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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF NEUTRAL NONLINEAR IMPULSIVE PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this article, we have discussed the oscillatory and asymptotic behavior of solutions of a class of neutral nonlinear impulsive partial differential equations. Some new sufficient conditions are derived by using Riccati transform and impulsive differential inequalities. Our results extend a number of results reported in the literature. An example is given to demonstrate the validity of our results.

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***Key Words**: Oscillation, Asymptotic, Impulse, Partial differential equations, Forcing term.*

1. INTRODUCTION

The notion of neutral delay impulsive differential equations, that is the impulsive equations in which the highest order derivative of the unknown function appears both with and without delays, are more appropriate to model the dynamical systems with discontinuities trajectories. These types of models are emerging in nonlinear mechanics dealing with the process in nonlinear oscillating system. Neutral delay differential equations appear in modeling of networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar, Euler equation in some variational problems in the theory of automatic control and in neuromechanical systems in which the inertia plays an important role, we can refer in [4, 9].

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses, but also represents a more natural framework for the mathematical modeling of many real world phenomena, see the monographs [1, 5, 10, 19, 20]. The problem of oscillatory and asymptotic behavior of neutral differential equations is of both theoretical and practical interest. In the last few decades, the oscillatory and asymptotic behavior of solutions of differential equations with impulses studied by many authors and the references [2, 3, 6–8, 11–17] cited therein. To the present time, it seems that only very little is known on the oscillatory and asymptotic behavior of neutral nonlinear impulsive partial differential equations. The above observation is motivated us to consider the following model whose governing equation is of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}\left(u(x,t)+c(t)u\big(x,\tau(t)\big)\right)\right]+q(t)f\left(u\big(x,\rho(t)\big)\right)= a(t)\Delta \mathrm{u}(\mathrm{x},\mathrm{t})\\
&\qquad -\sum_{i=1}^{n} b_i(t)\Delta u\big(x,\mu_i(t)\big)+F(x,t), \quad t\neq t_k, \quad (x,t)\in\Omega\times\mathbb{R}_+\equiv G\\
&u(x,t_k^+)= \alpha_k\left(x,t_k,u(x,t_k)\right)\\
&u_t(x,t_k^+)= \beta_k\big(x,t_k,u_t(x,t_k)\big), \quad k=1,2,\ldots
\end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$ and $\mathbb{R}_+ = [0,+\infty)$.

Equation (1.1) is enhancement with Dirchlet boundary condition

$$u(x,t)= 0, \qquad (x,t)\in\partial\Omega\times\mathbb{R}_+ \tag{1.2}$$

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This work is planned as follows: In Section 2, we present the definitions and some lemmas that will be needed in the sequel. In Section 3, we discussed the oscillatory and asymptotic behavior of the problem (1.1) and (1.2). In Section 4, we present an example is to illustrate our main results.

2 PRELIMINARIES

In this paper, we assume that the following assumptions (A) hold:

(A_1) $r(t) \in C'\big(\mathbb{R}_+, (0,+\infty)\big)$, $q(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\tau(t)$, $\rho(t)$, $\mu_i(t) \in C'(\mathbb{R}_+, \mathbb{R})$, $c(t) \in C^2(\mathbb{R}_+, \mathbb{R}_+)$, $\int_{t_0}^{\infty} \frac{1}{r(s)} ds = \infty$ and $\lim_{t\to+\infty} \tau(t) = \lim_{t\to+\infty} \rho(t) = \lim_{t\to+\infty} \mu_i(t) = +\infty$, $i = 1,2,\dots,n$.

(A_2) $f \in C(\mathbb{R},\mathbb{R})$ *is convex in* $\mathbb{R}_+$ *with* $uf(u) > 0$ *and* $\frac{f(u)}{u} \geq \epsilon > 0$ *for* $u \neq 0, F \in C(\bar{G}, \mathbb{R})$ *with* $\int_\Omega F(x,t)dx < 0$.

(A_3) $a(t)$, $b_i(t) \in PC(\mathbb{R}_+, \mathbb{R}_+)$ where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, and left continuous at $t = t_k$, $k = 1,2,\dots$.

(A_4) $u(x,t)$ and its derivative $u_t(x,t)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1,2,\dots$, and left continuous at $t = t_k$, $u(x,t_k) = u(x,t_k^-)$, $u_t(x,t_k) = u_t(x,t_k^-)$, $k = 1,2,\dots$.

(A_5) α_k, $\beta_k \in PC(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $k = 1,2,\dots$ and there exist positive constants a_k, a_k^*, b_k, b_k^* such that $a_k^* \leq a_k \leq b_k^* \leq b_k$ *for* $k = 1,2,\dots$,

$$a_k^* \leq \frac{\alpha_k\big(x,t_k,u(x,t_k)\big)}{u(x,t_k)} \leq a_k, \quad b_k^* \leq \frac{\beta_k(x,t_k,u_t(x,t_k))}{u_t(x,t_k)} \leq b_k.$$

Definition 2.1: *A solution* u *of the problem (1.1) and (1.2) is a function* $u \in C^2(\bar{\Omega} \times [t_{-1},+\infty),\mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1},+\infty),\mathbb{R})$ *that satisfies (1.1), where*

$$t_{-1} := min\{0, min_{1\leq i\leq n}\{inf_{t\geq 0}\mu_i(t)\}, \{inf_{t\geq 0}\tau(t)\}\}, \qquad \hat{t}_{-1} := min\,\{0, inf_{t\geq 0}\,\rho(t)\}.$$

Definition 2.2: *The solution u of the problem (1.1) and (1.2) is said to be eventually positive (negative) if it is positive (negative) for all sufficiently large* t*. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is non-oscillatory.*

It is identified that [18], the smallest eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\Delta\omega(x) + \lambda\omega(x) = 0, \quad \text{in} \quad \Omega$$
$$\omega(x) = 0, \quad \text{on} \quad \partial\Omega,$$

and the consequent eigenfunction $\Phi(x) > 0$ in Ω.

For convenience, we introduce the following notations:

$$v(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx, \qquad K_\Phi = \left(\int_\Omega \Phi(x)dx\right)^{-1}.$$

The following lemmas are useful for the main results.

Lemma 2.3: *If* X *and* Y *are non negative, then*

$$X^\lambda - \alpha XY^{\lambda-1} + (\lambda-1)Y^\lambda \geq 0, \qquad \lambda > 1$$
$$X^\lambda - \alpha XY^{\lambda-1} - (1-\lambda)Y^\lambda \leq 0, \qquad 0 < \lambda < 1$$

where the equality holds if and only if $X = Y$.

Lemma 2.4: *If* u *is a positive solution of the problem (1.1) - (1.2) in* G*, then the function* $z(t)$ *satisfies the following impulsive differential inequality*

$$(r(t)z'(t))' + cq(t)z\big(\tau(t)\big) \leq 0, \quad t \neq t_k$$
$$a_k^* \leq \frac{z(t_k^+)}{z(t_k)} \leq a_k, \quad b_k^* \leq \frac{z'(t_k^+)}{z'(t_k)} \leq b_k, \ k = 1,2,\dots \tag{2.1}$$

Proof: Let u be a positive solution of the problem (1.1) − (1.2) in G. We may assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0,+\infty)$, $t_0 \geq 0$. By assumption that there exists a $t_1 > t_0$ such that

$$u\big(x,\tau(t)\big) > 0, \ u\big(x,\rho(t)\big) > 0 \ \text{and} \ u\big(x,\mu_i(t)\big) > 0, \ \text{for } (x,t) \in \Omega \times [t_1,+\infty), \ i = 1,2,\dots,n.$$

For $t \geq t_0$, $t \neq t_k$, $k = 1,2,\dots$, multiplying both sides of Equation (1.1) by $K_\Phi\Phi(x) > 0$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned}\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(K_\Phi\int_\Omega u(x,t)\,\Phi(x)dx + c(t)K_\Phi\int_\Omega u\big(x,\tau(t)\big)\,\Phi(x)dx\right)\right] + K_\Phi\int_\Omega q(t)f(u(x,\rho(t)))\,\Phi(x)dx\\ = a(t)K_\Phi\int_\Omega \Delta u(x,t)\,\Phi(x)dx - \sum_{i=1}^{n} b_i(t)\,K_\Phi\int_\Omega \Delta u\big(x,\mu_i(t)\big)\,\Phi(x)dx + K_\Phi\int_\Omega F(x,t)\,\Phi(x)dx.\end{aligned}\right\} \quad (2.2)$$

Using Green's formula and boundary condition (1.2), we see that

$$\begin{aligned}K_\Phi\int_\Omega \Delta u(x,t)\,\Phi(x)dx &= K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + K_\Phi\int_\Omega u(x,t)\,\Delta\Phi(x)dx\\ &= -\lambda_0 v(t) \le 0 \end{aligned}\quad (2.3)$$

and for $i = 1,2,\dots,n$, we have

$$\begin{aligned}K_\Phi\int_\Omega \Delta u(x,\mu_i(t))\,\Phi(x)dx &= K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,\mu_i(t))}{\partial\gamma} - u(x,\mu_i(t))\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + K_\Phi\int_\Omega u(x,\mu_i(t))\,\Delta\Phi(x)dx\\ &= -\lambda_0 v(\mu_i(t)) \le 0\end{aligned}\quad (2.4)$$

where dS is surface element on $\partial\Omega$. Applying Jensen's inequality, from (A_2) and assumptions, it follows that

$$K_\Phi\int_\Omega q(t)f(u(x,\rho(t)))\,\Phi(x)dx \ge \epsilon q(t)K_\Phi\int_\Omega u(x,\rho(t)))\,\Phi(x)dx. \quad (2.5)$$

In view of (2.2) - (2.5), we obtain

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\Big(v(t)+c(t)v\big(\tau(t)\big)\Big)\right] + \epsilon q(t)v\big(\tau(t)\big) \le 0.$$

Let $z(t) = v(t) + c(t)v\big(\tau(t)\big)$. Then

$$(r(t)z'(t))' + \epsilon q(t)v\big(\tau(t)\big) \le 0. \quad (2.6)$$

It is easy to obtain that $z(t) > 0$ for $t \ge t_1$. Next we prove that $z'(t) > 0$ for $t \ge t_2$. Assume the contrary that there exists $T \ge t_2$ such that $z'(T) \le 0$.

$$(r(t)z'(t))' \le 0, \qquad t \ge t_2.$$

From this we have $r(t)z'(t) \le r(T)z'(T) \le 0,\ t \ge T$. Thus

$$z(t) \le z(T) + r(T)z'(T)\int_T^t \frac{ds}{r(s)}, \quad t \ge T.$$

From the hypothesis (A_1), we have $\lim_{t\to+\infty} z(t) = -\infty$. This contradicts that $z(t) > 0$ for $t \ge 0$. Thus $z'(t) > 0$, $\tau(t) \le t$ for $t \ge t_1$, we have

$$v(t) = z(t) - c(t)v(\tau(t))$$
$$v(t) \ge z(t)(1 - c(t))$$

and

$$v\big(\tau(t)\big) \ge c_0 z\big(\tau(t)\big).$$

Therefore from (2.6), we have

$$(r(t)z'(t))' + c_0\epsilon q(t)z\big(\tau(t)\big) \le 0, \quad t \ge t_1.$$
$$(r(t)z'(t))' + cq(t)z\big(\tau(t)\big) \le 0, \quad \text{where} \quad c = \epsilon c_0.$$

For $t \ge t_0,\ t = t_k,\ k = 1,2,3,\dots,$ multiplying both sides of the Equation (1.1) by $K_\Phi\Phi(x) > 0$, integrating with respect to x over the domain Ω, and from (A_5), we obtain

$$a_k^* \le \frac{u\big(x,t_k^+\big)}{u(x,t_k)} \le a_k, \quad b_k^* \le \frac{u_t(x,t_k^+)}{u_t(x,t_k)} \le b_k.$$

From assumptions we have,

$$a_k^* \le \frac{v(t_k^+)}{v(t_k)} \le a_k, \quad b_k^* \le \frac{v'(t_k^+)}{v'(t_k)} \le b_k$$

and

$$a_k^* \le \frac{z(t_k^+)}{z(t_k)} \le a_k, \qquad b_k^* \le \frac{z'(t_k^+)}{z'(t_k)} \le b_k.$$

Hence we obtain that $z(t)$ is a solution of impulsive inequality (2.1). This completes the proof.

Lemma 2.5: *Assume that conditions $(A_1)-(A_5)$ holds and let $u(x,t)$ be a positive solution of (1.1) and (1.2). Then for sufficiently large t, either*

(i) $z(t)>0,\ z'(t)>0,\ \ (r(t)z'(t))'<0$ *or*

(ii) $z(t)>0,\ z'(t)<0,\ (r(t)z'(t))'<0.$

Lemma 2.6: *Assume that conditions $(A_1)-(A_5)$ holds and let $u(x,t)$ be an eventually positive solution of (2.1) with $z(t)$ satisfying case (ii) of Lemma 2.5. If*

$$\int_{t_1}^{\infty}\frac{1}{r(y)}\int_{y}^{\infty}cq(s)z\big(\tau(s)\big)dsdy=\infty \tag{2.7}$$

then $lim_{t\to\infty} z(t)=0$.

Proof: Let $u(x,t)$ be an eventually positive solution of (1.1) and (1.2). Then $z(t)$ satisfies the inequality (2.1) and

$$(r(t)z'(t))'\le -cq(t)z\big(\tau(t)\big)\le 0.$$

By Lemma 2.5, there exists a constant l such that $\lim_{t\to\infty} z(t)=l<\infty$.

Integrating the above inequality from t to ∞, we get

$$r(t)z'(t)\ge\int_{t}^{\infty}cq(s)z\big(\tau(s)\big)ds\,,$$

$$z'(t)\ge\frac{1}{r(t)}\int_{t}^{\infty}cq(s)z\big(\tau(s)\big)ds.$$

Again integrating from t_1 to ∞, we obtain

$$z(t)\le-\int_{t_1}^{\infty}\frac{1}{r(y)}\int_{y}^{\infty}cq(s)z\big(\tau(s)\big)dsdy,$$

which contradicts (2.7) and so we have $l=0$. Therefore $\lim_{t\to\infty} z(t)=0$. This complete the proof.

3. MAIN RESULTS

In this section, by using Riccati transformation and impulsive differential inequality, we investigate the oscillatory and asymptotic behavior of all solutions of neutral nonlinear partial differential equations with impulse effects and obtained the following two theorems.

Theorem 3.1: *Assume that $(A_1)-(A_5)$ holds and there exists $\phi(t)\in C'([t_0,\infty),\mathbb{R})$ such that for all sufficiently large T and for $t_1\ge T$,*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\prod_{t_0\le t_k\le s}\left(\frac{b_k}{a_k^*}\right)^{-1}\left[c\phi(s)q(s)-\frac{(\phi'(s))^2}{4\tau'(s)r(s)\phi(s)}\right]ds=\infty \tag{3.1}$$

then every solution u of (1.1) and (1.2) is either oscillatory or converges to zero as $t\to\infty$.

Proof: Let $u(x,t)$ be a non oscillatory solution of (1.1) and (1.2). Without loss of generality, we may assume that there exists $t_1\ge t_0$ such that $u(x,t)>0, u\big(x,\tau(t)\big)>0$ and $u\big(x,\rho(t)\big)>0$ for $t\ge t_1$.

Let

$$w(t)=\phi(t)\frac{r(t)z'(t)}{z(\tau(t))}$$

$$w'(t)\le -c\phi(t)q(t)-\frac{w^2(t)}{r(t)\phi(t)}\tau'(t)+\frac{\phi'(t)}{\phi(t)}w(t). \tag{3.2}$$

Also

$$w(t_k^+)\le\frac{b_k}{a_k^*}w(t_k). \tag{3.3}$$

Define

$$V(t)=\prod_{t_0\le t_k<t}\left(\frac{b_k}{a_k^*}\right)^{-1}w(t).$$

In fact, $w(t)$ is a continuous on each interval $(t_k,t_{k+1}]$ and it follows that for $t\ge t_0$,

$$V(t_k^+)=\prod_{t_0\le t_j\le t_k}\left(\frac{b_k}{a_k^*}\right)^{-1}w(t_k^+)\le\prod_{t_0\le t_j<t}\left(\frac{b_k}{a_k^*}\right)^{-1}w(t_k)=V(t_k)$$

and for all $t \geq t_0$

$$V(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left(\frac{b_k}{a_k^*}\right)^{-1} w(t_k^-) \leq \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} w(t_k) = V(t_k),$$

Which implies that $V(t)$ is continuous on $[t_0, +\infty)$, from (3.2), we get

$$\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) V'(t) \leq -c\phi(t)q(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^2 \frac{\tau'(t)V^2(t)}{r(t)\phi(t)} + \frac{\phi'(t)}{\phi(t)} \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) V(t)$$

$$V'(t) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) \frac{\tau'(t)V^2(t)}{r(t)\phi(t)} + \frac{\phi'(t)}{\phi(t)} V(t) - c \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \phi(t)q(t) \tag{3.4}$$

Applying Lemma 2.3, we have

$$X = \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) \frac{\tau'(t)V(t)}{r(t)\phi(t)}} \qquad and \qquad Y = \frac{\phi'(t)}{2} \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{r(t)}{\phi(t)\tau'(t)}}$$

We have

$$\frac{\phi'(t)}{\phi(t)} V(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) \frac{\tau'(t)V^2(t)}{r(t)\phi(t)} \leq \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{(\phi'(t))^2}{4\tau'(t)r(t)\phi(t)}.$$

Thus

$$V'(t) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \left[c\phi(t)q(t) - \frac{(\phi'(t))^2}{4\tau'(t)r(t)\phi(t)}\right].$$

Integrating both sides from t_1 to t, we have

$$V(t) \leq V(t_1) - \int_{t_1}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \left[c\phi(s)q(s) - \frac{(\phi'(s))^2}{4\tau'(s)r(s)\phi(s)}\right] ds.$$

Letting $t \to \infty$, from (3.1), we have $\lim_{t\to\infty} V(t) = -\infty$, which is a contradiction. Then by Lemma 2.6, we have $\lim_{t\to\infty} z(t) = 0$. Since $0 < v(t) < z(t)$ on (t_1, ∞), we get that $\lim_{t\to\infty} u(x,t) = 0$. The proof of the theorem is complete.

Next we obtain some new oscillatory and asymptotic results for (1.1) and (1.2), by using integral average condition of Philos type. Let $D = \{(t,s): t_0 \leq s \leq t\}, H \in C^1(D, \mathbb{R})$. If $H \in \mathcal{H}$, then $H(t,t) = 0$ and $H(t,s) > 0$ for $t > s$ and $h \in L_{loc}(D, \mathbb{R})$ such that

$$\frac{\partial H(t,s)}{\partial t} = h(t,s)\sqrt{H(t,s)}, \qquad \frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)}\,.$$

Theorem 3.2: *Assume that conditions $(A_1) - (A_5)$ holds and there exist $\phi(t), \psi(t) \in C'\big([0,\infty), (0,+\infty)\big)$, if*

$$\limsup_{t\to+\infty} \frac{c}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \Bigg\{ \phi(s)q(s)H(t,s)\psi(s) - \frac{r(s)\phi(s)}{4} \frac{[h(t,s)\psi(s) - \sqrt{H(t,s)}\psi'(s) - \frac{\phi'(s)}{\phi(s)}\sqrt{H(t,s)}\psi(s)]^2}{\psi(s)\tau'(s)} \Bigg\} ds = +\infty, \tag{3.5}$$

then every solution of the boundary value problem (1.1) and (1.2) is oscillatory or converges to zero as $t \to \infty$.

Proof: Assume that the boundary value problem (1.1) and (1.2) has a non oscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t) > 0, (x,t) \in \Omega \times [0,+\infty)$. As in the proof of the Theorem 3.1, we obtain

$$V'(t) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) \frac{\tau'(t)V^2(t)}{r(t)\phi(t)} + \frac{\phi'(t)}{\phi(t)} V(t) - c \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \phi(t)q(t).$$

Multiplying the above inequality by $H(t,s)\psi(s)$ for $t \geq s \geq T$, and integrating from T to t, we have

$$\int_T^t V'(s)H(t,s)\psi(s)ds \leq -\int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right) \frac{\tau'(s)V^2(s)}{r(s)\phi(s)} H(t,s)\psi(s)\,ds + \int_T^t \frac{\phi'(s)}{\phi(s)} V(s)H(t,s)\psi(s)ds$$

$$-c\int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \phi(s)q(s)H(t,s)\psi(s)ds.$$

$$c\int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \phi(s)q(s)H(t,s)\psi(s)ds \leq V(t)H(t,T)\psi(T) - \int_T^t \{h(t,s)\sqrt{H(t,s)}\,\psi(s)$$

$$-H(t,s)\psi'(s) - \frac{\phi'(s)}{\phi(s)} H(t,s)\psi(s)\}V(s)ds - \int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right) \frac{\tau'(s)V^2(s)}{r(s)\phi(s)} H(t,s)\psi(s)\,ds \tag{3.6}$$

From this,

$$c\int_T^t \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \left\{\phi(s)q(s)H(t,s)\psi(s) - \frac{r(s)\phi(s)}{4} \frac{\left[h(t,s)\psi(s) - \sqrt{H(t,s)}\psi'(s) - \frac{\phi'(s)}{\phi(s)}\sqrt{H(t,s)}\psi(s)\right]^2}{\psi(s)\tau'(s)}\right\} ds$$
$$\leq V(T)H(t,T)\psi(T). \tag{3.7}$$

Letting $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{c}{H(t,t_0)} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \left\{\phi(s)q(s)H(t,s)\psi(s)\right.$$
$$\left. - \frac{r(s)\phi(s)}{4} \frac{[h(t,s)\psi(s) - \sqrt{H(t,s)}\psi'(s) - \frac{\phi'(s)}{\phi(s)}\sqrt{H(t,s)}\psi(s)]^2}{\psi(s)\tau'(s)}\right\} ds < +\infty \tag{3.8}$$

which is a contradiction with (3.5). Then by Lemma 2.6, we have $\lim_{t\to\infty} z(t) = 0$. Since $0 < v(t) < z(t)$ on (t_1, ∞), we get $\lim_{t\to\infty} u(x,t) = 0$. The proof of the theorem is complete.

4 EXAMPLE

In this section, we present an example to illustrate our results established in Section 3

Example 4.1: Consider the following impulsive neutral nonlinear partial differential equation is of the form

$$\frac{\partial}{\partial t}\left[t^2 \frac{\partial}{\partial t}\left(u(x,t) + \frac{1}{9}u\left(x, t-\frac{1}{3}\right)\right)\right] + \frac{2}{(3t-1)^2} u\left(x, t-\frac{1}{3}\right) = 3\Delta u(x,t)$$
$$-\frac{2t^2+2}{(3t-1)^2}\,\Delta u\left(x, t-\frac{1}{3}\right) + \frac{3\sin x}{t} - \frac{2t\sin x}{(3t-1)^2},\quad t \neq 2^k, (x,t) \in \Omega \times \mathbb{R}_+ \equiv G$$
$$u(x, t_k^+) = \frac{k}{k+1}\,u\,(x, 2^k)$$
$$u_t(x, t_k^+) = u_t(x, 2^k), \quad k = 1,2,\ldots, \tag{4.1}$$

for $(x,t) \in (0,\pi) \times \mathbb{R}_+$, with the boundary condition

$$u(0,t) = u(\pi,t) = 0,\ (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{4.2}$$

Here $r(t) = t^2$, $q(t) = \frac{1}{(3t-1)^2}$, $a(t) = 3$, $b_1(t) = \frac{2t^2+2}{(3t-1)^2}$, $c(t) = \frac{1}{9}$, $f(u) = 2u$, $\tau(t) = \rho(t) = \mu_1(t) = t - \frac{1}{3}$, $F(x,t) = \frac{3\sin x}{t} - \frac{2t\sin x}{(3t-1)^2}$. Let $\alpha_k = \alpha_k^* = \frac{k}{k+1}$, $\beta_k = \beta_k^* = 1$, $t_0 = 1$, $t_k = 2^k$. Here, it is easy to see that all conditions of Theorem 3.1 are satisfied. In fact $u(x,t) = \frac{\sin x}{t}$ is one such solution.

CONCLUSION

In this paper, we have studied the oscillatory and asymptotic behavior of solutions to impulsive neutral nonlinear partial differential equations. Our obtained results are essentially new and which generalize the results already existing in the literature. Moreover an example is also given to illustrate the effectiveness of our main results.

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