International Journal of Mathematical Archive-9(10), 2018, 85-88
Available online through www.ijma.info ISSN 2229 – 5046

ON THE CLASSES OF ANALYTIC FUNCTIONS DEFINED BY USING AL - OBOUDI OPERATOR

[1]YOGISH KUMAR M V AND [2]DILEEP L.

[1]Lecturer,
Department of Mathematics,
JSS PU College For Women, Mysuru- 570 009, INDIA.

[2]Associate Professor,
Department of Mathematics,
Vidyavardhaka College of Engineering, Mysuru - 570 002, INDIA.

(Received On: 19-01-18; Revised & Accepted On: 19-09-18)

ABSTRACT

In this paper, we define the new subclasses of analytic functions using the Al - Oboudi operator. For functions belonging to these classes we determine coefficient inequalities, extreme points and integral means inequalities.

Key words and phrases: *Analytic functions, Al - Oboudi operator, Coefficient inequalities, Extreme points and Integral means.*

2000 Mathematics Subject Classification: *30C45.*

1. INTRODUCTION

Let A denote the class of analytic functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disc $U = \{z; |z| < 1\}$.

Definition1.1: [1] Let $n \in \mathbb{N}$ and $\lambda \geq 0$, the Al0Oboudi operator $D_\lambda^n : A \rightarrow A$, is defined as

$$D_\lambda^0 f(z) = f(z), \quad D_\lambda^1 f(z) = (1 - \lambda) f(z) + z f'(z) = D_\lambda f(z) \ and$$
$$D_\lambda^n f(z) = D_\lambda \left(D_\lambda^{n-1} f(z) \right).$$

Further, if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ $, then\ we\ have,$

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)] a_k z^k, \ \ n \in \mathbb{N}_0. \tag{1.2}$$

Remarks 1.2: It is easy to observe that for $\lambda = 1$, we get the Săl$\breve{a}$gean operator [8].

Definition 1.3: A function $f \in A$ is said to be in the class $N_{m,n}(\alpha, \beta, \lambda)$ if

$$\mathbb{R}\left\{\frac{D_\lambda^m f(z)}{D_\lambda^n f(z)}\right\} > \beta \left|\frac{D_\lambda^m f(z)}{D_\lambda^n f(z)} - 1\right| + \alpha, \tag{1.3}$$

for some $0 \leq \alpha < 1$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ $, \lambda \geq 0$ and $z \in U$.

The following are the special cases of the class $N_{m,n}(\alpha, \beta, \lambda)$:

i. $N_{m,n}(\alpha, \beta, 1) = N_{m,n}(\alpha, \beta),$ $the\ class\ introducd\ by\ Eker\ and\ Owa$ [3].
ii. $N_{1,0}(\alpha, \beta, 1) = SD(\alpha, \beta, \lambda) and\ N_{2,1}(\alpha, \beta, \lambda) =$ $KD(\alpha, \beta),$ $the\ classes\ studied\ by\ Shams, Kulkarni\ and\ Jahangiri$ [9].
iii. $N_{m,n}(\alpha, 0,1) = K_{m,n}(\alpha),$ $be\ the\ class\ studied\ by\ Eker\ and\ Owa$ [4].
iv. $N_{1,0}(\alpha, 0,1) = S^*(\alpha) and\ N_{2,1}(\alpha, 0,1) = K(\alpha),$ $the\ classes\ introduced\ by\ Robertson$ [7].

Corresponding Author: [1]Yogish Kumar M V,
[1]Lecturer, Department of Mathematics,
JSS PU College For Women, Mysuru- 570 009, INDIA.

2. COEFFICIENT INEQUALITIES FOR THE CLASS $N_{m,n}(\alpha,\beta,\lambda)$

Theorem 2.1: *If $f \in A$ satisfies,*

$$\sum_{k=2}^{\infty}\psi(\lambda,m,n,k,\alpha,\beta)|a_k| \le 2(1-\alpha) \tag{2.1}$$

where

$$\begin{aligned}\psi(\lambda,m,n,k,\alpha,\beta) = &|(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m| \\ &+((1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m|) \\ &+2\beta|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n|\end{aligned}$$

for some $0 \le \alpha < 1,\ \beta \ge 0,\ m \in \mathbb{N},\ n \in \mathbb{N}_0, \lambda \ge 0,\ then\ f \in N_{m,n}(\alpha,\beta,\lambda)$.

Proof: Let the expression (2.1) be true for $0 \le \alpha < 1,\ \beta \ge 0,\ m \in \mathbb{N},\ n \in \mathbb{N}_0\ and\ \lambda \ge 0$. Hence to show that,

$$\begin{aligned}&\left|(1-\alpha)D_\lambda^n f(z)+D_\lambda^m f(z)-\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| \\ &\quad-\left|(1+\alpha)D_\lambda^n f(z)-D_\lambda^m f(z)+\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| > 0.\end{aligned}$$

So, we have

$$\begin{aligned}&\left|(1-\alpha)D_\lambda^n f(z)+D_\lambda^m f(z)-\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| \\ &\quad-\left|(1+\alpha)D_\lambda^n f(z)-D_\lambda^m f(z)+\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| \\ &=\left|\begin{array}{l}(2-\alpha)z+\sum_{k=2}^{\infty}\{(1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m\}a_k z^k \\ \quad -\beta e^{i\theta}\left|\sum_{k=2}^{\infty}\{[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n\}a_k z^k\right|\end{array}\right| \\ &\quad-\left|\begin{array}{l}\alpha z+\sum_{k=2}^{\infty}\{(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m\}a_k z^k \\ \quad +\beta e^{i\theta}\left|\sum_{k=2}^{\infty}\{[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n\}a_k z^k\right|\end{array}\right| \\ &\ge (2-\alpha)|z|-\sum_{k=2}^{\infty}\{(1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m\}|a_k|\,|z^k| \\ &\quad -\beta|e^{i\theta}|\sum_{k=2}^{\infty}|\{[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n\}||a_k|\,|z^k| \\ &\quad -\alpha|z|-\sum_{k=2}^{\infty}|\{(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m\}|\,|a_k|\,|z^k| \\ &\quad -\beta|e^{i\theta}|\sum_{k=2}^{\infty}|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n||a_k|\,|z^k| \\ &\ge 2(1-\alpha) \\ &\quad -\left\{\begin{array}{l}\sum_{k=2}^{\infty}|(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m| \\ +\left((1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m\right) \\ +2\beta|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n|\end{array}\right\}|a_k| \ \ge 0.\end{aligned}$$

3. RELATION FOR $\widetilde{N}_{m,n}(\alpha,\beta,\lambda)$

By Theorem 2.1, we introduce the class $\widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ as the subclass of $N_{m,n}(\alpha,\beta,\lambda)$ consisting of f satisfying

$$\sum_{k=2}^{\infty}\psi(\lambda,m,n,k,\alpha,\beta)|a_k| \le 2(1-\alpha) \tag{3.1}$$

Where

$$\begin{aligned}\psi(\lambda,m,n,k,\alpha,\beta) = &|(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m| \\ &+((1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m|) \\ &+2\beta|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n|\end{aligned}$$

for some $0 \le \alpha < 1,\ \beta \ge 0,\ m \in \mathbb{N},\ n \in \mathbb{N}_0\ and\ \lambda \ge 0$.

Theorem 3.1: *If* $f \in A, then\ \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta_2},\boldsymbol{\lambda}) \subset \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta_1},\boldsymbol{\lambda})$ ***for some*** $\boldsymbol{\beta_1}$ ***and*** $\boldsymbol{\beta_2}$ ***such that*** $\boldsymbol{0 \le \beta_1 \le \beta_2}$.

Proof: For $0 \le \beta_1 \le \beta_2$, we have

$$\sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta_1)|a_k| \le \sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta_2)|a_k|.$$

Therefore, if $f \in \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta_2},\boldsymbol{\lambda})\boldsymbol{then}$ if $f \in \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta_1},\boldsymbol{\lambda})$.

4. EXTREME POINTS

The determination of the extreme points of a family $\mathcal{F}$ of univalent functions enables us to solve many external problems for $\mathcal{F}$.

Theorem 4.1: *Let* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k, \qquad (k = 1,2,\dots; |\boldsymbol{\varepsilon}_k| = 1).$$

Then, $f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z), \quad where\ \boldsymbol{\lambda}_k \ge 0\ and \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k = 1.$$

Proof: Let $f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z), \quad \boldsymbol{\lambda}_k \ge 0 \ \ , k = 1,2,\dots \ \ with\ \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k = 1$. Then, we have

$$f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z) = \boldsymbol{\lambda}_1 z + \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k \left(z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k \right)$$
$$= z + \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k,$$

That is,

$$\sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta) \left| \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} \boldsymbol{\lambda}_k \right| = \sum_{k=2}^{\infty} 2(1-\alpha)\boldsymbol{\lambda}_k$$
$$= 2(1-\alpha)(1-\boldsymbol{\lambda}_1) \le 2(1-\alpha),$$

which is the condition (3.1) for $f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z)$. Thus $f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda). Conversly, let\ f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda)$.

Since

$$|a_k| \le \frac{2(1-\alpha)}{\psi(\lambda,m,n,k,\alpha,\beta)}, \quad (k = 2,3,\cdots)$$

We put

$$\boldsymbol{\lambda}_k = \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} a_k, \qquad (|\boldsymbol{\varepsilon}_k| = 1)$$

and $\boldsymbol{\lambda}_1 = 1 - \sum_{k=2}^{\infty} \boldsymbol{\lambda}_k$. Then $f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z)$.

Corollary 4.2: *The extreme points of* $\widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ *are the functions* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k, \qquad (k = 1,2,\dots; |\boldsymbol{\varepsilon}_k| = 1).$$

5. INTEGRAL MEANS INEQUALITIES

For any two functions $f\ and\ g$ analytic in U, f is said to be subordinate to $g\ in\ U$, denoted by $f \prec g$ if there exists an analytic function $\omega\ defined\ in\ U$ satisfying $\omega(0) = 0\ and\ |\omega(z)| < 1$ such that $f(z) = g(\omega(z)),\ z \in U$.

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)\ and\ f(U) \subset g(U)$. In 1925, Littlewood [6] proved the following subordination theorem.

Theorem 5.1: If $f\ and\ g$ are any two functions, analytic in U, with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$, (0 < r < 1),

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \le \int_0^{2\pi} |g(z)|^{\mu}\, d\theta.$$

Theorem 5.2: *Let* $f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ *and* f_k *be defined by*

$$f_k(z) = z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k, \qquad (k = 1,2,\dots; |\boldsymbol{\varepsilon}_k| = 1).$$

If there exists an analytic function $\omega(z)$ *given by*

$$|\omega(z)|^{k-1} = \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} a_k z^{k-1},$$

Then for $z = re^{i\theta}$ *and* $0 < r < 1$.

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \le \int_0^{2\pi} \left|f_k(re^{i\theta})\right|^{\mu} d\theta, \qquad (\mu > 0).$$

Proof: We have to prove that

$$\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k z^{k-1}\right|^{\mu} d\theta \le \int_0^{2\pi} \left|1 + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^{k-1}\right|^{\mu} d\theta.$$

By Theorem 5.1, it suffices to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^{k-1}.$$

By taking

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} [\omega(z)]^{k-1}$$

we get

$$[\omega(z)]^{k-1} = \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} a_k z^{k-1}.$$

Clearly, $\omega(0) = 0. By$ (3.1), we have

$$\begin{aligned} |[\omega(z)]|^{k-1} &= \left|\frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} |a_k||z^{k-1}|\right| \\ &\le \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} |a_k||z^{k-1}| \\ &\le |z| < 1. \end{aligned}$$

REFERENCES

1. F.M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Ind. J. Math. Math. Sci., 2004, No. 25-28, 1429-1436.
2. P. L.Duren, Univalent functions, Springer-Verlag, 1983.
3. S. S. Eker and S. Owa, Certain Classes of analytic functions involving Sălăgean operator, J. Inequal. Pure Appl. Math., (in coure of publication).
4. S.S. Eker and S. Owa, New applications of classes of analytic functions involving Sălăgean operator, International Symposium on Complex Function Theory and Apllications, Brasov, Romania, 1-5 (2006).
5. S.S.Eker and B. Seker, On a class of multivalent functions defined by Sălăgean operator, General Mathematics, Vol. 15, No. 2-3 (2007), 154-163.
6. J.E. Littlewood, On inequalities in the theory of functions, Proc. London Math. Soc., 23(1925), 481-519.
7. M.S Robertson, On the theory of univalent functions, Annals of Math., 37(1936), pp. 374-406.
8. G.S. Sălăgean, On some classes of univalent functions, Seminar of Geometric Function Theory, Cluj-Napoca, 1983.
9. S.Shams, S.R.Kulkarni and J.M. Jahangiri, Classes of uniformly starlike and convex functions, International Journal of Mathematics and Mathematical Science, 55 (2004), 2959-2961.

Source of support: Nil, Conflict of interest: None Declared.