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ON THE OSCILLATION OF CONFORMABLE FRACTIONAL NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

In this article, we investigate the oscillatory behavior of solutions of a class of conformable fractional nonlinear homogeneous differential equations of the form

$$T_\alpha\left(t\,r(t)\big(T_\alpha x(t)+\lambda t^{1-\alpha}x(t)\big)\right)+p(t)g\big(x(t)\big)=0,\ t\geq t_0,$$

where T_α denotes the conformable fractional derivative of order α with 0< $\alpha \leq 1$. We establish some new sufficient conditions by using the equivalence transformation and associated Riccati technique. These newly obtained results extend the known results for the differential equations of integer order. A suitable example is given to illustrate the effectiveness of our main results.

Key words: *Oscillation, Nonlinear differential equation, Conformable fractional differential equation.*

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1. INTRODUCTION

Differential equations are one of the most frequently used tools for mathematical modeling in engineering and life sciences. The explicit solution does not exist for the nonlinear differential equation. In the absence of closed form solutions to many of linear and nonlinear differential equations a rewarding alternative is to be resorted to the qualitative study of the solutions of the equations, without actually constructing or approximating them. In the qualitative study of differential equations oscillatory behavior of solutions plays a major role. For the basic theory and applications, see the monographs [2, 6, 8, 9, 15] and the references cited therein.

In the recent years, fractional calculus and fractional differential equations are the most rapidly growing area of research. A rigorous and encyclopedic study of fractional differential equations can be found in [1, 7, 12, 13, 17, 18]. Even though there are different concepts of fractional derivatives such as Riemann-Liouville and Caputo fractional derivatives are widely used, which are based on integrals and nonlocal. In 2014, Khalil *et al.* introduced the conformable fractional derivative based on the limit definition analogous to that of standard derivatives [4, 5, 11].

However, a huge volume of literature, see [3, 10, 14] on the oscillation and nonoscillation of self-adjoint second order differential equations of the form

$$\left(a(t)x'(t)\right)'+p(t)x(t)=0,\ t>t_0,$$

subject to the conditions

$$\int_{t_0}^{\infty}p(s)ds<\infty \text{ and } \int_{t_0}^{\infty}p(s)ds=\infty.$$

In 2012, Tariboon *et al.* [16] studied the class of second order linear impulsive differential equations of the form

$$(\mathrm{a(t)}(x'(t)+\lambda x(t)))'+p(t)x(t)=0,\ t\geq t_0,\ t\neq t_k,$$
$$x(t_k^+)=b_kx(t_k), x'(t_k^+)=c_kx'(t_k), k=1,2,\ldots$$

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To the best of the author's knowledge, it seems that there has been no work done on oscillation of conformable fractional differential equations. Motivated by this gap, we propose to initiate the following model of the form

$$T_\alpha\left(t\,r(t)\left(T_\alpha \mathrm{x(t)} + \lambda t^{1-\alpha}\mathrm{x(t)}\right)\right) + p(t)g\left(x(t)\right) = 0,\ t \geq t_0, \tag{1.1}$$

where T_α denotes the conformable fractional derivative of order α, $0< \alpha \leq 1$.

We assume throughout this paper that

(A_1) $t\,r(t) \in C^\alpha([t_0,\infty),(0,\infty))$, $p(t) \in C([t_0,\infty),\mathbb{R}_+)$, and λ is a real number;

(A_2) $g: R \longrightarrow R$ is a continuous function such that $\frac{g(x)}{x} \geq \mu$, for $x \neq 0$ and for certain constant $\mu > 0$.

It will be assumed that the equation (1.1) has the solutions which are nontrivial for large t.

A nontrivial solution $x(t)$ of differential equation (1.1) is said to be oscillatory if it has arbitrarily large zeros otherwise it said to be nonoscillatory. The equation (1.1) is oscillatory if all its solutions are oscillatory.

The main aim of this paper is to present new oscillation criteria for (1.1) by making use of the equivalence transformation and associated Riccati technique.

This paper is organized as follows: In section 2, we recall the basic definitions of conformable fractional derivative. In section 3, we present some new results of oscillation of solutions of (1.1). In section 4, an example is provided to illustrate the main results.

2. PRELIMINARIES

In this section, we shall present some preliminary results on conformable fractional derivatives. First we shall start with the definition.

Definition 2.1: [Khalil, 11] Given a function $f: [0,\infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha\,(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1]$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$.

We will sometimes write $f^{(\alpha)}(t)$ for $T_\alpha\,(f)(t)$, to denote the conformable fractional derivatives of f of order α.

Some properties of conformable fractional derivative [Khalil, 11]:

Let $\alpha \in (0,1]$ and f *and* g be α-differentiable at a point $t > 0$ .Then

(P_1) $T^\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.

(P_2) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

(P_3) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(P_4) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(P_5) If, in addition, f is differentiable, then $T_\alpha\,(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

3. MAIN RESULTS

In this section, we establish some new conditions for the oscillation of all solutions of the problem (1.1).

Theorem 3.1: Suppose that $(A_1) - (A_2)$ hold. If

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\mu s^{\alpha-1}p(s) - \frac{\lambda^2}{4}r(s)s^{2-\alpha}\right)ds = \infty. \tag{3.1}$$

Then every solution of (1.1) is oscillatory.

Proof: Assume that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t)$ is an eventually positive solution of (1.1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$.

Define the function w by the generalized Riccati substitution

$$w(t) = \frac{t\,r(t)T_\alpha\left(e^{\lambda t}x(t)\right)}{e^{\lambda t}x(t)}, t \geq t_1 \tag{3.2}$$

$$= \frac{t\,r(t)(T_\alpha x(t) + \lambda t^{1-\alpha}x(t))}{x(t)}, t \geq t_1. \tag{3.3}$$

Then $w(t) > 0$ for $t \geq t_1$.

Differentiating (3.2) α times with respect to t, using (1.1), (P_4) and (P_5), we have

$$T_\alpha\, w(t) = t^{1-\alpha} \frac{\left(e^{\lambda t}x(t)\left((t\, r(t) T_\alpha(e^{\lambda t}x(t))\right)' - t\, r(t) T_\alpha\left(e^{\lambda t}x(t)\right)\left(e^{\lambda t}x(t)\right)'\right)}{\left(e^{\lambda t}x(t)\right)^2}$$
$$\leq -\mu p(t) + \lambda t^{1-\alpha} w(t) - \frac{w^2(t)}{t r(t)}$$

Then

$$w'(t) \leq -\mu t^{\alpha-1} p(t) + \lambda w(t) - \frac{t^{\alpha-2}}{r(t)} w^2(t) \tag{3.4}$$

By using the inequality, $Bu - Au^2 \leq B^2/4A$,

$$w'(t) \leq -\mu t^{\alpha-1} p(t) + \frac{\lambda^2}{4} r(t) t^{2-\alpha}, \ t \geq t_1. \tag{3.5}$$

Integrating (3.5) on both sides from t_1 to t, we have

$$\int_{t_1}^{t} w'(s)\, ds \leq -\int_{t_1}^{t} \left(\mu s^{\alpha-1} p(s) - \frac{\lambda^2}{4} r(s) s^{2-\alpha}\right) ds.$$

Letting $t \to \infty$ we get, $\lim_{t\to\infty} w(t) \leq -\infty$, which contradicts to (3.1) and completes the proof.

For the following theorem, we introduce a class of functions $\mathcal{P}$.

Let $\mathcal{D}_0 = \{(t,s): t > s \geq t_0\}$ and $\mathcal{D} = \{(t,s): t \geq s \geq t_0\}$. Then function $H \in \mathcal{C}(\mathcal{D}; \mathbb{R})$ is said belong to the class $\mathcal{P}$, if
(H_1) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on $\mathcal{D}_0$;
(H_2) H has a continuous and non positive partial derivative on $\mathcal{D}_0$ with respect to s.

Theorem 3.2: Suppose that the conditions $(A_1) - (A_2)$ hold. Let $h,\ H : D \to \mathbb{R}$ be continuous function such that $H \in \mathcal{P}$ and

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \ \text{ for all } (t,s) \in D_0.$$

Furthermore, assume that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\, \mu s^{\alpha-1} p(s) - \frac{1}{4} G^2(t,s) s^{2-\alpha} r(s)\right] ds = \infty, \tag{3.6}$$

where $G(t,s) = h(t,s) - \lambda\sqrt{H(t,s)}$. Then every solution of (1.1) is oscillatory.

Proof: Assume that x(t) is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x(t) is an eventually positive solution of (1.1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we have

$$w'(t) \leq -\mu t^{\alpha-1} p(t) + \lambda w(t) - \frac{t^{\alpha-2}}{r(t)} w^2(t).$$

Multiplying both sides of above by $H(t,s)$ and integrating from t_1 to t for $t \geq t_1$, we get

$$\int_{t_1}^{t} H(t,s) \mu s^{\alpha-1} p(s)\, ds \leq -\int_{t_1}^{t} H(t,s) w'(s) ds + \int_{t_1}^{t} H(t,s) \lambda w(s) ds - \int_{t_1}^{t} H(t,s) \frac{s^{\alpha-2}}{r(s)} w^2(s) ds \tag{3.7}$$
$$= H(t,t_1) w(t_1) - \int_{t_1}^{t} \left[\frac{-\partial H(t,s)}{\partial s} w(s) - H(t,s)\lambda w(s) + \frac{H(t,s) s^{\alpha-2}}{r(s)} w^2(s)\right] ds$$
$$= H(t,t_1) w(t_1) - \int_{t_1}^{t} \left[h(t,s)\sqrt{H(t,s)}\, w(s) - H(t,s)\lambda w(s) + \frac{H(t,s) s^{\alpha-2}}{r(s)} w^2(s)\right] ds$$
$$= H(t,t_1) w(t_1) - \int_{t_1}^{t} \left[\sqrt{\frac{H(t,s) s^{\alpha-2}}{r(s)}}\, w(s) + \frac{1}{2} G(t,s) \sqrt{s^{2-\alpha} r(s)}\right]^2 ds + \int_{t_1}^{t} \frac{1}{4} G^2(t,s) s^{2-\alpha} r(s) ds.$$

Thus for all $t \geq t_1 \geq t_0$, we conclude that

$$\int_{t_1}^{t} \left[H(t,s) \mu s^{\alpha-1} p(s) - \frac{1}{4} G^2(t,s) s^{2-\alpha} r(s)\right] ds \leq H(t,t_1) w(t_1).$$

It follows that,

$$\int_{t_0}^{t} \left[H(t,s) \mu s^{\alpha-1} p(s) - \frac{1}{4} G^2(t,s) s^{2-\alpha} r(s)\right] ds$$
$$\leq \int_{t_0}^{t_1} \left[H(t,s) \mu s^{\alpha-1} p(s) - \frac{1}{4} G^2(t,s) s^{2-\alpha} r(s)\right] ds + H(t,t_1) w(t_1)$$
$$\leq H(t,t_0) \int_{t_0}^{t_1} \mu s^{\alpha-1} p(s) ds + H(t,t_0) |w(t_0)|$$

$$\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s) \mu s^{\alpha-1} p(s) - \frac{1}{4} G^2(t,s) s^{2-\alpha} r(s)\right] ds \leq \int_{t_0}^{t_1} \mu s^{\alpha-1} p(s) ds + |w(t_0)|.$$

This inequality yields that,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\mu s^{\alpha-1}p(s)\ -\frac{1}{4}G^2(t,s)s^{2-\alpha}r(s)\right]\mathrm{ds}\ \le\ \int_{t_0}^{t_1}\mu s^{\alpha-1}p(s)ds+|w(t_0)|<\infty,$$

which contradicts to (3.6) and completes the proof.

Let H(t,s) =$(t-s)^{n-1}$, (t,s) ∈ D for some integer n > 2. Then, Theorem 3.2 gives the following result.

Corollary 3.1: Let assumption (3.6) in Theorem 3.2 be replaced by

$$\limsup_{t\to\infty}(t-t_0)^{1-n}\int_{t_0}^{t}\left[\mu s^{\alpha-1}(t-s)^{n-1}p(s)-\ \frac{1}{4}\left((n-1)(t-s)^{\frac{n-3}{2}}-\lambda(t-s)^{\frac{n-1}{2}}\right)^2\ s^{2-\alpha}r(s)\right]ds, \tag{3.8}$$

for some integer n > 2. Then every solution x(t) of (1.1) is oscillatory.

4. EXAMPLES

In this section, we present an example to illustrate our main results.

Example 4.1: Consider the conformable fractional differential equation

$$T_{\frac{1}{2}}\left(\frac{1}{\sqrt{t}}\left(T_{\frac{1}{2}}\mathrm{x(t)}-t^{\frac{1}{2}}\mathrm{x(t)}\right)\right)+t^{\frac{1}{2}}g(x(t))=0,t\ge t_0\ , \tag{4.1}$$

Here $\alpha=\frac{1}{2}$, r(t) = $\frac{1}{t^{\frac{3}{2}}}$, λ = -1, p(t) = $t^{\frac{1}{2}}$, g(u) = u +$\sqrt{1-u^2}$ and $\frac{g(x(t))}{x(t)}\ \ge 1=\mu$,
where t > cosec^{-1}(1). It is easy to see that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\mu s^{\alpha-1}p(s)-\frac{\lambda^2}{4}r(s)s^{2-\alpha}\right)ds=\limsup_{t\to\infty}\int_{t_1}^{t}\left(s^{-\frac{1}{2}}s^{\frac{1}{2}}-\frac{1}{4}\frac{1}{s^{\frac{3}{2}}}s^{\frac{3}{2}}\right)ds\to\infty.$$

Thus all the conditions of Theorem 3.1 are satisfied. Hence every solution of (4.1) is oscillatory. In fact, x(t)= sin t is one such solution.

CONCLUSION

In this study, we have obtained some new oscillation results for some class of conformable fractional nonlinear homogeneous differential equations by using the equivalence transformation and associated Riccati technique. This work extends some of the results in the exiting literature with integer order and the equation (1.1) has found to be an effective tool to describe the evolution if physical phenomena in fluctuating environments where the memory effects are taken into consideration.

REFERENCES

1. Abbas S., Benchohra M. and N'Guerekata G.M., Topics in fractional differential equations, Springer, Newyork, 2012.
2. Agarwal R.P., Grace S.R. and O'Regan D., Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations, Kluwer academic publishers, Dordrecht, 2002.
3. Agarwal R.P. and O'Regan D., An introduction to ordinary differential equations, Springer, New York, 2008.
4. Anderson D.R. and Ulness D.J., Newly defined conformable derivatives, Advances in Dynamical Systems and Applications, 10 (2) (2015), 109–137.
5. Atangana A., Baleanu D. and Alsaedi A., New properties of conformable derivative, Open math, (2015), 889-898.
6. Bainov D. and Mishev D., Oscillatory theory of operator-differential equations, World scientific, Singapore, 1987.
7. Daftardar-Gejji V., Fractional calculus theory and applications, Narosa publishing house Pvt. Ltd., 2014.
8. Erbe L. H., Kong Q. K., and Zhang B. G., Oscillation theory for functional differential equations, Marcel Dekker, New York, 1995.
9. Gyori I. and Ladas G., Oscillation theory of delay differential equations, Claendon Press, Oxford, 1991.
10. Kelley W.G. and Peterson A.C., The theory of differential equations: Classical and Qualitative, Springer, New York, 2010.
11. Khalil R., Al Horani M., Yousef A. and Sababheh M., A new definition of fractional derivative, Journal of computational and applied mathematics, 264 (2014), 65-70.
12. Kilbas A.A., Srivastava H.M. and Trujillo J.J., Theory and applications of fractional differential equations, Elsevier science, B.V. Amsterdam, 2006.
13. Miller K.S. and Ross B., An introduction to fractional calculus and fractional differential equations, J.Wiley and Sons, Newyork, 1993.
14. Saker S. H., Oscillation theory of delay differential and difference equations, VDM Verlag Dr.Muller Aktiengesellschaft and Co, USA, 2010.

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15. Swanson G.A., Comparison and oscillation theory of linear differential equations, Academic press, New York, 1968.
16. Tariboon J. and Thiramanus P., Oscillation of a class of second order linear impulsive differential equations, Advances in difference equations, (2012), 2012:205.
17. Tarasov V. E., Fractional dynamics, Springer, 2009.
18. Zhou Y., Basic theory of fractional differential equations, World scientific, Singapore, 2014.

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