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SOME RESULT ON FIXED POINT OF EXPANSIVE MAPPING OVER MODULAR METRIC SPACE

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ABSTRACT
In the present paper, we introduce the notion of expansive mapping over a modular metric space and wish to study fixed point of expansive type mapping over such spaces.

1. INTRODUCTION

In 1984, Wang et.al in (see [4]) had initiated a fixed point theorem for an expansive mapping T over a metric space. A mapping T over a metric space (X, d) is said to be expansive if there is a real constant $c > 1$ such that $d(T(x), T(y)) \geq cd(x, y)$ for all $x, y \in X$ (see [4]). In 2007, M. Saha (see[3]) had proved a fixed point theorem for a class of expansive mappings strictly larger than those of Wang *et. al* (see [4]). In 2008, Chistyakov (see[1]) introduced the notion of modular metric spaces generated by F-modular and obtained some results on the spaces. Keeping on the same idea Chistyakov (see[2]) had been able to define the notion of a modular on an arbitrary set and develop the theory of metric spaces generated by modular called the modular metric spaces.In this paper we have been able to investigate some results on fixed points of expansive mapping over a modular metric space.

2. SOME BASIC IDEAS AND DEFINITIONS

In order to prove our main results, we need to recall the following definition.

Definition 2.1: (see [1]) Let X be a nonempty set. A function $\omega: (0, \infty) \times X \times X \to [0, \infty]$ is said to be metric modular on X denoted by ω_λ if satisfying the following conditions for all $x, y, z \in X$.

(i) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ iff $x = y$
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$for all $\lambda, \mu > 0$.

The pair (X, ω_λ) is called a modular metric space and symbolically we write X_ω.

The function $0 < \lambda \to \omega_\lambda(x, y) \in [0, \infty]$ is a non-increasing on $(0, \infty)$. If $0 < \mu < \lambda$ then from (i)-(iii) implies that $\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) \leq \omega_\mu(x, y)$.

Let's recall the following set $X_\omega \equiv X_\omega(x_0) = \{x \in X: \omega_\lambda(x, x_0) \to 0 \ as\ \lambda \to \infty\}$.

Definition 2.2: Let X_ω be a modular metric space

(i) The sequence (x_n) in X_ω is said to be convergent to $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$, as $n \to \infty$,for all $\lambda > 0$.
(ii) The sequence (x_n) in X_ω is said to be cauchy if $\omega_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$ for all $\lambda > 0$.
(iii) A modular metric space X_ω is said to be complete if every cauchy sequence is converges to an element of X_ω.

Theorem 2.3: *(see [1]) If ω is a metric modular on X, then the modular set X_ω is a metric space with metric given by $d_\omega^0(x, y) = inf\{\lambda > 0: \omega_\lambda(x, y) < \lambda\}$, $x, y \in X_\omega$.*

Theorem 2.4: *(see [1]) Let ω be a modular on a set X, Given a sequence $(x_n) \subset X_\omega$ and $x \in X_\omega$ we have $d_\omega^0(x_n, x) \to 0$ as $n \to \infty$ iff $\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$. A similar assertion holds for a cauchy sequences.*

3. MAIN RESULTS

Theorem 3.1: *Let X_ω be a ω-complete modular metric space and $T: X_\omega \to X_\omega$ is a continuous map which is onto itself such that $\omega_\lambda(Tx, Ty) \geq a\omega_\lambda(y, Ty) + b\omega_\lambda(x, y)$ holds for all $x, y \in X_\omega$ where a, b are non-negative reals with $a + b > 1$. Then T has a fixed point and the fixed point is unique only when $b > 1$.*

Proof: Since T is onto. We define a sequence(x_n) in X_ω s.t $x_{n-1} = Tx_n$.If we assume $x_n = x_{n+1}$ for all $n \in \mathbb{N}$, then T has a fixed point. We therefore assume $x_n \neq x_{n+1}$. Now from the given condition we have

$$\begin{aligned}\omega_\lambda(x_0, x_1) &= \omega_\lambda(Tx_1, Tx_2)\\ &\geq a\omega_\lambda(x_2, Tx_2) + b\omega_\lambda(x_1, x_2)\\ &\geq a\omega_\lambda(x_2, x_1) + b\omega_\lambda(x_1, x_2)\\ &= (a + b)\omega_\lambda(x_2, x_1).\end{aligned}$$

Therefore we have $\omega_\lambda(x_2, x_1) \leq \frac{1}{a+b}\omega_\lambda(x_0, x_1) = r\omega_\lambda(x_0, x_1)$ where $r = \frac{1}{a+b}$.

Similarly $\omega_\lambda(x_2, x_3) \leq r\omega_\lambda(x_1, x_2) \leq r^2\omega_\lambda(x_0, x_1)$.Proceding in this way we get $\omega_\lambda(x_n, x_{n+1}) \leq r^n\omega_\lambda(x_0, x_1)$ for all $\lambda > 0$ Therefore, $\lim\omega_\lambda(x_n, x_{n+1}) = 0$. For each $\varepsilon > 0$ and $\lambda > 0\, \exists\, n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n, x_{n+1}) < \varepsilon$, for all $n \in \mathbb{N}$ with $n \geq n_0$ .Without loss of generality we assume $m, n \in \mathbb{N}$ and $m > n$. Since $\frac{\lambda}{(m-n)} > 0$, therefore $\exists\, n_{\frac{\lambda}{(m-n)}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) < \frac{\varepsilon}{m-n}$ for all $n \geq n_{\frac{\lambda}{(m-n)}}$.

Now we have $m, n \geq n_{\frac{\lambda}{(m-n)}}$,

$$\begin{aligned}\omega_\lambda(x_n, x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \ldots\ldots\ldots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m)\\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \ldots\ldots\ldots + \frac{\varepsilon}{m-n}\\ &= \varepsilon.\end{aligned}$$

This implies that (x_n) is a cauchy sequence in X_ω.By completeness of X_ω, $\lim x_n = z \in X_\omega$(say). Now by using the continuity of T we have $z = \lim x_n = \lim Tx_{n+1} = T(\lim(x_{n+1})) = Tz$, which shows that z is a fixed point of T.

For uniqueness of z, if possible, let z_1 be another fixed point of T such that $Tz_1 = z_1$.

Now $\omega_\lambda(z, z_1) = \omega_\lambda(Tz, Tz_1) \geq a\omega_\lambda(z_1, Tz_1) + b\omega_\lambda(z, z_1) = b\omega_\lambda(z, z_1)$.

Therefore we have $(b - 1)\omega_\lambda(z, z_1) \leq 0 \Rightarrow \omega_\lambda(z, z_1) = 0$ if $b > 1 \Rightarrow \omega_\lambda(z, z_1) = 0\ \forall\ \lambda > 0 \Rightarrow z = z_1$.

Similarly we can prove the following theorem.

Theorem 3.2: *Let X_ω be a ω-complete modular metric space and $T: X_\omega \to X_\omega$ is a continuous map which is onto itself such that $\omega_\lambda(Tx, Ty) \geq a\omega_\lambda(x, Tx) + b\omega_\lambda(x, y)$ holds for all $x, y \in X_\omega$ where a, b are non-negative reals with $a + b > 1$. Then T has a fixed point and the fixed point is unique only when $b > 1$.*

Theorem 3.3: *Let X_ω be a ω-complete modular metric space and $T: X_\omega \to X_\omega$ is a continuous mapping which is onto such that $\omega_\lambda(Tx, Ty) \geq a\omega_\lambda(x, Tx) + b\omega_\lambda(y, Ty) + c\omega_\lambda(x, y)$ holds for all $x, y \in X_\omega$ where a, b, c are non-negative reals with $a < 1$ and $a + b + c > 1$. Then T has a fixed point and the fixed point is unique only when $c > 1$.*

Proof: Since T is onto, we define a sequence(x_n) in X_ω s.t $x_{n-1} = Tx_n$.If we assume $x_n = x_{n+1}$ for all $n \in \mathbb{N}$ then T has a fixed point. We therefore assume $x_n \neq x_{n+1}$. Now from the given condition we have

$$\begin{aligned}\omega_\lambda(x_0, x_1) &= \omega_\lambda(Tx_1, Tx_2)\\ &\geq a\omega_\lambda(x_1, Tx_1) + b\omega_\lambda(x_2, Tx_2) + c\omega_\lambda(x_1, x_2)\\ &\geq a\omega_\lambda(x_1, x_0) + b\omega_\lambda(x_2, x_1) + c\omega_\lambda(x_1, x_2)\\ &= a\omega_\lambda(x_1, x_0) + (b + c)\omega_\lambda(x_2, x_1)\end{aligned}$$

Therefore $\omega_\lambda(x_2, x_1) \leq \frac{1-a}{b+c}\omega_\lambda(x_0, x_1) = r\omega_\lambda(x_0, x_1)$, where $r = \frac{1-a}{b+c}$.

Similarly $\omega_\lambda(x_2, x_3) \leq r\omega_\lambda(x_1, x_2) \leq r^2\omega_\lambda(x_0, x_1)$. Proceding in this way we get $\omega_\lambda(x_n, x_{n+1}) \leq r^n\omega_\lambda(x_0, x_1)$ for all $\lambda > 0$ Therefore $\lim\omega_\lambda(x_n, x_{n+1}) = 0$. For each $\varepsilon > 0$ and $\lambda > 0 \exists n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n, x_{n+1}) < \varepsilon$, for all $n \in \mathbb{N}$ with $n \geq n_0$.Without loss of generality we assume $m, n \in \mathbb{N}$ and $m > n$.
Since $\frac{\lambda}{(m-n)} > 0$, therefore $\exists n_{\frac{\lambda}{(m-n)}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) < \frac{\varepsilon}{m-n}$ for all $n \geq n_{\frac{\lambda}{(m-n)}}$.

Now we have $m,n \geq n_{\frac{\lambda}{(m-n)}}$,

$$\begin{aligned}\omega_\lambda(x_n,x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n,x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1},x_{n+2}) + \ldots\ldots\ldots + \omega_{\frac{\lambda}{m-n}}(x_{m-1},x_m) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \ldots\ldots\ldots + \frac{\varepsilon}{m-n} \\ &= \varepsilon.\end{aligned}$$

This implies that (x_n) is a cauchy sequence in X_ω.By completeness of X_ω, $\lim x_n = z \in X_\omega$ (say). Now by using the continuity of T we have $z = \lim x_n = \lim Tx_{n+1} = T(\lim(x_{n+1}) = Tz$, which shows that z is a fixed point of T.

For uniqueness of z, if possible let z_1 be another fixed point of T such that $Tz_1 = z_1$.

Now $\omega_\lambda(z,z_1) = \omega_\lambda(Tz,Tz_1 \geq a\omega_\lambda(z,Tz) + b\omega_\lambda(z_1,Tz_1) + c\omega_\lambda(z,z_1) = c\omega_\lambda(z,z_1)$.

Therefore $\forall\lambda > 0$ we have $(c-1)\omega_\lambda(z,z_1) \leq 0 \Rightarrow \omega_\lambda(z,z_1) = 0$ if $c > 1 \Rightarrow \omega_\lambda(z,z_1) = 0\ \forall\lambda > 0 \Rightarrow z = z_1$.

Theorem 3.4: *Let X_ω be a ω-complete modular metric space and and $T: X_\omega \to X_\omega$ is a continuous map which is onto itself such that $\omega_\lambda(Tx,Ty) \geq a\omega_\lambda(x,Tx) + b\omega_{\lambda\lambda}(y,Ty) + c\omega_\lambda(x,y) + d\omega_\lambda(x,Ty) + e\omega_\lambda(y,Tx)$ holds for all $x,y \in X_\omega$ where a,b,c are non-negative reals with $a < 1$ and $a+b+c > 1$. Then T has a fixed point and the fixed point is unique only when $c+d+e > 1$.*

Proof: Since T is onto. We define a sequence(x_n) in X_ω s.t $x_{n-1} = Tx_n$. Now from the given condition we have

$$\begin{aligned}\omega_\lambda(x_0,x_1) &= \omega_\lambda(Tx_1,Tx_2) \\ &\geq a\omega_\lambda(x_1,Tx_1) + b\omega_\lambda(x_2,Tx_2) + c\omega_\lambda(x_1,x_2) + d\omega_\lambda(x_1,Tx_2) + e\omega_\lambda(x_2,Tx_1) \\ &= a\omega_\lambda(x_1,x_0) + b\omega_\lambda(x_2,x_1) + c\omega_\lambda(x_1,x_2) + d\omega_\lambda(x_1,x_1) + e\omega_\lambda(x_2,x_0) \\ &\geq a\omega_\lambda(x_1,x_0) + b\omega_\lambda(x_2,x_1) + c\omega_\lambda(x_1,x_2)\end{aligned}$$

Therefore we have $\omega_\lambda(x_2,x_1) \leq \frac{1-a}{b+c}\omega_\lambda(x_0,x_1) = r\omega_\lambda(x_0,x_1)$ where $r = \frac{1-a}{b+c}$.

Similarly $\omega_\lambda(x_2,x_3) \leq r\omega_\lambda(x_1,x_2) \leq r^2\omega_\lambda(x_0,x_1)$. Proceding in this way we get $\omega_\lambda(x_n,x_{n+1}) \leq r^n\omega_\lambda(x_0,x_1)$ for all $\lambda > 0$ Therefore $\lim\omega_\lambda(x_n,x_{n+1}) = 0$. For each $\varepsilon > 0$ and $\lambda > 0\ \exists\ n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n,x_{n+1}) < \varepsilon$, for all $n \in \mathbb{N}$ with $n \geq n_0$.Without loss of generality we assume $m,n \in \mathbb{N}$ and $m > n$. Since $\frac{\lambda}{(m-n)} > 0$, therefore $\exists n_{\frac{\lambda}{(m-n)}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{m-n}}(x_n,x_{n+1}) < \frac{\varepsilon}{m-n}$ for all $n \geq n_{\frac{\lambda}{(m-n)}}$.

Now we have $m,n \geq n_{\frac{\lambda}{(m-n)}}$,

$$\begin{aligned}\omega_\lambda(x_n,x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n,x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1},x_{n+2}) + \ldots\ldots\ldots + \omega_{\frac{\lambda}{m-n}}(x_{m-1},x_m) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \ldots\ldots\ldots + \frac{\varepsilon}{m-n} \\ &= \varepsilon.\end{aligned}$$

This implies that (x_n) is a cauchy sequence in X_ω.By completeness of X_ω, $\lim x_n = z \in X_\omega$ (say). Now by using the continuity of T we have $z = \lim x_n = \lim Tx_{n+1} = T(\lim(x_{n+1}) = Tz$, which shows that z is a fixed point of T.

For uniqueness of z, if possible let z_1 be another fixed point of T such that $Tz_1 = z_1$. Now,
$\omega_\lambda(z,z_1) = \omega_\lambda(Tz,Tz_1) \geq a\omega_\lambda(z,Tz) + b\omega_\lambda(z_1,Tz_1) + c\omega_\lambda(z,z_1) + d\omega_\lambda(z,Tz_1) + e\omega_\lambda(z_1,Tz) = c\omega_\lambda(z,z_1) + {}$
$+d\omega_\lambda(z,z_1) + e\omega_\lambda(z_1,z)$.
Therefore we have $(c+d+e-1)\omega_\lambda(z,z_1) \leq 0 \Rightarrow \omega_\lambda(z,z_1) = 0 \Rightarrow \omega_\lambda(z,z_1) = 0\forall\lambda > 0 \Rightarrow z = z_1$.

Theorem 3.5: *Let X_ω be a ω-complete modular metric space and and T and S is a continuous mapping of X onto itself such that $\omega_\lambda(Sx,Ty) \geq a\omega_\lambda(x,Sx) + b\omega_\lambda(y,Ty) + c\omega_\lambda(x,y)$ holds for all $x,y \in X_\omega$ where a,b,c are non-negative reals with $a < 1$ and $a+b+c > 1$. Then both T and S have a fixed point and the fixed point is unique only when $c > 1$.*

Proof: Let $x_0 \in X_\omega$. Since T and S is onto. We define a sequence(x_n) in X_ω s.t $x_{2n} = Sx_{2n+1}$ and $x_{2n+1} = Tx_{2n+2}$.

Now from the given condition we have

$$\begin{aligned}\omega_\lambda(x_{2n},x_{2n+1}) &= \omega_\lambda(Sx_{2n+1},Tx_{2n+2}) \\ &\geq a\omega_\lambda(x_{2n+1},Sx_{2n+1}) + b\omega_\lambda(x_{2n+2},Ty_{2n+2}) + c\omega_\lambda(x_{2n+1},x_{2n+2}) \\ &= (a+b)\omega_\lambda(x_2,x_1).\end{aligned}$$

Therefore $\omega_\lambda(x_{2n+1}, x_{2n+2}) \le \frac{1-a}{b+c}\omega_\lambda(x_{2n}, x_{2n+1}) = r\omega_\lambda(x_{2n}, x_{2n+1})$, where $r = \frac{1-a}{b+c}$.

Similarly $\omega_\lambda(x_{2n}, x_{2n+1}) \le r\omega_\lambda(x_{2n-1}, x_{2n})$.

So for arbitrary n we get $\omega_\lambda(x_n, x_{n+1}) \le r\omega_\lambda(x_{n+1}, x_n)$.

Therefore $\omega_\lambda(x_n, x_{n+1}) \le r^n\omega_\lambda(x_2, x_1)$. Therefore $\lim\omega_\lambda(x_n, x_{n+1}) = 0$ For each $\varepsilon > 0$ and $\lambda > 0\ \exists\ n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n, x_{n+1}) < \varepsilon$, for all $n \in \mathbb{N}$ with $n \ge n_0$. Without loss of generality we assume $m, n \in \mathbb{N}$ and $m > n$. Since $\frac{\lambda}{(m-n)} > 0$, therefore $\exists n_{\frac{\lambda}{(m-n)}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) < \frac{\varepsilon}{m-n}$ for all $n \ge n_{\frac{\lambda}{(m-n)}}$.

Now we have $m, n \ge n_{\frac{\lambda}{(m-n)}}$,

$$\begin{aligned}\omega_\lambda(x_n, x_m) &\le \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \dots\dots\dots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m)\\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \dots\dots\dots + \frac{\varepsilon}{m-n}\\ &= \varepsilon.\end{aligned}$$

This implies that (x_n) is a cauchy sequence in X_ω

Since X_ω is complete, there exist $z \in X_\omega$ such that $x_n \to z \in X_\omega$. Now by using the continuity of S and T we have $z = \lim x_{2n} = \lim Sx_{2n+1} = S(\lim(x_{2n+1}) = Sz$. Similarly $Tz = z$ which shows that z is a common fixed point of S and T.

For uniqueness of z, if possible let z_1 be another common fixed point. Now we have $\omega_\lambda(z, z_1) = \omega_\lambda(Sz, Tz_1 \ge a\omega_\lambda(z, Sz) + b\omega_\lambda(z_1, Tz_1) + c\omega_\lambda(z, z_1) = c\omega_\lambda(z, z_1)$. Therefore we have $(c-1)\omega_\lambda(z, z_1) \le 0 \Rightarrow \omega_\lambda(z, z_1) = 0$ $.\Rightarrow \omega_\lambda(z, z_1) = 0 \forall \lambda > 0 \Rightarrow z = z_1$.

Theorem 3.6: *Let X_ω be a ω-complete modular metric space and T and S is a continuous mapping of X onto it itself such that $\omega_\lambda(Sx, Ty) \ge a\omega_\lambda(x, Sx) + b\omega_\lambda(y, Ty) + c\omega_\lambda(x, y)$ holds for all $x, y \in X_\omega$ where a, b, c are non-negative reals with $a < 1$ and $a + b + c > 1$. Then both T and S have a common fixed point and the fixed point is unique only when $c > 1$.*

Proof: Let $x_0 \in X_\omega$.Since T and S is onto. We define a sequence(x_n) in X_ω s.t $x_{2n} = Sx_{2n+1}$ and $x_{2n+1} = Tx_{2n+2}$. Now from the given condition we have

$$\begin{aligned}\omega_\lambda(x_{2n}, x_{2n+1}) &= \omega_\lambda(Sx_{2n+1}, Tx_{2n+2})\\ &\ge a\omega_\lambda(x_{2n+1}, Sx_{2n+1}) + b\omega_\lambda(x_{2n+2}, Ty_{2n+2}) + c\omega_\lambda(x_{2n+1}, x_{2n+2})\\ &= a\omega_\lambda(x_{2n+1}, x_{2n}) + b\omega_\lambda(x_{2n+2}, y_{2n+1}) + c\omega_\lambda(x_{2n+1}, x_{2n+2})\\ &= (b+c)\omega_\lambda(x_{2n+1}, x_{2n+2})\end{aligned}$$

Therefore $\omega_\lambda(x_{2n+1}, x_{2n+2}) \le \frac{1-a}{b+c}\omega_\lambda(x_{2n}, x_{2n+1}) = r\omega_\lambda(x_{2n}, x_{2n+1})$ where $r = \frac{1-a}{b+c}$.

Similarly $\omega_\lambda(x_{2n}, x_{2n+1}) \le r\omega_\lambda(x_{2n-1}, x_{2n})$.

So for arbitrary n we get $\omega_\lambda(x_n, x_{n+1}) \le r\omega_\lambda(x_{n+1}, x_n)$. Therefore $\omega_\lambda(x_n, x_{n+1}) \le r^n\omega_\lambda(x_{n+1}, x_n)$. Since X_ω is complete, there exist $z \in X_\omega$ such that $x_n \to z \in X_\omega$.Now by using the continuity of S and T we have

$$z = \lim x_{2n} = \lim Sx_{2n+1} = S(\lim(x_{2n+1}) = Sz.$$

Similarly $Tz = z$, which shows that z is a common fixed point of$S$ and T. For uniqueness of z, If possible let z_1 be another common fixed point.Now we have

$$\omega_\lambda(z, z_1) = \omega_\lambda(Sz, Tz_1) \ge a\omega_\lambda(z, Sz) + b\omega_\lambda(z_1, Tz_1) + c\omega_\lambda(z, z_1) = c\omega_\lambda(z, z_1).$$

Therefore we have $(c-1)\omega_\lambda(z, z_1) \le 0 \Rightarrow \omega_\lambda(z, z_1) = 0$ $.\Rightarrow \omega_\lambda(z, z_1) = 0\ \forall\ \lambda > 0 \Rightarrow z = z_1$.

Theorem 3.7: *Let X_ω be a ω-complete modular metric space and (T_n) be a sequence of continuous mapping of X onto itself such that $\omega_\lambda(T_i^{m_i}x, T_j^{m_j}y) \ge a\omega_\lambda(x, T_i^{m_i}x) + b\omega_\mu(y, T_j^{m_j}y) + c\omega_\lambda(x, y)$ holds for all $x, y \in X_\omega$ where a, b, c are non-negative reals with $a < 1$ and $a + b + c > 1$ and (m_n) is the sequence of non-negative integers. Then (T_n) has a unique fixed point in X_ω.*

Proof: Define $S_k = T_k^{m_k}$. We get $\omega_\lambda(S_ix, S_jy) \ge a\omega_\lambda(x, S_ix) + b\omega_\mu(y, S_jy) + c\omega_\lambda(x, y)$. Since each (T_n) is continuous and onto itself. So we define $x_n = S_nx_{n+1}$. Now by routine calculation we can see that (S_n) have a unique common fixed point z (say) i.e $S_kz = z$ for all $z = 1,2, \dots$

Therefore $T_n z = T_n(S_n z) = T_n(T_n^{m_n} z) = T_n^{m_n}(T_n z)$. So it follows that T_n have a unique fixed point z.

Theorem 3.8: *Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. If X_ω be a complete modular metric space and T is a continuous mapping of X_ω onto itself such that $\omega_\lambda(Tx,Ty) \geq a\omega_\lambda(x,y) + b\{\omega_\lambda(x,Tx) + \omega_\lambda(y,Ty)\} + cmax\{\omega_\lambda(x,Ty), \omega_\lambda(y,Tx)\}$ holds for all $x, y \in X_\omega$ where a, b, c are non-negative reals with $b < 1$ and $a + 2b > 1$. Then T has a fixed point and the fixed point is unique only when $a + c > 1$.*

Proof: Since T is onto. We define a sequence(x_n) in X_ω s.t $x_{n-1} = Tx_n$.Now from the given condition we have

$$\begin{aligned}\omega_\lambda(x_0,x_1) &= \omega_\lambda(Tx_1,Tx_2)\\ &\geq a\omega_\lambda(x_1,x_2) + b\{\omega_\lambda(x_1,Tx_1) + \omega_\lambda(x_2,Tx_2)\} + c\max\{\omega_\lambda(x_1,Tx_2), \omega_\lambda(x_2,Tx_1)\}\\ &\geq a\omega_\lambda(Tx_1,x_2) + b\{\omega_\lambda(x_1,x_0) + \omega_\lambda(x_2,x_1)\} + c\max\{\omega_\lambda(x_1,x_1), \omega_\lambda(x_2,x_0)\}\\ &\geq a\omega_\lambda(x_1,x_2) + b\omega_\lambda(x_1,x_0) + b\omega_\lambda(x_2,x_1) + c\omega_\lambda(x_2,x_0)\\ &\geq a\omega_\lambda(x_1,x_2) + b\omega_\lambda(x_1,x_0) + b\omega_\lambda(x_2,x_1).\end{aligned}$$

Therefore we have $\omega_\lambda(x_1,x_2) \leq \frac{1-b}{a+b}\omega_\lambda(x_1,x_0) = r\omega_\lambda(x_1,x_0$. where $r = \frac{1-b}{a+b}$ and $0 < r < 1$. Therefore, $\lim\omega_\lambda(x_n,x_{n+1}) = 0$. So for each $\varepsilon > 0$ and $\lambda > 0\exists\ n_0 \in \mathbb{N}$ such that $\omega_\lambda(x_n,x_{n+1}) < \varepsilon$, for all $n \in \mathbb{N}$ with $n \geq n_0$. Without loss of generality we assume $m, n \in \mathbb{N}$ and $m > n$. Since $\frac{\lambda}{(m-n)} > 0$, therefore $\exists n_{\frac{\lambda}{(m-n)}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{m-n}}(x_n,x_{n+1}) < \frac{\varepsilon}{m-n}$ for all $n \geq n_{\frac{\lambda}{(m-n)}}$.

Now we have $m, n \geq n_{\frac{\lambda}{(m-n)}}$,

$$\begin{aligned}\omega_\lambda(x_n,x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n,x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1},x_{n+2}) + \ldots\ldots\ldots + \omega_{\frac{\lambda}{m-n}}(x_{m-1},x_m)\\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \ldots\ldots\ldots + \frac{\varepsilon}{m-n}\\ &= \varepsilon.\end{aligned}$$

This implies that (x_n) is a cauchy sequence in X_ω. Therefore z is a fixed point of T.

For uniqueness of z, if possible let z_1 be another fixed point of T such that $Tz_1 = z_1$.
Then

$$\begin{aligned}\omega_\lambda(z,z_1) &= \omega_\lambda(Tz,Tz_1)\\ &\geq a\omega_\lambda(z_1,z_2) + b\{\omega_\lambda(z_1,Tz_1) + \omega_\lambda(z_2,Tz_2)\} + c\max\{\omega_\lambda(z_1,Tz_2), \omega_\lambda(z_2,Tz_1)\}\\ &\geq a\omega_\lambda(z_1,z_2) + c\omega_\lambda(z_1,z_2)\end{aligned}$$

Therefore we have

$$(a + c - 1)\omega_\lambda(z,z_1) \leq 0 \Rightarrow \omega_\lambda(z,z_1) = 0\ .\Rightarrow \omega_\lambda(z,z_1) = 0\forall\lambda > 0 \Rightarrow z = z_1.$$

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