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FACE MAGIC $\mathbb{A}$- LABELING OF GRAPHS

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ABSTRACT

In this paper, we introduce the Face magic $\mathbb{A}$-labeling of graphs, where the labels are the elements of an arbitrary abelian group $\mathbb{A}$, of order $p+q$ and the weight of each face is the product of the labels of all vertices and edges in the boundary of that face, an assignment of labels to the vertices and edges which induces an assignment of labels to the faces of a graph such that the weight of each face is constant. We discuss the Face magic $\mathbb{A}$-labeling of some graphs and the relationship between face, vertex and edge magic constants.

Key Words. *Labeling, Face magic $\mathbb{A}$-labeling of graphs, Face magic graph over $\mathbb{A}$.*

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1 INTRODUCTION

By a graph, we mean a finite, connected, undirected planar graph without loops and multiple edges. By a planar graph, we mean that it can be drawn in a plane such that no two edges intersect.The group $\mathbb{A}$ is always a finite abelian group. Let us consider $\mathbb{A} = \langle a \rangle$ where $a^n = 1_{\mathbb{A}}$ or simply 1, the identity element of $\mathbb{A}$.

A graph G is magic if the edges of G can be labeled by the numbers $1,2,3,\ldots,|E(G)|$ so that the sum of the labels of all the edges incident with any vertex is the same [1]. A group distance magic labeling of a graph $G(V,E)$ with $|V| = n$ is an injection form V to an abelian group Γ of order n such that the sum of labels of all neighbors of every vertex x in V is equal to the same element μ in Γ. A graph that admits a distance magic labeling is called a distance magic graph [4].

There are some graphs with an embedding in the plane such that all vertices belong to the unbounded face of the embedding. Such planar graph is an outer planar graph.

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. Duplication of a vertex v of a graph G by a vertex v' produces a new graph G' with $N(v') = N(v)$ where $N(v)$ is the set of all neighbors of v. Duplication of an edge $e = uv$ of a graph G by an edge $e' = u'v'$ produces a new graph G' where $N(u)$ and $N(v)$. Duplication of an edge $e = uv$ by a vertex v' in a graph G produces a new graph G' where $V(G') = V(G) \cup \{v'\}$, $E(G') = E(G) \cup \{uv', v'v\}$. Duplication of a vertex v of a graph G by an edge $e' = u'v'$ produces a new graph G' where $N(u') = \{v, v'\}$ and $N(v') = \{v, u'\}$.

The $(m,n) -$ tadpole graph G is a graph obtained by identifying a vertex of a cycle of length m and a vertex of a path of length n. A Barbell graph $D_{a,b,c}$ is a graph obtained by connecting two disjoint cyles on a and b number of vertices respectively by a path of c number of vertices.

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Let λ be a labeling over a group $\mathbb{A}$. D.Combe *et al.* [5] define the λ-weight, $\omega = \omega_\lambda$ of the graph as follows
(i) for every vertex $x \varepsilon V$, the weight is the product of the label of x and the labels of the edges incident with x, that is $\omega(x) = \lambda(x)$ x $\prod_{y \in V:xy} \lambda(xy)$; (ii) for every edge $xy \in E$, the weight is the product of the labels of xy and the labels of x and y, that is $\omega(xy) = \lambda(x)$ x $\lambda(xy)$ x $\lambda(y)$.

The labeling λ is a vertex magic $\mathbb{A}$-labeling of G if there is an element $h = h(\lambda)$ of $\mathbb{A}$ such that for every $x \in V, \omega(x) = h$. The element h is the vertex constant. The labeling λ is an edge magic $\mathbb{A}$-labeling of G if there is an element $k = k(\lambda)$ of $\mathbb{A}$ such that for every $xy \in E, \omega(xy) = k$. The element k is the edge constant.

The labeling λ is a totally magic $\mathbb{A}$-labeling of G if it is both a vertex magic $\mathbb{A}$-labeling and an edge magic $\mathbb{A}$-labeling. A graph G is said to be vertex $\mathbb{A}$-magic if there exists a vertex magic labeling over $\mathbb{A}$. Similarly G is edge $\mathbb{A}$-magic if there exists an edge magic labeling over $\mathbb{A}$; and G is totally $\mathbb{A}$-magic if there exists a totally magic labeling over $\mathbb{A}$ [5].

Two $\mathbb{A}$-labelings λ_1, λ_2 of G are translation equivalent, denoted by $\lambda_1 \sim_T \lambda_2$ if there exists $a \varepsilon \mathbb{A}$ such that $\forall z \in V \cup E, \lambda_1(z) = a\lambda_2(z)$. The labeling λ_1 is the translation of λ_2 by a and is denoted by $\lambda_1 = a\lambda_2$. Clearly, translation equivalence is an equivalence relation on labelings [5].

Motivated by these works, we introduce face magic $\mathbb{A}$-labeling as follows:

The labeling ψ is a face magic $\mathbb{A}$-labeling of G if there is an element $l = l(\psi)$ of $\mathbb{A}$ such that for every face $f \in F, \omega(f) = l$. We define for every $f \in F$, the weight is the product of the labels of all vertices and edges in the boundary of the face f, that is $\omega(f) = \prod_{x in f} \psi(x)$ x $\prod_{xy in f} \psi(xy)$

The element l is the face constant. A graph G is said to be face magic graph over $\mathbb{A}$ if there exists a face magic labeling over $\mathbb{A}$.

In this paper, We introduce face magic $\mathbb{A}$-labeling of G and derive the relation between vertex, edge and face constants of a graph G. Also we discuss the face magic $\mathbb{A}$-labeling of some graphs obtained by duplication of any one of a vertex or an edge of a graph.

2 MAIN RESULTS

Proposition 2.1: *Let G be a graph having $\mathbb{A}$-labeling which is both edge magic $\mathbb{A}$-labeling with edge constant k and a face magic $\mathbb{A}$-labeling with face constant l. Then the product of the vertex labels of any s-sided face is $k^s l^{-1}$.*

Proof: Let f be a s-sided face in G which admits a $\mathbb{A}$-labeling ψ . Product of all edge constants of all edges in f is

$$= k^s = \left[\prod_{x \varepsilon f} \psi(x)\right]^2 . \left[\prod_{xy \varepsilon f} \psi(xy)\right]$$
$$= \left[\prod_{x \varepsilon f} \psi(x)\right] . \left[\prod_{xy \varepsilon f} \psi(xy)\right] . \left[\prod_{x \varepsilon f} \psi(x)\right]$$
$$= l . \prod_{x \varepsilon f} \psi(x)$$

Hence the result follows.

Proposition 2.2: *Let G be a graph having $\mathbb{A}$-labeling which is both vertex magic $\mathbb{A}$-labeling with vertex constant h and a face magic $\mathbb{A}$-labeling with face constant l. Then the product of all the edge labels incident with every vertex of any s-sided face is $h^s l^{-1}$.*

Proof: Let f be a s-sided face in G which admits a $\mathbb{A}$-labeling ψ . Product of all vertex constants of all vertices in s-sided face f is

$$= h^s = \prod_{x \varepsilon f} \omega(x)$$
$$= \prod_{x \varepsilon f} \left\{\psi(x)\left[\prod_{xy \varepsilon f} \psi(xy)\right]^2 \left[\prod_{xz \varepsilon f} \psi(xz)\right]^{d_x - 2}\right\}$$
$$= \prod_{x \varepsilon f} \left\{\psi(x) \prod_{xy \varepsilon f} \psi(xy)\right\} . \prod_{x \varepsilon f} \left\{\prod_{xy \varepsilon f} \psi(xy) \prod_{xz \varepsilon f} \psi(xz)\right\}$$
$$= l . \prod_{x \varepsilon f} \left(\prod_{xy \varepsilon E(G)} \psi(xy)\right)$$

Thus the product of all the edge labels incident with every vertex of the s-sided face is $h^s l^{-1}$.

Proposition 2.3: *Let G be a planar graph in which all its edges are in the boundary of its exterior face. If G has a face magic $\mathbb{A}$-labeling with face constant l, then*

$$l = \begin{cases} 1_{\mathbb{A}}, & \text{if } p + q \text{ is odd} \\ a^{(p+q)/2}, & \text{if } p + q \text{ is even} \end{cases}$$

Proof: Let G has a face magic $\mathbb{A}$-labeling ψ with face constant l.Let f_0 be the face in G which covers all vertices and edges of G. Then

$$l = \prod_{x\varepsilon f_0} \psi(x).\prod_{xy\varepsilon f_0} \psi(xy)$$
$$= 1.a.a^2 \ldots a^{p+q-1}$$
$$= a^k, \text{where } k = \frac{(p+q)(p+q-1)}{2}$$

when $p+q$ is odd, $a^k = (a^{p+q})^{\frac{p+q-1}{2}} = 1_{\mathbb{A}}$. Also,

$$\left(a^{\frac{p+q}{2}}\right)^r = \begin{cases} a^{\frac{p+q}{2}}, & \text{if } p+q \text{ is even and r is odd} \\ 1_{\mathbb{A}}, & \text{if } p+q \text{ is even and r is even} \end{cases}$$

So, $a^k = a^{\frac{p+q}{2}}$ when $p+q$ is even. Hence the result follows.

Consider the outer planar graph G isomorphic to (m,n)-Tadpole graph.

Let $\{u_1, u_2, \ldots, u_m\}$ be the vertices on cycle and $\{v_0, v_1, v_2, \ldots, v_n\}$ be the vertices on path where $u_1 = v_0$. In G, $|V(G)| = m+n, |E(G)| = m+n$ and $|V(G)| + |E(G)| = 2m+2n$, $F(G) = \{f_0, f_1\}$ where f_0 is the outer face having all vertices and edges and f_1 is the cycle C_m.

Consider any cyclic group $\langle a\rangle$ of order $2m+2n$. Then $\langle a\rangle = \{1, a, a^2, \ldots, a^{2m+2n-1}\}$ where $a^{2m+2n} = 1$. The labels of the path in the sequence vertex - edge - vertex - edge - vertex... are given as $a^{2m+n}, a^2, a^{2m+n+1}, a^3, a^{2m+n+2}, a^4, \ldots, a^{2m+2n-1}$. The labels for the cycle C_n starting with identified vertex in the sequence vertex - edge - vertex - edge - vertex... are given as $1, a^{n+m}, a^{n+1}, a^{n+m+1}, a^{n+2}, \ldots, a^{n+m-1}, a^{2m+n-1}$. The above labels are the labels in the face f_1. Therefore,Weight of the face f_1 is

$$= 1.a^{n+m}.a^{n+1}.a^{n+m+1}.a^{n+2} \ldots a^{n+m-1}.a^{2m+n-1}$$
$$= a^{m(n+m)}.a^{2(1+2+\ldots+(m-1))}.a^{n(m-1)}$$
$$= a^{m(2n+2m)-(m+n)}$$
$$= a^{(m+n)}, \text{where } p+q = 2n+2m \text{ is even and } a^{2n+2m} = 1.$$

Now, Weight of the outer face f_0 is the product of all elements in the cyclic group which is

$$= 1.a.a^2.a^3 \ldots a^{2m+2n-1}$$
$$= a^{(2m+2n)(2m+2n-1)/2}$$
$$= a^{m+n}, \text{where } p+q = 2n+2m \text{ is even and } a^{2n+2m} = 1.$$

This example shows that (m,n)-Tadpole graph is a face magic graph over $\mathbb{A}$ with face constant a^{m+n} and illustrates proposition 2.3 when $p+q$ is even.

Consider the outer planar graph G isomorphic to a Barbell graph $D_{n,n,2}$.

Let $V(G) = V(C_1) \cup V(C_2) = \{u_1, u_2, \ldots, u_3\} \cup \{v_1, v_2, \ldots, v_n\}$, $E(G) = \{u_i u_{i+1}: 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{v_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{u_n v_n\}$ and $F(G) = \{f_0, f_1, f_2\}$ where f_1, f_2 are cycles C_1, C_2 respectively and f_0 is the face having all the vertices and edges of G. In G, $p = |V(G)| = 2n,\ q = |E(G)| = 2n+1,\ deg(f_i) = n,\ i = 1,2,$ $deg(f_0) = 4n+1$. So, $p+q = 4n+1$ which is odd.

Consider any cyclic group $\langle a\rangle$ of order $4n+1$. Then $\langle a\rangle = \{1, a, a^2, \ldots, a^{4n}\}$ where $a^{4n+1} = 1$. Label the bridge of G by the identity element 1 of $\langle a\rangle$. Label the first cycle C_1starting with vertex u_1 and end with the edge $u_n u_1$ in the sequence vertex - edge - vertex - edge - vertex...edge as $a, a^4, a^5 a^8, a^9, a^{12}, \ldots, a^{4n-3}, a^{4n}$. Label the second cycle C_2 starting with vertex v_1 and end with the edge $v_n v_1$ in the sequence vertex - edge - vertex - edge - vertex...edge as $a^2, a^3 a^6, a^7, a^{10}, \ldots, a^{4n-6}, a^{4n-2}, a^{4n-1}$.

Now, the weight of the face f_1 is the product of all the labels of vertices and edges in C_1 which is

$$= [a.a^5.a^9 \ldots a^{4n-3}].[a^4.a^8.a^{12} \ldots a^{4n}]$$
$$= [a^{4(1+2+\ldots+n)-3n}].[a^{4(1+2+\ldots+n)}]$$
$$= a^{4n^2+n} = a^{n(4n+1)} = 1_{\mathbb{A}}.$$

Also the weight of the face f_2 is the product of all the labels of vertices and edges in C_2 which is

$$= [a^2.a^6.a^{10} \ldots a^{4n-2}].[a^3.a^7 \ldots a^{4n-6}.a^{4n-1}]$$
$$= [a^{2(1+3+\ldots+(2n-1))}].[a^{4(1+2+\ldots+n)-n}]$$
$$= a^{4n^2+n} = a^{n(4n+1)} = 1_{\mathbb{A}}.$$

Now, the weight of the outer face f_0 is the product of all elements in the cyclic group which is

$$= 1.a.a^2 \ldots .a^{4n}$$
$$= a^{2n(4n+1)} = 1_{\mathbb{A}}.$$

This example shows that $D_{n,n,2}$ is a face magic graph over $\mathbb{A}$ with face constant $1_{\mathbb{A}}$, and illustrates proposition 2.3 when $p+q$ is odd.

Corollary 2.4: *Every Tree is a face magic graph over $\mathbb{A}$ where $\mathbb{A}$ is a cyclic group of order $2n-1$.*

Proposition 2.5: *If any $2-$connected graph which is totally magic with vertex constant h and an edge constant k admits a face magic $\mathbb{A}$- labeling ψ of G with a face constant $l_{\mathbb{A}}$,then the product of all face constants of G is $i^{\mathbb{A}}.h^p.k^q$, where p,q and r are the number of vertices, edges and faces of G respectively and $i_{\mathbb{A}} = \prod_{g\varepsilon\mathbb{A}} g$.*

Proof: Product of all face constants of G is l^r which is

$$= \prod_{e\varepsilon E} \psi(e).\prod_{v\varepsilon V} (\psi(v))^{d_v}$$
$$= [\prod_{e\varepsilon E} \psi(e).\prod_{v\varepsilon V} \psi(v)].[\prod_{e\varepsilon E} \psi(e)].[\prod_{v\varepsilon V} (\psi(v))^{d_v-1}]$$
$$= i_{\mathbb{A}}.h^p.i_{\mathbb{A}}.i_{\mathbb{A}}^{-1}.k^q$$
$$= i_{\mathbb{A}}.h^p.k^q$$

Hence the result follows.

Proposition 2.6: *If $\mathbb{A}$ is of even order and ψ is a face magic $\mathbb{A}$-labeling of G, then there exists an element a such that $a\psi$ is also a face magic $\mathbb{A}$-labeling of G.*

Proof: If $\mathbb{A}$ is of even order, then there exists a in G such that $a^2 = e$. For each s-sided face f, $(a\psi)f = a^{2s}\psi(f) = \psi(f)$. Hence the result follows.

Proposition 2.7: *Let $\mathbb{A}$ be an abelian group in which every element is an involution and ψ be a face magic $\mathbb{A}$-labeling of G. Let T_ψ be the set of all face magic $\mathbb{A}$ labelings obtained from the translations of ψ, that is, $T_\psi = \{a\psi: a\varepsilon\mathbb{A}\}$. Then T_ψ forms an abelian group under the operation defined by $(a_1\psi * a_2\psi)(x) = (a_1a_2)\psi(x), x\varepsilon V\cup E$*

Proof: The set T_ψ is closed under the operartion $*$, since if $a_1\psi, a_2\psi \in T_\psi, (a_1\psi * a_2\psi)(x) = (a_1a_2)\psi(x) \in T_\psi$, by proposition 2.6.

If $a_1\psi, a_2\psi \in T_\psi$, then$(a_1\psi * a_2\psi)*(a_3\psi) = (a_1a_2)\psi * a_3\psi = ((a_1a_2)a_3)\psi = (a_1(a_2a_3))\psi = a_1\psi * (a_2\psi * a_3\psi)$ and hence associativity of T_ψ follows.

Also, the element $e\psi\varepsilon T_\psi, e\varepsilon\mathbb{A}$ is considered as an identity element, since $(a\psi * e\psi)(x) = (ae)\psi(x) = (a\psi)(x) = ((ea)\psi)(x) = (e\psi * a\psi)(x)$.

Since $(a\psi * a\psi)(x) = (a^2)\psi(x) = (e\psi)(x)$, every element in T_ψ has a self inverse in T_ψ.

As $(a_1\psi * a_2\psi)(x) = (a_1a_2)\psi(x) = (a_2a_1)\psi(x)$,by the commutative property in $\mathbb{A}$, equals $(a_2\psi * a_1\psi)(x)$. Hence $(T_\psi,*)$ is an abelian group.

Proposition 2.8: *The graph obtained by duplication of any vertex in cycle C_n by a vertex admits a face magic $\mathbb{A}$-labeling with face constant $1_{\mathbb{A}}$, for $n \geq 3$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n, for $n \geq 3$. By the graph isomorphism, duplicate the vertex v_1 by a vertex $v_1{'}$. Let the resultant graph be G whose vertex set is $V(C_n)\cup\{v_1{'}\}$,edge set is $E(C_n)\cup\{v_1{'}v_n, v_1{'}v_2\}$ and face set is $\{f_0, f_1, f_2\}$ where f_1 is the cycle C_n, f_2 is the face bounded by the edges $v_1v_2, v_2v_1{'}, v_1{'}v_n, v_nv_1$ and f_0 is the outer face bounded by all the edges of G except v_1v_2, v_nv_1.

Consider any cyclic group $\langle a\rangle$ of order $p+q = 2n+3$.

Define a labeling $\psi: V(G)\cup E(G) \to \langle a\rangle$ as follows.

$$\psi(v_i) = \begin{cases} a^n, & i=1 \\ a^{2n+2}, & i=2 \\ 1, & i=3 \\ a^{i-1}, & 4\leq i\leq n-1 \\ a, & i=n, \end{cases}$$
$$\psi(v_1{'}) = a^{2n+1},$$
$$\psi(v_1v_n) = 2,$$
$$\psi(v_1{'}v_2) = a^{n+2},$$
$$\psi(v_1{'}v_n) = a^{n+3} \text{ and}$$
$$\psi(v_iv_{i+1}) = \begin{cases} a^{n+1}, & i=1 \\ a^{n-1}, & i=2 \\ n+i+1, & 3\leq i\leq n-1 \end{cases}$$

Then the induced face labeling is given by

$$\psi^*(f_1) = \prod_{i=1}^{n} \psi(v_i).\prod_{i=1}^{n} \psi(v_i v_{i+1}).\psi(v_n v_1)$$
$$= a^n.a^{2n+2}.1.a^3.a^4 \dots a^{n-2}.a.a^{n+1}.a^{n-1}.a^2.a^{n+4}.a^{n+5} \dots a^{2n}$$
$$= [1.a.a^2.a^3 \dots a^{n-2}.a^{n-1}.a^n.a^{n+1}].[a^{n+4}.a^{n+5} \dots a^{2n}].a^{2n+2}$$
$$= a^{[2n(2n+1)/2]-(n+2)-(n+3)+(2n+2)}$$
$$= a^{(2n+3)(n-1)} = 1_{\mathbb{A}},$$
$$\psi^*(f_0) = \prod_{i=2}^{n} \psi(v_i).\psi(v_1{}').\prod_{i=3}^{n-1} \psi(v_i v_{i+1}).\psi(v_1{}'v_2).\psi(v_1{}'v_n).\psi(v_2 v_3)$$
$$= a^{2n+1}.a^{2n+2}.1.a^3.a^4 \dots a^{n-2}.a^{n+2}.a^{n+3}.a^{n-1}.a^{n+4}.a^{n+5} \dots a^{2n}.a$$
$$= [a.a^3.a^4 \dots a^{n-2}.a^{n-1}.a^{n+2}.a^{n+3}].[a^{n+4}.a^{n+5} \dots a^{2n}].a^{2n+1}.a^{2n+2}$$
$$= a^{[(2n+2)(2n+3)/2]-2-n-(n+1)}$$
$$= a^{(2n+3)(n)} = 1_{\mathbb{A}} \text{ and}$$
$$\psi^*(f_2) = \psi(v_1).\psi(v_2).\psi(v_n).\psi(v_1{}'v_2).\psi(v_1{}'v_n).\psi(v_1{}').\psi(v_1{}'v_2).\psi(v_1{}'v_n).\psi(v_1 v_2).\psi(v_1 v_n)$$
$$= a^n.a^{2n+2}.a.a^{n+2}.a^{n+3}.a^{2n+1}.a^{n+1}.a^2$$
$$= a^{8n+12} = a^{(2n+3)4} = 1_{\mathbb{A}}.$$

Thus ψ admits a face magic $\mathbb{A}$-labeling with face constant $1_{\mathbb{A}}$.

A Face magic $\mathbb{A}$-labeling of a vertex duplication of C_9 by a vertex is shown in Figure 1.

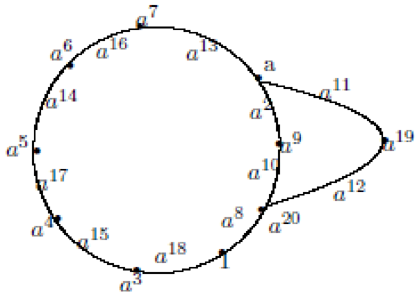

Figure-1: Face magic $\mathbb{A}$-labeling of a vertex duplication of C_9 by a vertex with face constant $1_{\mathbb{A}}$.

Proposition 2.9: *The graph G obtained by duplication of any edge in cycle C_n by an edge admits a face magic $\mathbb{A}$-labeling with face constant $1_{\mathbb{A}}$ for $n \geq 3$.*

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n, for $n \geq 3$. By the graph isomorphism, duplicate the edge $v_1 v_2$ by a vertex $v_1{}'v_2{}'$. Let the resultant graph be G whose vertex set is $V(C_n) \cup \{v_1{}'\} \cup \{v_2{}'\}$, edge set is $E(C_n) \cup \{v_1{}'v_n, v_1{}'v_2{}', v_2{}'v_3\}$ and face set is $\{f_0, f_1, f_2\}$ where f_1 is the cycle C_n, f_2 is the face bounded by the edges $v_1v_2, v_2v_3, v_3v_2{}', v_1{}'v_2{}', v_1{}'v_n, v_nv_1$ and f_0 is the outer face bounded by all the edges of G except v_1v_2, v_2v_3, v_nv_1.

Consider any cyclic group $\langle a \rangle$ of order $p + q = 2n + 5$.

Define a labeling $\psi: V(G) \cup E(G) \to \langle a \rangle$ as follows.

$$\psi(v_i{}') = \begin{cases} a^n, & i = 1 \\ a^{n+2}, & i = 2, \end{cases}$$
$$\psi(v_i) = \begin{cases} a^{n+3}, & i = 1 \\ a^{n+5}, & i = 2 \\ a^{2n+4}, & i = 3 \\ a^{n+i+3}, & 4 \leq i \leq n-1 \\ a, & i = n, \end{cases}$$
$$\psi(v_i v_{i+1}) = \begin{cases} a^{n-1}, & i = 1 \\ a^{n+1}, & i = 2 \\ a^i, & 3 \leq i \leq n-2 \\ 1, & i = n-1, \end{cases}$$
$$\psi(v_n v_1) = a^2, \psi(v_{1'} v_n) = a^{n+4},$$
$$\psi(v_1{}'v_2{}') = a^{n+6} \text{ and } \psi(v_2{}'v_3) = a^{2n+3}.$$

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Then the induced face labeling is given by

$$\psi^*(f_1) = \prod_{i=1}^{n} \psi(v_i).\prod_{i=1}^{n-1} \psi(v_i v_{i+1}).\psi(v_n v_1)$$
$$= a^{n+3}.a^{n+5}.a^{2n+4}.a^{n+7}.a^{n+8} \dots a^{2n+2}.a.a^{n-1}.a^{n+1}.a^3.a^4 \dots a^{n-2}.1.a^2$$
$$= 1.a.a^2.a^3 \dots a^{n-2}.a^{n-1}.a^{n+1}.a^{n+3}.a^{n+5}.a^{n+7}.a^{n+8} \dots a^{2n+2}.a^{2n+4}$$
$$= a^{[(2n+4)(2n+5)/2]-n-(n+2)-(n+4)-(n+6)-(2n+3)}$$
$$= a^{2n^2+3n-5} = a^{(2n+5)(n-1)} = 1_{\mathbb{A}},$$

$$\psi^*(f_0) = \prod_{i=3}^{n} \psi(v_i).\psi(v_1{}').\psi(v_2{}').\prod_{i=3}^{n-1} \psi(v_iv_{i+1}).\psi(v_1{}'v_2).\psi(v_1{}'v_n).\psi(v_2v_3)$$
$$= a^{2n+4}.a^{n+7}.a^{n+8}\ldots a^{2n+2}.a.a^3.a^4\ldots a^{n-2}.1.a^{n+6}.a^{n+4}.a^{2n+3}.a^n.a^{n+2}$$
$$= 1.a.a^3.a^4\ldots a^{n-2}.a^n.a^{n+2}.a^{n+4}.a^{n+6}.a^{n+7}.a^{n+8}\ldots a^{2n+2}.a^{2n+3}.a^{2n+4}$$
$$= a^{[(2n+4)(2n+5)/2]-(n-1)-(n+1)-(n+3)-(n+5)-2}$$
$$= a^{(2n+5)(n)} = 1_{\mathbb{A}} \text{ and}$$
$$\psi^*(f_2) = \prod_{i=1}^{3} \psi(v_i).\psi(v_n).\psi(v_1).\psi(v_2).\psi(v_1v_2).\psi(v_2v_3).\psi(v_1v_n).\psi(v_1{}'v_2).\psi(v_1{}'v_n).\psi(v_2{}'v_3)$$
$$= a^{n+3}.a^{n+5}.a^{2n+4}.a.a^n.a^{n+2}.a^{n-1}.a^{n+1}.a^2.a^{n+6}.a^{n+4}.a^{2n+3}$$
$$= a^{12n+50} = a^{(2n+5)6} = 1_{\mathbb{A}}.$$

Thus ψ admits a face magic 𝔸-labeling with face constant $1_{\mathbb{A}}$.

A face magic 𝔸-labeling of edge duplication of C_8 by an edge is shown in Figure 2.

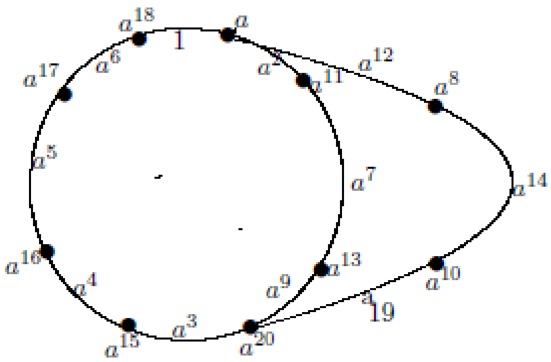

Figure-2: Face magic 𝔸-labeling of edge duplication of C_8 by an edge with face constant $1_{\mathbb{A}}$.

Proposition 2.10: *The graph G obtained by duplication of any edge in cycle C_n by a vertex admits a face magic 𝔸-labeling with face constant $1_{\mathbb{A}}$ for $n \geq 3$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n, for $n \geq 3$. By the graph isomorphism, duplicate the edge v_1v_2 by a vertex u.This vertex u may be placed inside the cycle C_n or outside the cycle C_n.

Case i: Assume that vertex u is placed outside the cycle C_n. Let the resultant graph be G whose vertex set is $V(C_n) \cup \{u\}$, edge set is $E(G) = E(C_n) \cup \{uv_1, uv_2\}$ and face set is $\{f_0, f_1, f_2\}$ where f_1 is the cycle C_n, f_2 is the face bounded by the edges v_1v_2, uv_1, uv_2 and f_0 is the outer face bounded by all the edges of G except v_1v_2.

Consider any cyclic group $\langle a \rangle$ of order $p + q = 2n + 3$.

Define a labeling $\psi: V(G) \cup E(G) \to \langle a \rangle$ as follows.

$$\psi(u) = a^{2n+1},$$
$$\psi(v_i) = \begin{cases} a, & i = 1 \\ a^{2n+2}, & i = 2 \\ a^{n+1}, & i = 3 \\ a^{n+i}, & 4 \leq i \leq n, \end{cases}$$
$$\psi(v_iv_{i+1}) = \begin{cases} 1, & i = 1 \\ a^i, & 2 \leq i \leq n-1, \end{cases}$$
$$\psi(v_nv_1) = a^n,$$
$$\psi(uv_1) = a^{n+2} \text{ and}$$
$$\psi(uv_2) = a^{n+3}.$$

Then the induced face labeling is given by

$$\psi^*(f_1) = \prod_{i=1}^{n} \psi(v_i).\prod_{i=1}^{n-1} \psi(v_iv_{i+1}).\psi(v_nv_1)$$
$$= a.a^{2n+2}.a^{n+1}.a^{n+4}.a^{n+5}\ldots a^{2n}.1.a^2.a^3\ldots a^{n-1}.a^n$$
$$= 1.a.a^2.a^3\ldots a^n.a^{n+1}.a^{n+4}.a^{n+5}.a^{n+6}\ldots a^{2n}.a^{2n+2}$$
$$= a^{(2n+2)(2n+3)/2}$$
$$= a^{2n^2+n-3} = a^{(2n+3)(n-1)} = 1_{\mathbb{A}},$$
$$\psi^*(f_0) = \prod_{i=1}^{n} \psi(v_i).\psi(u).\prod_{i=1}^{n-1} \psi(v_iv_{i+1}).\psi(v_nv_1)$$
$$= a.a^{2n+2}.a^{n+1}.a^{n+4}.a^{n+5}\ldots a^2n.a^{2n+1}.a^2.a^3\ldots a^{n-1}.a^na^{n+2}.a^{n+3}$$
$$= a.a^2.a^3\ldots a^{n-1}.a^n.a^{n+1}.a^{n+2}\ldots a^{2n}.a^{2n+1}.a^{2n+2}$$
$$= a^{(2n+2)(2n+3)/2}$$
$$= a^{(2n+3)(n+1)} = 1_{\mathbb{A}} \text{ and}$$
$$\psi^*(f_2) = \psi(v_1).\psi(v_2).\psi(u).\psi(uv_2).\psi(v_1v_2).\psi(uv_1)$$
$$= a.a^{2n+2}.a^{2n+1}.1.a^{n+2}.a^{n+3}$$
$$= a^{6n+9} = a^{(2n+3)3} = 1_{\mathbb{A}}.$$

Thus ψ admits a face magic 𝔸-labeling with face constant $1_{\mathbb{A}}$.

Case ii: Assume that vertex u is placed inside the cycle C_n. In this case, the resultant graph have the same faces as in case (i), but f_1 is an outer face and f_0 is an inner face. By the same labeling defined in case (i), the result follows.

A face magic 𝔸-labeling of G obtained by duplicating an edge of C_7 by a vertex is shown in Figure 3.

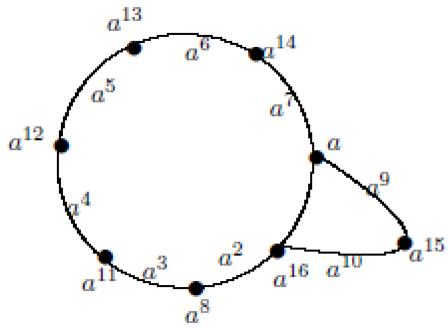

Figure-3: Face magic 𝔸-labeling of G obtained by duplicating an edge of C_7 by a vertex with face constant $1_{\mathbb{A}}$.

Proposition 2.11: *The graph G obtained by duplication of any vertex in cycle C_n by an edge admits a face magic 𝔸-labeling with face constant $1_{\mathbb{A}}$, for $n \geq 3$.*

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n, for $n \geq 3$. By the graph isomorphism, duplicate the vertex v_1 by an edge uv. Let the resultant graph be G whose vertex set is $V(C_n) \cup \{u, v\}$, edge set is $E(C_n) \cup \{uv, uv_1, vv_1\}$ and face set is $\{f_0, f_1, f_2\}$ where f_1 is the cycle C_n, f_2 is the face bounded by the edges uv, uv_1, vv_1 and f_0 is the outer face bounded by all the edges of G.

Consider any cyclic group $\langle a \rangle$ of order $p + q = 2n + 5$.

Define a labeling $\psi: V(G) \cup E(G) \to \langle a \rangle$ as follows.

$$\psi(u) = a^{n+1},$$
$$\psi(v) = a^{n+2}$$
$$\psi(v_i) = \begin{cases} 1, & i = 1 \\ a^{i+1}, & 2 \leq i \leq n-1 \\ a^{n+3}, & i = n, \end{cases}$$
$$\psi(uv) = a^{2n+4},$$
$$\psi(uv_1) = a,$$
$$\psi(vv_1) = a^2,$$
$$\psi(v_i v_{i+1}) = a^{n+3+i}, 1 \leq i \leq n-1 \text{ and}$$
$$\psi(v_n v_1) = a^{2n+3}.$$

Then the induced face labeling is given by

$$\begin{aligned} \psi^*(f_1) &= \prod_{i=1}^{n} \psi(v_i). \prod_{i=1}^{n-1} \psi(v_i v_{i+1}). \psi(v_n v_1) \\ &= 1. a^3. a^4 \dots a^n. a^{n+3}. a^{n+4}. a^{n+5} \dots a^{2n+2}. a^{2n+3} \\ &= a^{[(2n+3)(2n+4)/2]-1-2-(n+1)-(n+2)} \\ &= a^{2n^2+5n} = a^{(2n+5)(n)} = 1_{\mathbb{A}} \text{ and} \end{aligned}$$
$$\begin{aligned} \psi^*(f_2) &= \psi(u). \psi(v). \psi(v_1). \psi(uv_1). \psi(vv_1). \psi(uv) \\ &= a^{n+1}. a^{n+2}. 1. a. a^2. a^{2n+4} \\ &= a^{4n+10} = a^{(2n+5)2} = 1_{\mathbb{A}}. \end{aligned}$$

Since G is outer planar and $p + q = 2n + 5$ is odd, by proposition 2.3, $\psi^*(f_0) = 1_{\mathbb{A}}$.

Thus ψ admits a face magic 𝔸-labeling with face constant $1_{\mathbb{A}}$.

A face magic 𝔸-labeling of G obtained by duplicating a vertex of C_6 by an edge is shown in Figure 4.

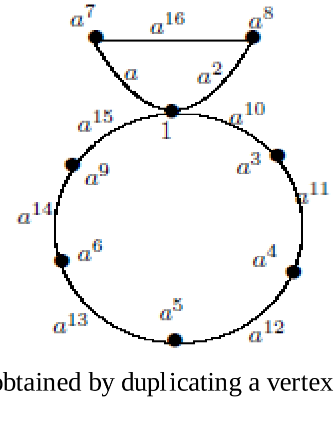

Figure-4: Face magic 𝔸-labeling of G obtained by duplicating a vertex of C_6 by an edge with face constant $1_{\mathbb{A}}$.

Proposition 2.12: *The graph G obtained by duplication of any vertex in path P_n by a vertex admits a face magic $\mathbb{A}$-labeling with face constant $a^{(p+q)/2}$ if $p+q$ is even or $1_{\mathbb{A}}$ if $p+q$ is odd, for $n \geq 2$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n, for $n \geq 2$. Let the vertex v_i be duplicated by a vertex v_i'. Then the resultant graph is G which is outer planar with all its edges in the boundary of exterior face.

Case i. $i = 1$ *or* n
If any one of the end vertices is duplicated, then G is isomorphic to a tree graph having $p + q = 2n + 1$ which is odd. By corollary 2.4, G is a Face magic graph over $\mathbb{A}$ with face constant $1_{\mathbb{A}}$, for $n \geq 1$.
Case ii. $2 \leq i \leq n - 1$

Let the resultant graph G has vertex set $V(G) = V(P_n) \cup \{v_i'\}$, edge set $E(P_n) \cup \{v_i' v_{i-1}, v_i' v_{i+1}\}$ and face set $\{f_0, f_1\}$ where f_1 is the face bounded by the edges $v_{i-1}v_i, v_i v_{i+1}, v_i' v_{i-1}, v_i' v_{i+1}$ and f_0 is the outer face bounded by all the edges of G.

Consider the cyclic group $\langle a \rangle$ of order $p + q = 2n + 2$. Assign the labels $1, a, a^2, a^3, a^{n-1}, a^n, a^{n+1}, a^{2n-1}$ to the vertices and edges of the face f_1 and assign the remaining labels to the remaining vertices and edges of the graph G. Now, the weight of the face $f_1 = a^{5n+5} = a^{2(2n+2)}.a^{n+1} = a^{n+1}$, since $a^{2n+2} = 1_{\mathbb{A}}$. Also the weight of the outer face $f_0 = 1.a.a^2.a^3 \ldots a^{2n+1} = a^{n+1}$.

Thus G obtained by duplication of any vertex in path P_n by a vertex is a face magic graph over $\mathbb{A}$.

A face magic $\mathbb{A}$-labeling of G obtained by duplicating a vertex of P_n by a vertex is shown in Figure 5.

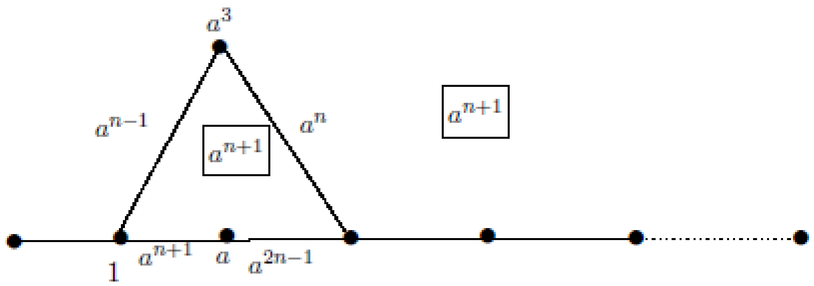

Figure-5: Face magic $\mathbb{A}$-labeling of G obtained by duplicating a vertex of P_n by a vertex.

Proposition 2.13: *The graph G obtained by duplication of any arbitrary edge in path P_n admits a face magic $\mathbb{A}$-labeling with face constant $a^{(p+q)/2}$ if $p+q$ is even or $1_{\mathbb{A}}$ if $p+q$ is odd, for $n \geq 3$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n, for $n \geq 3$. Let the edge $e = v_i v_{i+1}$ be duplicated by the edge $e' = v_i' v_{i+1}'$. Then the resultant graph is G which is outer planar with all its edges in the boundary of exterior face.

Case i. $i = 1$ *or* n
If any one of the edges $v_1 v_2$ or $v_{n-1} v_n$ is duplicated, then G is isomorphic to a tree graph having $p + q = 2n + 3$ which is odd. By corollary 2.4, G is a Face magic $\mathbb{A}$-graph with face constant $1_{\mathbb{A}}$, for $n \geq 3$.

Case ii. $2 \leq i \leq n - 2$
Let the resultant graph be G has vertex set $V(P_n) \cup \{v_i', v_{i+1}'\}$, edge set $E(P_n) \cup \{v_i' v_{i+1}', v_i' v_{i-1}, v_{i+1}' v_{i+2}\}$ and face set $\{f_0, f_1\}$ where f_1 is the face bounded by the edges $\{v_{i-1}v_i, v_i v_{i+1}, v_{i+1}v_{i+2}, v_{i+2}v_{i+1}', v_i' v_{i+1}', v_i' v_{i-1}\}$ and f_0 is the outer face bounded by all the edges of G.

Consider the cyclic group $\langle a \rangle$ of order $p + q = 2n + 4$. Assign the labels $1, a, a^2, a^3, a^4, a^{n-1}, a^n, a^{n+2}, a^{n+3}, a^{n+4}, a^{2n-1}, a^{2n+1}$ to the vertices and edges of the face f_1 and assign the remaining labels to the remaining vertices and edges of the graph. Now, the weight of the face $f_1 = a^{9n+18} = a^{4(2n+4)}.a^{n+2} = a^{n+2}$, since $a^{2n+4} = 1_{\mathbb{A}}$. Also, the weight of the outer face $f_0 = 1.a.a^2.a^3 \ldots a^{2n+3} = a^{n+2}$.

Thus G obtained by duplication of any edge in path P_n by an edge is a face magic graph over $\mathbb{A}$.

A face magic $\mathbb{A}$-lableing of G obtained by duplicating an edge of path P_n by an edge is shown in Figure 6.

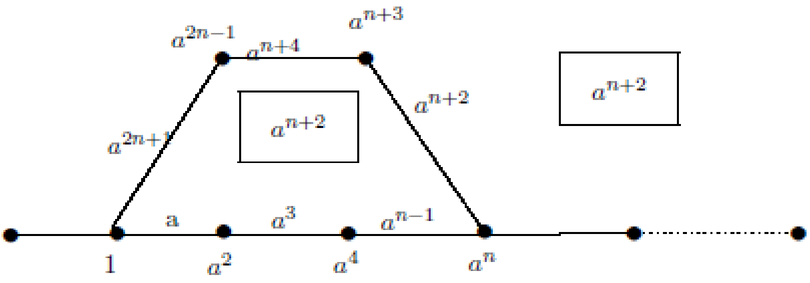

Figure-6: Face magic $\mathbb{A}$-lableing of G obtained by duplicating an edge of path P_n by an edge.

Proposition 2.14: *The graph G obtained by duplication of any edge by a vertex in path P_n admits a face magic $\mathbb{A}$-labeling with face constant a^{n+1}, for $n \geq 2$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n, for $n \geq 2$.

By duplicating any edge $v_i v_{i+1}$ by a vertex u, $1 \leq i \leq n-1$, the resultant graph G has vertex set $V(P_n) \cup \{u\}$, edge set $E(P_n) \cup \{uv_i, uv_{i+1}\}$ and face set $\{f_0, f_1\}$ where f_1 is the face bounded by the edges $v_i v_{i+1}, uv_{i+1}, uv_i$ and f_0 is the outer face bounded by all the edges of G.

Consider the cyclic group $\langle a \rangle$ of order $p + q = 2n + 2$. Assign the labels $1, a, a^2, a^{n-1}, a^n, a^{n+1}$ to the vertices and edges of the face f_1 and assign the remaining labels to the remaining vertices and edges of the graph. Now, the weight of the face $f_1 = a^{3n+3} = a^{2n+2}.a^{n+1} = a^{n+1}$, since $a^{2n+2} = 1_{\mathbb{A}}$. Also the weight of the outer face $f_0 = 1.a.a^2.a^3 \ldots a^{2n+1} = a^{n+1}$.
Thus G obtained by duplication of any edge by a vertex in path P_n is a face magic graph over $\mathbb{A}$.

A face magic $\mathbb{A}$-labeling of G obtained by duplication of any edge by a vertex in path P_n is shown in Figure 7.

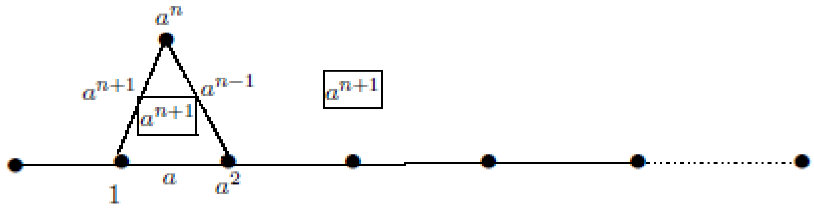

Figure-7: Face magic $\mathbb{A}$-labeling of G obtained by duplication of any edge by a vertex in path P_n with face constant a^{n+1}.

Proposition 2.15: *The graph G obtained by duplication of any vertex by an edge in path P_n admits a face magic $\mathbb{A}$-labeling with face constant a^{n+2}.*

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n, for $n \geq 1$.
By duplicating any vertex v_i by an edge uv, $1 \leq i \leq n$, the resultant graph G has vertex set $V(P_n) \cup \{u, v\}$, edge set $E(P_n) \cup \{uv, uv_i, vv_i\}$ and face set $\{f_0, f_1\}$ where f_1 is the face bounded by the edges uv, uv_i, vv_i and f_0 is the outer face bounded by all the edges of G.

Consider the cyclic group $\langle a \rangle$ of order $p + q = 2n + 4$. Assign the labels $1, a, a^2, a^n, a^{n+1}, a^{n+2}$ to the vertices and edges of the face f_1 and assign the remaining labels to the remaining vertices and edges of the graph. Now, the weight of the face $f_1 = a^{3n+6} = a^{2n+4}.a^{n+2} = a^{n+2}$, since $a^{2n+4} = 1_{\mathbb{A}}$. Also, the weight of the outer face $f_0 = 1.a.a^2.a^3 \ldots a^{2n+3} = a^{n+2}$.
Thus G obtained by duplication of any vertex by an edge in path P_n is a face magic graph over $\mathbb{A}$.

A face magic $\mathbb{A}$-labeling of G obtained by duplication of any vertex by an edge in path P_n is shown in Figure 8.

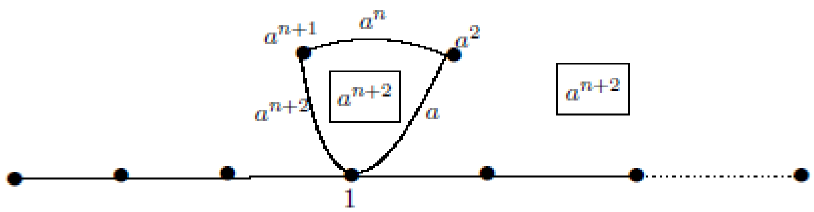

Figure-8: Face magic $\mathbb{A}$-labeling of G obtained by duplication of any vertex by an edge in path P_n with face constant a^{n+2}.

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