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# ON SYSTEMS OF DOUBLE EQUATIONS WITH SURDS

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***ABSTRACT***

*This paper concerns with 6 different systems of double equations involving surds to obtain their solutions in real numbers respectively.*

***Keywords:*** *System of indeterminate quadratic equations, pair of quadratic equations, system of double quadratic equation, irrational solutions.*

***2010 Mathematics Subject Classification:*** *11D99.*

## INTRODUCTION

Systems of indeterminate quadratic equations of the form $ax+c=u^2$, $bx+d=v^2$ where a, b, c, d are non-zero distinct constants, have been investigated for solutions by several authors [1, 2] and with a few possible exceptions, most of the them were primarily concerned with rational solutions. Even those existing works wherein integral solutions have been attempted, deal essentially with specific cases only and do not exhibit methods of finding integral solutions in a general form. In [3], a general form of the integral solutions to the system of equations $ax+c=u^2$, $bx+d=v^2$ where a, b, c, d are non-zero distinct constants is presented when the product ab is a square free integer whereas the product cd may or may not a square integer. For other forms of system of double diophantine equations, one may refer [4-12].

In the above references, the equations are polynomial equations with integer coefficients which motivated us to search for solutions to system of equations with surds. This communication concerns with the problem of obtaining solutions a, b in real numbers satisfying each of the system of double equations with surds represented by

i) $a\sqrt{a}+b\sqrt{b}=N$ , $a\sqrt{b}+b\sqrt{a}=N-1$

ii) $a\sqrt{a}+b\sqrt{b}=N+1$ , $a\sqrt{b}+b\sqrt{a}=N$

iii) $a\sqrt{a}+b\sqrt{b}=N+4$ , $a\sqrt{b}+b\sqrt{a}=N$

iv) $a\sqrt{a}+b\sqrt{b}=N+24$ , $a\sqrt{b}+b\sqrt{a}=N$

v) $a\sqrt{a}+b\sqrt{b}=2N+1$ , $a\sqrt{b}+b\sqrt{a}=N+1$

vi) $a\sqrt{a}+b\sqrt{b}=k^2+k+1$ $a\sqrt{b}+b\sqrt{a}=3k+8$

where N is an integer. In each case, a few interesting relations among the solutions are presented.

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**Notations:**

- Pyramidal number of rank n with size m

$$P_n^m = \frac{1}{6}[n(n+1)][(m-2)n+(5-m)]$$

- Star number of rank n

$$S_n = 6n(n-1)+1$$

- Gnomonic number of rank n

$$GNO_n = 2n-1$$

- Pronic number of rank n

$$\Pr_n = n(n+1)$$

## METHOD OF ANALYSIS

**System-1:** Let a, b be two positive distinct real numbers and N be a positive integer such that

$$a\sqrt{a}+b\sqrt{b}=N \quad (1.1)$$

$$a\sqrt{b}+b\sqrt{a}=N-1 \quad (1.2)$$

Consider the identity

$$\left(\sqrt{a}+\sqrt{b}\right)^3 = \left(a\sqrt{a}+b\sqrt{b}\right)+3\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right) \quad (1.3)$$

The substitution of (1.1), (1.2) in (1.3) gives

$$\left(\sqrt{a}+\sqrt{b}\right)^3 = 4N-3 \quad (1.4)$$

After performing some algebra, it is seen that (1.4) is satisfied by

$$N = 16k^3-36k^2+27k-6 \quad (1.5)$$

$$\sqrt{a}+\sqrt{b} = 4k-3 \quad (1.6)$$

From (1.2), note that

$$\sqrt{ab} = \frac{16k^3-36k^2+27k-7}{4k-3}$$

Employing the identity

$$\left(\sqrt{a}-\sqrt{b}\right)^2 = \left(\sqrt{a}+\sqrt{b}\right)^2 - 4\sqrt{ab} \quad (1.7)$$

$$\Rightarrow \sqrt{a}-\sqrt{b} = \frac{1}{\sqrt{4k-3}} \quad (1.8)$$

From (1.6) and (1.8), it is obtained that

$$a = \frac{1}{4(4k-3)}\left((4k-3)\sqrt{4k-3}+1\right)^2$$

$$b = \frac{1}{4(4k-3)}\left((4k-3)\sqrt{4k-3}-1\right)^2$$

**Properties:**

- $24(4k-3)\left(a^2-b^2\right)^2$ is a Nasty number
- $8(4k-3)^2\left(a^2+b^2\right)$ is written as a sum of square and cube
- Each of the following expressions represents a cubical integer
  - $2(4k-3)(a+b)-1$
  - $4(4k-3)^3\left(8\left(a^3+b^3\right)-3(4k-3)^3-3\right)$

**System-2:** Let a, b be two positive distinct real numbers and N be a positive integer such that

$$a\sqrt{a}+b\sqrt{b}=N+1 \quad (2.1)$$

$$a\sqrt{b}+b\sqrt{a}=N \quad (2.2)$$

 

The substitution of (2.1), (2.2) in (1.3) gives

$$\left(\sqrt{a}+\sqrt{b}\right)^3 = 4N+1 \tag{2.3}$$

After performing some algebra, it is seen that (2.3) is satisfied by

$$N = 16k^3 + 12k^2 + 3k \tag{2.4}$$

$$\sqrt{a}+\sqrt{b} = 4k+1 \tag{2.5}$$

From (2.2), note that

$$\sqrt{ab} = \frac{16k^3+12k^2+3k}{4k+1}$$

The identity (1.7) leads to

$$\sqrt{a}-\sqrt{b} = \frac{1}{\sqrt{4k+1}} \tag{2.6}$$

From (2.5) and (2.6), it is obtained that

$$a = \frac{1}{4(4k+1)}\left((4k+1)^3 + 1 + 2\sqrt{4k+1}(4k+1)\right)$$

$$b = \frac{1}{4(4k+1)}\left((4k+1)^3 + 1 - 2\sqrt{4k+1}(4k+1)\right)$$

**Properties:**

- Each of the following expressions represents a Perfect Square
  - $4(4k+1)\left(a^2-b^2\right)^2$
  - $8(4k+1)^2\left(a^2+b^2\right)+8$
- $2(4k+1)(a+b)-1$ is a cubical integer

**System-3:** Let a, b be two positive distinct real numbers and N be a positive integer such that

$$a\sqrt{a}+b\sqrt{b} = N+4 \tag{3.1}$$

$$a\sqrt{b}+b\sqrt{a} = N \tag{3.2}$$

The substitution of (3.1), (3.2) in (1.3) gives

$$\left(\sqrt{a}+\sqrt{b}\right)^3 = 4N+4 \tag{3.3}$$

After performing some algebra, it is seen that (3.3) is satisfied by

$$N = 2k^3 - 1 \tag{3.4}$$

$$\sqrt{a}+\sqrt{b} = 2k \tag{3.5}$$

From (3.2), note that

$$\sqrt{ab} = \frac{2k^3-1}{2k}$$

The identity (1.7) gives

$$\sqrt{a}-\sqrt{b} = \sqrt{\frac{2}{k}} \tag{3.6}$$

From (3.5) and (3.6), it is obtained that

$$a = \frac{1}{4k^2}\left(2k^2+\sqrt{2k}\right)^2 \;,\; b = \frac{1}{4k^2}\left(2k^2-\sqrt{2k}\right)^2$$

**Properties:**

- $2k^2\left(a^2+b^2\right)-8k^3$ is a perfect square
- $12k^2\left(a^2+b^2\right)-48k^3$ is a Nasty number
- Each of the following expressions represents a cubical integer
  - $4k\left(a+b\right)-4$
  - $k\left(a^2-b^2\right)^2\left(2k^3+1\right)$
  - $8k^3\left(a^3+b^3\right)+6\left(4k^6-1\right)\left(2k^3-1\right)$
- $2k\left(a+b\right)$ is written as sum of k squares

**Illustration:**

- $k=2 \Rightarrow 4\left(a+b\right)=3^2+5^2$
- $k=3 \Rightarrow 6\left(a+b\right)=1^2+3^2+10^2$
- $k=4 \Rightarrow 8\left(a+b\right)=2^2+3^2+7^2+14^2$
- $k=5 \Rightarrow 10\left(a+b\right)=1^2+4^2+6^2+7^2+20^2$

**System-4:** Let a, b be two positive distinct real numbers and N be a positive integer such that

$$a\sqrt{a}+b\sqrt{b}=N+24 \tag{4.1}$$

$$a\sqrt{b}+b\sqrt{a}=N \tag{4.2}$$

The substitution of (4.1), (4.2) in (1.3) gives

$$\left(\sqrt{a}+\sqrt{b}\right)^3=4N+24 \tag{4.3}$$

After performing some algebra, it is seen that (4.3) is satisfied by

$$N=16k^3-6 \tag{4.4}$$

$$\sqrt{a}+\sqrt{b}=4k \tag{4.5}$$

From (4.2), note that

$$\sqrt{ab}=\frac{8k^3-3}{2k}$$

The identity (1.7) leads to

$$\sqrt{a}-\sqrt{b}=\sqrt{\frac{6}{k}} \tag{4.6}$$

From (4.5) and (4.6), it is obtained that

$$a=\frac{1}{4k}\left(4k\sqrt{k}+\sqrt{6}\right)^2\ ,\ b=\frac{1}{4k}\left(4k\sqrt{k}-\sqrt{6}\right)^2$$

**Properties:**

- $\dfrac{a-b}{4k}=\sqrt{a}-\sqrt{b}$
- $2k^2\left(a^2+b^2\right)-96k^3$ is a perfect square
- $k\left(a+b\right)-3$ is a cubical integer
- Each of the following expressions represents a Nasty number
  - $k\left(a^2-b^2\right)^2$
  - $32k^3\left(a^2+b^2\right)-16k\left(8k^3+3\right)^2$

Equation (4.3) is also satisfied by

$$N=16k^3+24k^2+12k-4 \tag{4.7}$$

$$\sqrt{a}+\sqrt{b}=4k+2 \tag{4.8}$$

 

From (4.2), note that

$$\sqrt{ab}=\frac{8k^3+12k^2+6k-2}{2k+1}$$

The identity (1.7) gives

$$\sqrt{a}-\sqrt{b}=\frac{2\sqrt{3}}{\sqrt{2k+1}} \tag{4.9}$$

From (4.8) and (4.9), it is obtained that

$$a=\frac{1}{(2k+1)}\left((2k+1)^3+3+2\sqrt{3}\sqrt{2k+1}(2k+1)\right)$$

$$b=\frac{1}{(2k+1)}\left((2k+1)^3+3-2\sqrt{3}\sqrt{2k+1}(2k+1)\right)$$

**Properties:**

- $(a-b)^2\equiv 0(\mathrm{mod}(2k+1))$
- $4(2k+1)(a+b)-24$ is a cubical integer
- Each of the following expressions represents a Nasty number
  - $2(2k+1)(a^2-b^2)^2$
  - $3(2k+1)^2(a^2+b^2)+432$
  - $3(a^2+b^2-48(2k+1))$

Further equation (4.3) is satisfied by

$$N=2k^3+6k^2+6k-4,\ \sqrt{a}+\sqrt{b}=2k+2$$

In this case, the corresponding values of a and b satisfying the given system are represented by

$$a=\frac{1}{(k+1)}\left((k+1)^3+3+2\sqrt{3}\sqrt{k+1}(k+1)\right)$$

$$b=\frac{1}{(k+1)}\left((k+1)^3+3-2\sqrt{3}\sqrt{k+1}(k+1)\right)$$

**System-5:** Let a, b be two positive distinct real numbers and N be a positive integer such that

$$a\sqrt{a}+b\sqrt{b}=2N+1 \tag{5.1}$$

$$a\sqrt{b}+b\sqrt{a}=N+1 \tag{5.2}$$

The substitution of (5.1), (5.2) in (1.3) gives

$$\left(\sqrt{a}+\sqrt{b}\right)^3=5N+4 \tag{5.3}$$

After performing some algebra, it is seen that (5.3) is satisfied by

$$N=25k^3-15k^2+3k-1 \tag{5.4}$$

$$\sqrt{a}+\sqrt{b}=5k-1 \tag{5.5}$$

From (5.2), note that

$$\sqrt{ab}=\frac{25k^3-15k^2+3k}{5k-1}$$

The identity (1.7) leads to

$$\sqrt{a}-\sqrt{b}=\sqrt{\frac{25k^3-15k^2+3k-1}{5k-1}} \tag{5.6}$$

From (5.5) and (5.6), it is obtained that

$$a=\frac{1}{4(5k-1)}\left(150k^3-90k^2+18k-2+2(5k-1)\sqrt{5k-1}\sqrt{25k^3-15k^2+3k-1}\right)$$

$$b=\frac{1}{4(5k-1)}\left(150k^3-90k^2+18k-2-2(5k-1)\sqrt{5k-1}\sqrt{25k^3-15k^2+3k-1}\right)$$

 

**Properties:**

- $(5k-1)(a+b)-6P_k^5+3k$ is written as sum of two cubes
- $5(a-b)^2+1000P_k^5-108S_k-2\Pr_k-606k+103$ is a biquadratic integer
- $7(5k-1)^2(a^2+b^2)+2100P_k^5-250S_k-180\Pr_k-597GNO_k-354$ is a perfect square
- For k given, $(5k-1)(a+b)$ is written as sum of squares.

**Illustration:**

- $k=1 \Rightarrow 4(a+b)=2^2+3^2+5^2$
- $k=2 \Rightarrow 9(a+b)=1^2+6^2+20^2$
- $k=3 \Rightarrow 14(a+b)=1^2+3^2+6^2+40^2$
- $k=4 \Rightarrow 19(a+b)=2(3^2)+4^2+9^2+20^2+60^2$

**System-6:** Let a, b be two positive distinct real numbers and N be a positive integer such that

$$a\sqrt{a}+b\sqrt{b}=k^2+k+1 \tag{6.1}$$

$$a\sqrt{b}+b\sqrt{a}=3k+8 \tag{6.2}$$

The substitution of (6.1), (6.2) in (1.3) gives

$$\left(\sqrt{a}+\sqrt{b}\right)^3=(k+5)^2 \tag{6.3}$$

Assume $k=\alpha^3-5$ (6.4)

Substituting (6.4) in (6.3), we get

$$\sqrt{a}+\sqrt{b}=\alpha^2 \tag{6.5}$$

From (6.2), note that

$$\sqrt{ab}=\frac{3k+8}{\alpha^2}$$

The identity (1.7) gives

$$\sqrt{a}-\sqrt{b}=\frac{1}{\alpha}\sqrt{\alpha^6-12\alpha^3+28} \tag{6.6}$$

From (6.5) and (6.6), it is obtained that

$$a=\frac{1}{4\alpha^2}\left(2\alpha^6-12\alpha^3+28+2\alpha^3\sqrt{\alpha^6-12\alpha^3+28}\right)$$

$$b=\frac{1}{4\alpha^2}\left(2\alpha^6-12\alpha^3+28-2\alpha^3\sqrt{\alpha^6-12\alpha^3+28}\right)$$

**Properties:**

- Each of the following expressions represents a Nasty number
  - $6\left(\alpha^2(a+b)-5\right)$
  - $6\left((a-b)^2+8\alpha^2\right)$
- For $\alpha>1$, $\alpha^2(a+b)$ is written as sum of 3 squares

**Remark:** When $\alpha=2$, the values of a and b are in complex numbers.

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