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# ON THE CLASSES OF ANALYTIC FUNCTIONS DEFINED BY USING AL - OBOUDI OPERATOR

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***ABSTRACT***

*In this paper, we define the new subclasses of analytic functions using the Al - Oboudi operator. For functions belonging to these classes we determine coefficient inequalities, extreme points and integral means inequalities.*

***Key words and phrases:*** *Analytic functions, Al - Oboudi operator, Coefficient inequalities, Extreme points and Integral means.*

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## 1. INTRODUCTION

Let $A$ denote the class of analytic functions $f$ of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disc $U = \{z; |z| < 1\}$.

**Definition1.1: [1]** Let $n \in \mathbb{N}$ *and* $\lambda \geq 0$, the Al0Oboudi operator $D_\lambda^n: A \to A$, is defined as

$$D_\lambda^0 f(z) = f(z), \quad D_\lambda^1 f(z) = (1-\lambda) f(z) + z f'(z) = D_\lambda f(z) \ and$$
$$D_\lambda^n f(z) = D_\lambda \left( D_\lambda^{n-1} f(z) \right).$$

Further, if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then we have,*

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)] a_k z^k, \ \ n \in \mathbb{N}_{0.} \tag{1.2}$$

**Remarks 1.2:** It is easy to observe that for $\lambda = 1$, we get the Să̆lă̆gean operator [8].

**Definition 1.3:** A function $f \in A$ is said to be in the class $N_{m,n}(\alpha, \beta, \lambda)$ if

$$\mathbb{R}\left\{\frac{D_\lambda^m f(z)}{D_\lambda^n f(z)}\right\} > \beta \left|\frac{D_\lambda^m f(z)}{D_\lambda^n f(z)} - 1\right| + \alpha, \tag{1.3}$$

for some $0 \leq \alpha < 1,\ \beta \geq 0,\ m \in \mathbb{N},\ n \in \mathbb{N}_0,\ \lambda \geq 0$ *and* $z \in U$.

The following are the special cases of the class $N_{m,n}(\alpha, \beta, \lambda)$:

i. $N_{m,n}(\alpha, \beta, 1) = N_{m,n}(\alpha, \beta)$, *the class introducd by Eker and Owa* [3].
ii. $N_{1,0}(\alpha, \beta, 1) = SD(\alpha, \beta, \lambda)$ *and* $N_{2,1}(\alpha, \beta, \lambda) = KD(\alpha, \beta)$, *the classes studied by Shams, Kulkarni and Jahangiri* [9].
iii. $N_{m,n}(\alpha, 0, 1) = K_{m,n}(\alpha)$, *be the class studied by Eker and Owa* [4].
iv. $N_{1,0}(\alpha, 0, 1) = S^*(\alpha)$ *and* $N_{2,1}(\alpha, 0, 1) = K(\alpha)$, *the classes introduced by Robertson* [7].

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## 2. COEFFICIENT INEQUALITIES FOR THE CLASS $N_{m,n}(\alpha,\beta,\lambda)$

**Theorem 2.1:** *If $f \in A$ satisfies,*

$$\sum_{k=2}^{\infty}\psi(\lambda,m,n,k,\alpha,\beta)|a_k| \le 2(1-\alpha) \quad (2.1)$$

*where*

$$\psi(\lambda,m,n,k,\alpha,\beta) = |(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m| + ((1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m|) + 2\beta|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n|$$

*for some $0 \le \alpha < 1,\ \beta \ge 0,\ m \in \mathbb{N},\ n \in \mathbb{N}_0,\lambda \ge 0,\ then\ f \in N_{m,n}(\alpha,\beta,\lambda)$.*

**Proof:** Let the expression (2.1) be true for $0 \le \alpha < 1,\ \beta \ge 0,\ m \in \mathbb{N},\ n \in \mathbb{N}_0\ and\ \lambda \ge 0$. Hence to show that,

$$\left|(1-\alpha)D_\lambda^n f(z)+D_\lambda^m f(z)-\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| - \left|(1+\alpha)D_\lambda^n f(z)-D_\lambda^m f(z)+\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| > 0.$$

So, we have

$$\begin{aligned}
&\left|(1-\alpha)D_\lambda^n f(z)+D_\lambda^m f(z)-\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| - \left|(1+\alpha)D_\lambda^n f(z)-D_\lambda^m f(z)+\beta e^{i\theta}|D_\lambda^m f(z)-D_\lambda^n f(z)|\right| \\
&= \left|\begin{matrix}(2-\alpha)z+\sum_{k=2}^{\infty}\{(1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m\}a_kz^k \\ -\beta e^{i\theta}\left|\sum_{k=2}^{\infty}\{[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n\}a_kz^k\right|\end{matrix}\right| \\
&\quad -\left|\alpha z+\sum_{k=2}^{\infty}\{(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m\}a_kz^k + \beta e^{i\theta}\left|\sum_{k=2}^{\infty}\{[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n\}a_kz^k\right|\right| \\
&\ge (2-\alpha)|z| - \sum_{k=2}^{\infty}\{(1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m\}|a_k|\,|z^k| \\
&\quad -\beta|e^{i\theta}|\sum_{k=2}^{\infty}|\{[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n\}||a_k|\,|z^k| \\
&\quad -\alpha|z| - \sum_{k=2}^{\infty}|\{(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m\}|\,|a_k|\,|z^k| \\
&\quad -\beta|e^{i\theta}|\sum_{k=2}^{\infty}|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n||a_k|\,|z^k| \\
&\ge 2(1-\alpha) \\
&\quad -\left\{\sum_{k=2}^{\infty}|(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m| + \left((1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m\right) + 2\beta|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n|\right\}|a_k| \ge 0.
\end{aligned}$$

## 3. RELATION FOR $\widetilde{N}_{m,n}(\alpha,\beta,\lambda)$

By Theorem 2.1, we introduce the class $\widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ as the subclass of $N_{m,n}(\alpha,\beta,\lambda)$ consisting of $f$ satisfying

$$\sum_{k=2}^{\infty}\psi(\lambda,m,n,k,\alpha,\beta)|a_k| \le 2(1-\alpha) \quad (3.1)$$

Where

$$\psi(\lambda,m,n,k,\alpha,\beta) = |(1+\alpha)[1+(k-1)\lambda]^n-[1+(k-1)\lambda]^m| + ((1-\alpha)[1+(k-1)\lambda]^n+[1+(k-1)\lambda]^m|) + 2\beta|[1+(k-1)\lambda]^m-[1+(k-1)\lambda]^n|$$

for some $0 \le \alpha < 1,\ \beta \ge 0,\ m \in \mathbb{N},\ n \in \mathbb{N}_0\ and\ \lambda \ge 0$.

**Theorem 3.1:** *If* $f \in A, then\ \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta}_2,\boldsymbol{\lambda}) \subset \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta}_1,\boldsymbol{\lambda})$ ***for some*** $\boldsymbol{\beta}_1$ ***and*** $\boldsymbol{\beta}_2$ ***such that*** $\mathbf{0} \leq \boldsymbol{\beta}_1 \leq \boldsymbol{\beta}_2$.

**Proof:** For $0 \leq \beta_1 \leq \beta_2$, we have

$$\sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta_1)|a_k| \leq \sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta_2)|a_k|.$$

Therefore, if $f \in \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta}_2,\boldsymbol{\lambda})\boldsymbol{then}$ if $f \in \widetilde{\boldsymbol{N}}_{\boldsymbol{m,n}}(\boldsymbol{\alpha},\boldsymbol{\beta}_1,\boldsymbol{\lambda})$.

## 4. EXTREME POINTS

The determination of the extreme points of a family $\mathcal{F}$ of univalent functions enables us to solve many external problems for $\mathcal{F}$.

**Theorem 4.1:** *Let* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k, \qquad (k = 1,2,\ldots; |\boldsymbol{\varepsilon}_k| = 1).$$

*Then,* $f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z), \quad where\ \boldsymbol{\lambda}_k \geq 0\ and \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k = 1.$$

**Proof:** Let $f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z), \quad \boldsymbol{\lambda}_k \geq 0 \quad, k = 1,2,\ldots \quad with\ \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k = 1$. Then, we have

$$f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z) = \boldsymbol{\lambda}_1 z + \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k \left( z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k \right)$$
$$= z + \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k,$$

That is,

$$\sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta) \left| \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} \boldsymbol{\lambda}_k \right| = \sum_{k=2}^{\infty} 2(1-\alpha)\boldsymbol{\lambda}_k$$
$$= 2(1-\alpha)(1-\boldsymbol{\lambda}_1) \leq 2(1-\alpha),$$

which is the condition (3.1) for $f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z)$. Thus $f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda). Conversly, let\ f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda)$.

Since

$$|a_k| \leq \frac{2(1-\alpha)}{\psi(\lambda,m,n,k,\alpha,\beta)}, \quad (k = 2,3,\cdots)$$

We put

$$\boldsymbol{\lambda}_k = \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} a_k, \qquad (|\boldsymbol{\varepsilon}_k| = 1)$$

and $\boldsymbol{\lambda}_1 = 1 - \sum_{k=2}^{\infty} \boldsymbol{\lambda}_k$. Then $f(z) = \sum_{k=1}^{\infty} \boldsymbol{\lambda}_k f_k(z)$.

**Corollary 4.2:** *The extreme points of* $\widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ *are the functions* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k, \qquad (k = 1,2,\ldots; |\boldsymbol{\varepsilon}_k| = 1).$$

## 5. INTEGRAL MEANS INEQUALITIES

For any two functions $f\ and\ g$ analytic in $U$, $f$ is said to be subordinate to $g\ in\ U$, denoted by $f \prec g$ if there exists an analytic function $\omega\ defined\ in\ U$ satisfying $\omega(0) = 0\ and\ |\omega(z)| < 1$ such that $f(z) = g(\omega(z)),\ z \in U$.

In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)\ and\ f(U) \subset g(U)$. In 1925, Littlewood [6] proved the following subordination theorem.

**Theorem 5.1:** If $f\ and\ g$ are any two functions, analytic in U, with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$, (0 < r < 1),

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leq \int_0^{2\pi} |g(z)|^{\mu}\, d\theta.$$

 

**Theorem 5.2:** *Let* $f \in \widetilde{N}_{m,n}(\alpha,\beta,\lambda)$ *and* $f_k$ *be defined by*

$$f_k(z) = z + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^k, \qquad (k = 1,2,\dots; |\boldsymbol{\varepsilon}_k| = 1).$$

*If there exists an analytic function* $\omega(z)$ *given by*

$$|\omega(z)|^{k-1} = \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} a_k z^{k-1},$$

*Then for* $z = re^{i\theta}$ *and* $0 < r < 1$.

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu}\, d\theta \le \int_0^{2\pi} |f_k(re^{i\theta})|^{\mu}\, d\theta, \qquad (\mu > 0).$$

**Proof:** We have to prove that

$$\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k z^{k-1}\right|^{\mu} d\theta \le \int_0^{2\pi} \left|1 + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^{k-1}\right|^{\mu} d\theta.$$

By Theorem 5.1, it suffices to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} z^{k-1}.$$

By taking

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{2(1-\alpha)\boldsymbol{\varepsilon}_k}{\psi(\lambda,m,n,k,\alpha,\beta)} [\omega(z)]^{k-1}$$

we get

$$[\omega(z)]^{k-1} = \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} a_k z^{k-1}.$$

Clearly, $\omega(0) = 0. By$ (3.1), we have

$$\begin{aligned} |[\omega(z)]|^{k-1} &= \left|\frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} |a_k||z^{k-1}|\right| \\ &\le \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\boldsymbol{\varepsilon}_k} \sum_{k=2}^{\infty} |a_k||z^{k-1}| \\ &\le |z| < 1. \end{aligned}$$

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