{2n+1} + s^3\delta^{2n+2} + --- + (s\delta)^{2m-1}]|d(y_0, y_1)| \\ &\leq \left[\frac{s\delta^{2n}}{1 - s\delta}\right]|d(y_0, y_1)| \end{aligned}$$

and so

$$|d(y_{2n}, y_{2m})| \leq \left[\frac{s\delta^{2n}}{1-s\delta}\right]|d(y_0, y_1)| \to 0 \text{ as}, m, n \to \infty.$$

Hence $\{y_{2n}\}$ is a Cauchy sequence and since X is complete, sequence $\{y_{2n}\}$ converges to point u in X and its subsequences $Sx_{2n}, Tx_{2n+1}, fx_{2n+2}$ and gx_{2n+1} of sequence $\{y_{2n}\}$ also converges to point u.

Let f is continuous and since S and f are compatible on X. Then by Lemma (1.11), we have f^2x_{2n} *and* $Sfx_{2n} \to fu$ as $n \to \infty$.

Consider

$$d(Sfx_{2n}, Tx_{2n+1}) \precsim \propto d(f^2x_{2n}, gx_{2n+1}) + \beta\left[\frac{d(f^2x_{2n}, Sfx_{2n})d(gx_{2n+1}, Tx_{2n+1})}{d(f^2x_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sfx_{2n}) + d(f^2x_{2n}, gx_{2n+1})}\right].$$

Letting $n \to \infty$, we get

$$d(fu, u) \precsim \propto d(fu, u) + \beta\left[\frac{d(fu, fu)d(u, u)}{d(fu, u) + d(u, fu) + d(fu, u)}\right].$$
$$(1-\propto)d(fu, u) \precsim 0 \text{ so that } fu = u.$$

Again consider

$$d(Su, Tx_{2n+1}) \precsim \propto d(fu, gx_{2n+1}) + \beta\left[\frac{d(fu, Su)d(gx_{2n+1}, Tx_{2n+1})}{d(fu, Tx_{2n+1}) + d(gx_{2n+1}, Su) + d(fu, gx_{2n+1})}\right].$$

Letting $n \to \infty$, we get

$$d(Su, u) \precsim \propto d(u, u) + \beta\left[\frac{d(u, Su)d(u, u)}{d(u, u) + d(u, Su) + d(u, u)}\right].$$
$$d(Su, u) \precsim 0 \text{ so that } Su = u.$$

Now since $S \subset g$ and there exists another point w in X, such that $u = Su = gw$.

Consider

$$d(u,Tw) = d(Su,Tw)$$
$$\precsim \propto d(fu,gw) + \beta\left[\frac{d(fu,Su)d(gw,Tw)}{d(fu,Tw)+d(gw,Su)+d(fu,gw)}\right]$$
$$\precsim \propto d(u,u) + \beta\left[\frac{d(u,u)d(u,Tw)}{d(u,Tw)+d(u,u)+d(u,u)}\right]$$

$d(u,Tw) \precsim 0$ so that $Tw = u$.

Since T and g are weak compatible on X and $Tw = gw \text{ and } Tgw = gTw$.

Consider

$$d(u,gu) = d(Su,Tu)$$
$$\precsim \propto d(fu,gu) + \beta\left[\frac{d(fu,Su)d(gu,Tu)}{d(fu,Tu)+d(gu,Su)+d(fu,gu)}\right]$$
$$d(u,gu) \precsim \propto d(u,gu) + \beta\left[\frac{d(u,u)d(gu,Tu)}{d(u,Tu)+d(gu,u)+d(u,gu)}\right]$$

$(1-\propto)d(u,gu) \precsim 0$ so that $gu = u$.

Hence $fu = gu = Su = Tu = u$.

Thus u is a common fixed point of $f, g, S \text{ and } T$. similarly, we can show that u is a common fixed point of $f, g, S \text{ and } T$, when S is continuous. Next, we will prove the (iv) part of Theorem 2.1.

Let T is continuous and since T and g are compatible on X. Then by Lemma (1.11), we have T^2x_{2n} and $gTx_{2n} = Tu$ as $n \to \infty$.

Consider

$$d(Sx_{2n},T^2x_{2n}) \precsim \propto d(fx_{2n},gTx_{2n}) + \beta\left[\frac{d(fx_{2n},Sx_{2n})d(gTx_{2n},T^2x_{2n})}{d(fx_{2n},T^2x_{2n})+d(gTx_{2n},Sx_{2n})+d(fx_{2n},gTx_{2n})}\right].$$

Letting $n \to \infty$, we get

$$d(u,Tu) \precsim \propto d(u,Tu) + \beta\left[\frac{d(u,u)d(Tu,Tu)}{d(u,Tu)+d(Tu,u)+d(u,Tu)}\right]$$

$(1-\propto)d(u,Tu) \precsim 0$ so that $Tu = u$.

Now since $T \subset f$, there exists a point $v \text{ in } X$, such that $u = Tu = fv$.

Consider

$$d(Sv,T^2x_{2n}) \precsim \propto d(fv,gTx_{2n}) + \beta\left[\frac{d(fv,Sv)d(gTx_{2n},T^2x_{2n})}{d(fv,T^2x_{2n})+d(gTx_{2n},Sv)+d(fv,gTx_{2n})}\right].$$

Letting $n \to \infty$, we get

$$d(Sv,Tu) \precsim \propto d(u,Tu) + \beta\left[\frac{d(u,Sv)d(Tu,Tu)}{d(u,Tu)+d(Tu,Sv)+d(u,Tu)}\right]$$
$$d(Sv,u) \precsim \propto d(u,u)$$
$$d(Sv,u) \precsim 0 \text{ so that } Sv = u.$$

Since S and f are weakly compatible on X and $Sv = fv$ and $Sfv = fSv \Rightarrow Su = Sfv = fSv = fu$.

Now consider

$$d(Su,Tx_{2n+1}) \precsim \propto d(fu,gx_{2n+1}) + \beta\left[\frac{d(fu,Su)d(gx_{2n+1},Tx_{2n+1})}{d(fu,Tx_{2n+1})+d(gx_{2n+1},Su)+d(fu,gx_{2n+1})}\right].$$

Letting $n \to \infty$, we get

$$d(Su,u) \precsim \propto d(Su,u) + \beta\left[\frac{d(Su,Su)d(u,u)}{d(Su,u)+d(u,Su)+d(Su,u)}\right]$$

$(1-\propto)d(Su,u) \precsim 0 \text{ so that } Su = u.$

Now since $S \subset g$, there exists a point t in X, such that $u = Su = gt$. Now

$$d(u,Tt) = d(Su,Tt)$$
$$\precsim \propto d(fu,gt) + \beta\left[\frac{d(fu,Su)d(gt,Tt)}{d(fu,Tt)+d(gt,Su)+d(fu,gt)}\right]$$
$$\precsim \propto d(u,u) + \beta\left[\frac{d(u,u)d(u,Tt)}{d(t,Tt)+d(u,u)+d(u,u)}\right]$$
$$d(u,Tt) \precsim 0 \; so\ that\ u = Tt.$$

Since T and g are compatible on X and $Tt = gt = u$ and $d(gTt,Tgt) = 0 \Rightarrow gu = gTt = Tgt = Tu$.
Hence $Su = Tu = fu = gu = u$.

Therefore, u is common fixed point of $f, g, S\ and\ T$. Similarly, we can show that u is also common fixed point of $f, g, S\ and\ T$, when g is continuous.

To prove the uniqueness of fixed point u, assume that u^* is another common fixed point of $f, g, S\ and\ T$. Then

$$d(u,u^*) = d(Su,Tu^*)$$
$$\precsim \propto d(fu,gu^*) + \beta\left[\frac{d(fu,Su)d(gu^*,Tu^*)}{d(fu,Tu^*)+d(gu^*,Su)+d(fu,gu^*)}\right]$$
$$\precsim \propto d(u,u^*) + \beta\left[\frac{d(u,u)d(u^*,u^*)}{d(u,u^*)+d(u^*,u)+d(u,u^*)}\right]$$
$$d(u,u^*) \precsim \propto d(u,u^*)$$
$$(1-\propto)d(u,u^*) \precsim 0,$$ which is a contradiction.

Hence $u = u^*$.

Therefore, u is unique common fixed point of $f, g, S\ and\ T$.

By setting $f = g = I$ we get the following Corollary:

Corollary 2.2: Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $S,T: X \to X$ satisfy:

(i) $S \subset T$

(ii) $d(Sx,Ty) \precsim \propto d(x,y) + \beta\left[\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\right]$

for all $x, y\ in\ X$ such that $x \neq y, d(x,Ty) + d(y,Sx) + d(x,y) \neq 0$, where $\propto, \beta$ are nonnegative reals with $\propto + s\beta < 1$. If pair (S,T) is weakly compatible. Then S and T have unique common fixed point in X.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

REFERENCES

1. A. Azam, B. Fisher and M. Khan, Common Fixed Point Theorems in Complex Valued Metric Spaces, Numer. Funct. Anal. Optim, 3(3) (2011), 243-253.
2. Anil Kumar Dubey, Complex Valued b-Metric Spaces and Common Fixed Point Theorems under Rational Contractions, Journal of Complex analysis, Vol. 2016, Article ID 9786063, 7 pages, http//dx.doi.org/ 10.1155/2016/9786063.
3. Anil Kumar Dubey and Manjula Tripathi, Common Fixed Point Theorem in Complex Valued b-Metric Space for Rational Contractions, Joural of Informatics and Mathematical Sciences, 7(3)(2015), 149-161.
4. Anil Kumar Dubey, Manjula Tripathi and Ravi Prakash Dubey, Various Fixed Point Theorems in Complex Valued b-Metric Spaces, International Journal of Engineering Mathematics, Vol.2016, Article ID 7072606,7 pages, http//dx.doi.org/10.1155/2016/7072606.
5. A.K. Dubey, Manjula Tripathi, M.D. Pandey, Common Fixed Point Results for Rational Type Contraction in Complex Valued b-Metric Spaces, International Journal of Pure and Applied Mathematics, 116(2)(2017), 447-456.
6. Anil Kumar Dubey, Urmila Mishra and Manjula Tripathi, Common Fixed Points of Mappings satisfying Rational Inequality in Complex Valued b-Metric Spaces, Communications in Mathematics and Applications 8(3)(2017),289-300.
7. Anil Kumar Dubey, Shweta Bibay and R.P. Dubey, Fixed Point Theorems for Generalized Rational Contractions in Complex Valued b-Metric Spaces. International Journal of Advances in Mathematics, Vol. 2018, No. 3 (2018), 25-33.

8. A. Sindersiya, A. Pariya, N. Gupta and V.H. Badshah, Common Fixed Point Theorem in Complex Valued Metric Spaces, Adv. Fixed Point Theory, 7(4)(2017), 572-579.
9. F. Rouzkard and M. Imdad, Some Common Fixed Point Theorems on Complex Valued Metric Spaces, Comp. Math. Appl. 64(2012), 1866-1874.
10. G.S. Saluja, Some Fixed Point Theorems for Contractive type conditions in Complex Valued b-Metric Spaces, Nonlinear Functional Analysis and Applications, 22(2)(2017), 311-322.
11. K.P.R.Rao, P.R. Swamy and J.R. Prasad, A Common Fixed Point Theorem in Complex Valued b-Metric Space, Bull. Math. Stat. Res., 1(1) (2013),1-8.
12. R.K. Verma and H.K. Pathak, Common Fixed Point Theorems Using Property (E.A.) in Complex Valued Metric Spaces, Thai J. Math. 11(2) (2013), 347-355.
13. W. Sintunavarat and P. Kumam, Generalized Common Fixed Point Theorems in Complex Valued Metric Spaces and Applications, J. Inequal. Appl. doi: 10.1186/1029-242X-2012-84.

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