Volume 9, No. 3, March - 2018 (Special Issue)
International Journal of Mathematical Archive-9(3), 2018, 97-106
Available online through www.ijma.info ISSN 2229 – 5046

A STUDY ON FINE INTUITIONISTIC FUZZY TOPOLOGICAL RING SPACES

R. NANDHINI[1] AND DR. D. AMSAVENI[2]

[1]Department of Mathematics (Research Scholar),
Sri Sarada College for Women, Salem - 636 016, Tamilnadu, India.

[2]Department of Mathematics (Assistant Professor),
Sri Sarada College for Women, Salem - 636 016, Tamilnadu, India.

E-mail: nandhinirajamradha@gmail.com[1], d_amsaveni@rediffmail.com[2].

ABSTRACT

In this paper the notions of fine intuitionistic fuzzy topological spaces are introduced and the concepts of fine intuitionistic fuzzy ring structure spaces are defined also its characterizations are investigated. Finally, the fine intuitionistic fuzzy normal subrings are introduced and studied few of its properties.

Keywords: *Fine intuitionistic fuzzy set (FIfS), Fine intuitionistic fuzzy topological space (FIfTS), Fine intuitionistic fuzzy ring structure space(FIfTRSS).*

2010 Subject Classification Primary: *54A40, 03E72, 54F45.*

1. INTRODUCTION

Lofti A. Zadeh[10] introduced the fuzzy set theory in 1965. The notion of a fuzzy topology was introduced by Chang [3] in 1968. The concept of intuitionistic fuzzy set was first publishedby Krassimir T. Attanassov[1] as a generalization of the notion of fuzzy sets. Coker.D [4] developed the idea of an intuitionistic fuzzy topological spaces in 1997.Meena.K and Thomas.V[5]introduced, an intuitionisticL-fuzzy Subrings. Fine topological space was introduced by Power P. L. and RajakK [8]. Fine fuzzy topological spaces are introduced and studied the concept of fine fuzzy sp-closed sets by Nandhini. R [6]. In the present paper, we introduced a fine intuitionistic fuzzy set and fineintuitionistic fuzzy topological spaces and discussed some of its properties. The concepts of fine intuitionistic fuzzy ring structure spaces are defined and its characterizations are also examined. Finally, the fine intuitionistic fuzzy normal subrings are studied with few of its properties.

2. PRELIMINARIES

Definition 2.1 [3]: Let X be a non-empty set and I be the unit interval. A **fuzzy set** of X is an element of the set I^{X} of all functions from X to I .

Definition 2.2[4]: Let X be a nonempty fixed set. An **intuitionistic fuzzy set** A is an object having the form $A=\{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ where the functions $\mu_A(x): X \to I$ and $\gamma_A(x): X \to I$ denote the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A, respectively and $0 \le \mu_A + \gamma_A \le 1$ for each $x \in X$.

Definition 2.3 [4]: An intuitionistic fuzzy topology (IFT) on a nonempty set X is a family τ of IFSs in X satisfying the following conditions:

(a) $0_{\sim}, 1_{\sim} \in \tau$.

(b) if $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.

CONFERENCE PAPER
National Conference dated 26th Feb. 2018, on "New Trends in Mathematical Modelling" (NTMM - 2018), Organized by Sri Sarada College for Women, Salem, Tamil Nadu, India.

(c) $\bigcup G_i \in \tau$ for any arbitrary family $\{ G_i : i \in J \} \subseteq \tau$.

The pair (X, τ) is called an intuitionistic fuzzy topological space (abbreviated as IFTS) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS) in X.

Definition 2.4 [4]: Let A and B be intuitionistic fuzzy set of X. Then A is said to be **quasi coincident** with B is denoted byAqB if there exist an $x \in X$ such that $\gamma_B(x) < \mu_A(x)$ or $\gamma_A(x) < \mu_B(x)$.

Definition 2.5[8]: Let (X, τ) be a topological space and define $\tau(A_\alpha) = \tau_\alpha$ (*say*) = $\{G_\alpha(\neq X) : G_\alpha \cap A_\alpha \neq \phi$, for $A_\alpha \in \tau$ and $A_\alpha \neq \phi, X$, for some $\alpha \in J$, where J is the index set $\}$.Define $\tau_f = \{\phi, X, \bigcup_{\alpha \in J}\{T_\alpha\}\}$. The collection τ_f of subsets of Xis called the fine collection of subsets of X and (X, τ, τ_f) is said to be **fine space** X generated by the topology τ on X.

Definition 2.6 [6]: Let (X,T) be a fuzzy topological space and let $T(\lambda_\alpha) = T_\alpha = \{ \mu_\alpha(\neq 1_X) : \mu_\alpha \wedge \lambda_\alpha \neq 0_X$, for $\lambda_\alpha \in T$ and $\lambda_\alpha \neq 0x, 1x$ in some $\alpha \in J$, where Jis the index set$\}$be the collection of fine fuzzy sets of X. Then the collection $T_f = \{0_X, 1_X, \bigcup_{\alpha \in J}\{T_\alpha\}\}$ is said to be the fine fuzzy collection of subsets of X and (X, T, T_f) is called the **Fine fuzzy topological space** (abbreviated as) FfTS.

Definition 2.7[5]: An Intuitionistic L-fuzzy subset $A = \{x, \mu_A(x), \nu_A(x)/x \in R\}$ of R is said to be an **Intuitionistic L-fuzzy subring** of R (ILFSR) if for all x, y ∈ R

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ (ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ (iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$(iv) $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

Definition 2.8 [9]: Let $(R, +, .)$ be a ring. An intuitionistic fuzzy subring A of R is said to be **an intuitionistic fuzzy normal subring** of R if it satisfies the following axioms

(i) $\mu_A(xy) = \mu_A(yx)$

(ii) $\nu_A(xy) = \nu_A(yx) \; \forall \; x, y \in R$.

3. FINE INTUITIONISTIC FUZZY TOPOLOGICAL SPACE

Definition 3.1: Let X be a nonempty set. Let a ***fine intuitionistic fuzzy set***be defined as

$$A = \left\langle x, \tau(\mu_{A_\alpha}(x)), \tau(\gamma_{A_\alpha}(x)) : x \in X \right\rangle \text{ for all } x \in X$$

$$\tau_{\mu_A} = \tau(\mu_{A_\alpha}(x)) = \left\{ \begin{array}{l} \lambda_{A_\alpha}(\neq 1_\sim) : \lambda_{A_\alpha} \text{ quasi } \delta_{A_\alpha}, \text{ where } \lambda_{A_\alpha}(x) \in I^X \text{ is degree of membership} \\ \text{function with } \delta_{A_\alpha}(x) \in \tau_{\mu_A}, \text{ and} \neq 1_\sim, 0_\sim, \text{ for some } \alpha \in J, J \text{ is an index set} \end{array} \right\}$$

$$\tau_{\gamma_A} = \tau(\gamma_{A_\alpha}(x)) = \left\{ \begin{array}{l} \nu_{A_\alpha}(= 0_\sim) : \nu_{A_\alpha} \text{ not quasi } \theta_{A_\alpha}, \text{ where } \nu_{A_\alpha}(x) \in I^X \text{ is degree of non membership} \\ \text{function with } \theta_{A_\alpha}(x) \in \tau_{\gamma_A} \text{ and} = 0_\sim, 1_\sim, \text{ for some } \alpha \in J, J \text{ is an index set} \end{array} \right\}$$

Where $0_\sim \leq \tau_{\mu A} + \tau_{\gamma A} \leq 1_\sim$.

Definition 3.2: A fine intuitionistic fuzzy topology (FIfT) on a nonempty set X is a family τ_f of FIfSs in X satisfying the following axioms

1. $0_{X\sim}, 1_{X\sim} \in \tau_f$.
2. If for any $\mu_1, \mu_2 \in \tau_f$ then $\mu_1 \cap \mu_2 \in \tau_f$.
3. $\bigcup \mu_i \in I\tau_f$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau_f$.

The triplet$(X, \tau, \tau_f)$is called a ***Fine intuitionistic fuzzy topological space*** FIfTS (for short) and any FIfS in τ_f is known as afineintuitionistic fuzzy open set (FIfOS) in X and its complement is denoted by fine intuitionistic fuzzy closed set(FIfCS) in X .

Example 3.1: Let X = {a, b, c} and $A=\left\langle x,\left(\frac{a}{0.4}\ \frac{b}{0.7}\ \frac{c}{0.8}\right),\left(\frac{a}{0.4}\ \frac{b}{0.3}\ \frac{c}{0.2}\right)\right\rangle$,
$B=\left\langle x,\left(\frac{a}{0.6}\ \frac{b}{0.8}\ \frac{c}{0.8}\right),\left(\frac{a}{0.4}\ \frac{b}{0.2}\ \frac{c}{0.2}\right)\right\rangle$, $C=\left\langle x,\left(\frac{a}{0.7}\ \frac{b}{0.9}\ \frac{c}{0.85}\right),\left(\frac{a}{0.3}\ \frac{b}{0.1}\ \frac{c}{0.1}\right)\right\rangle$
Then the family $\tau=\{0_\sim, 1_\sim, A, B, C\}$ is an intuitionistic fuzzy topological space in X.

Let us define a fineintuitionistic fuzzy set as follows

$$\tau_{\mu_A}=\tau(\mu_{A_\alpha}(x))=\left\{\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\mu_{A_\alpha}\le\left(\frac{a}{0.39},\frac{b}{0.69},\frac{c}{0.79}\right)\right\}$$

$$\tau_{\gamma_A}=\tau(\gamma_{A_\alpha}(x))=\left\{0_\sim,1_\sim,\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\gamma_{A_\alpha}\le\left(\frac{a}{0.39},\frac{b}{0.29},\frac{c}{0.19}\right)\right\},$$

$$\tau_{\mu_B}=\tau(\mu_{B_\alpha}(x))=\left\{\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\mu_{B_\alpha}\le\left(\frac{a}{0.59},\frac{b}{0.79},\frac{c}{0.69}\right)\right\}$$

$$\tau_{\gamma_B}=\tau(\gamma_{B_\alpha}(x))=\left\{0_\sim,1_\sim,\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\gamma_{B_\alpha}\le\left(\frac{a}{0.39},\frac{b}{0.19},\frac{c}{0.19}\right)\right\},$$

$$\tau_{\mu_C}=\tau(\mu_{C_\alpha}(x))=\left\{\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\mu_{C_\alpha}\le\left(\frac{a}{0.69},\frac{b}{0.89},\frac{c}{0.84}\right)\right\}$$

$$\tau_{\gamma_C}=\tau(\gamma_{C_\alpha}(x))=\left\{0_\sim,1_\sim,\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\gamma_{C_\alpha}\le\left(\frac{a}{0.29},\frac{b}{0.09},\frac{c}{0.09}\right)\right\}$$

Where $0_\sim\le\tau_{\mu_A}+\tau_{\gamma_A}\le 1_\sim$
Then

$$\tau_f=\left\{0_{X_\sim},1_{X_\sim},\vee\left\{\begin{array}{l}\left\langle\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\mu_{A_\alpha}\le\left(\frac{a}{0.39},\frac{b}{0.69},\frac{c}{0.79}\right),0_\sim,1_\sim,\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\gamma_{A_\alpha}\le\left(\frac{a}{0.39},\frac{b}{0.29},\frac{c}{0.19}\right)\right\rangle\\ \left\langle\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\mu_{B_\alpha}\le\left(\frac{a}{0.59},\frac{b}{0.79},\frac{c}{0.69}\right),0_\sim,1_\sim,\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\gamma_{B_\alpha}\le\left(\frac{a}{0.39},\frac{b}{0.19},\frac{c}{0.19}\right)\right\rangle,\\ \left\langle\left(\frac{a}{0.1},,\frac{b}{0.1},\frac{c}{0.1}\right)\le\mu_{C_\alpha}\le\left(\frac{a}{0.69},\frac{b}{0.89},\frac{c}{0.84}\right),0_\sim,1_\sim,\left(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.1}\right)\le\gamma_{C_\alpha}\le\left(\frac{a}{0.29},\frac{b}{0.09},\frac{c}{0.09}\right)\right\rangle\end{array}\right\}\right\}$$

is a fine intuitionistic fuzzy topology defined on X
Then (X,τ,τ_f) is called a fine intuitionistic fuzzytopological space.

Definition 3.3: Let (X,τ_1,τ_{f1}) and (X,τ_2,τ_{f2}) be two fine intuitionistic fuzzy topological space on X. Then τ_{f1} is contained in τ_{f2} if $\delta\in\tau_{f2}$ for each $\delta\in\tau_{f1}$. In this case we say that $\delta\in\tau_{f1}$ is coarser than τ_{f2}.

Definition 3.4: Let (X,τ_1,τ_{f1}) be a fine intuitionistic fuzzy topological space and $\delta=\langle x,\tau(\lambda_\delta),\tau(\gamma_\delta)\rangle$ be afine intuitionistic fuzzy set in X and ***fineintuitionistic fuzzy interior*** **and** ***fine intuitionistic fuzzy closure*** of δ is defined by
FIfcl(δ) = $\bigcap\{\lambda:\lambda\ is\ an\ FIfCS\ in\ X\ and\ \delta\subseteq\lambda\}$
FIfint(δ) = $\bigcup\{\mu:\mu\ is\ an\ FIfOS\ in\ X\ and\ \mu\subseteq\delta\}$.

Remark 3.1:

a. δ is an FIfCS in X iffFIfcl(δ).
b. δ is an FIfOS in X iffFIfint(δ).

Proposition 3.1: Let $\{\tau_{f_i} : i \in J\}$ be a family of fine intuitionistic fuzzy topological space on X. Then $\bigcap \tau_{f_i}$ is afineintuitionistic fuzzy topological space on X. Furthermore, $\bigcap \tau_{f_i}$ is the coarsest fine intuitionistic fuzzy topological space on X containing all τ_{f_i}

Proof: It is simple by definition 3. 3.

Proposition 3.2: Let (X, τ, τ_f) be an FIfTS, then for any FIfS θ in (X, τ, τ_f) in we have

a) FIfcl($\overline{\theta}$)= $\overline{FIfInt(\theta)}$).
b) FIfInt($\overline{\theta}$) = $\overline{FIfcl(\theta)}$

Proof: Let $\theta = \left\langle x, \tau(\mu_{\theta_\alpha}(x)), \tau(\gamma_{\theta_\alpha}(x)) : x \in X \right\rangle$. Then we get

FIfInt(θ) = $\left\langle x, \vee \tau(\mu_{\theta_{\alpha_i}}(x)), \wedge \tau(\gamma_{\theta_{\alpha_i}}(x)) \right\rangle$ and hence, $\overline{FIfInt(\theta)} = \left\langle x, \wedge \tau(\gamma_{\theta_\alpha}), \vee \tau(\mu_{\theta_\alpha}) \right\rangle$.

Now $\overline{\theta} = \left\langle x, \tau(\gamma_{\theta_\alpha}), \tau(\mu_{\theta_\alpha}) \right\rangle$ and $\tau_{\mu_{G_i}} \le \tau_{\mu_\theta}, \tau_{\gamma_{G_i}} \ge \tau_{\gamma_\theta}$, $FIfcl(\overline{\theta}) = \overline{FIfInt(\theta)}$. Similarly we prove b.

Proposition 3.3: Let (X, τ, τ_f) be an FIfTSA and B be FIfSs in X. Then the following properties are hold.

a) $FIfint(A) \subseteq A$ and $A \subseteq FIfcl(A)$.
b) $A \subseteq B \Rightarrow FIfInt(A) \subseteq FIfint(B)$ and $A \subseteq B \Rightarrow FIfcl(A) \subseteq FIfcl(B)$.
c) $FIfInt(FIfInt(A)) = FIfint(A)$ and $FIfcl(FIfcl(A)) = FIfcl(A)$.
d) $FIfInt(A \cap B) = FIf\ \mathrm{int}(A) \cap FIfInt(B)$ and $FIfcl(A \cup B) = FIfcl(A) \cup FIfcl(B)$.
e) $FIfInt(1_{X_\sim}) = 1_{X_\sim}$, $FIfInt(0_{X_\sim}) = 0_{X_\sim}$.

Proof:

a) $FIfint(A) = \bigcup \{B : B \text{ is an FIfOS in } X \ni B \subseteq A\}$
Since $Fint(A) \subseteq A \subseteq Fcl(A)$ and by definition
$FIfcl(A) = \bigcap \{C : C \text{ is an FIfCS in } X \ni A \subseteq C\}$.
Hence, $FIfint(A) \subseteq A$ and $A \subseteq FIfcl(A)$. $FIfInt(A \cap B) \subseteq FIfInt(B)$
Similarly for b) and c).

d) $FIfInt(A \cap B) \subseteq FIfint(A)$ and $FIfInt(A \cap B) \subseteq FIfint(B)$
we get $FIfInt(A \cap B) \subseteq FIfint(A) \cap FIfint(B)$ --------------(1)
$FIfint(A) \subseteq A$ and $FIfint(B) \subseteq B$
$FIfint(A) \cap FIfint(B) \subseteq A \cap B$ and $FIfint(A) \cap FIfint(B) \in \tau_f$,
Hence, $FIfint(A) \cap FIfint(B) \subseteq FIfint(A \cap B)$ -------------(2)
From (1) and (2) we get $FIfInt(A \cap B) = FIfint(A) \cap FIfint(B)$.
Similarly for $FIfcl(A \cup B) = FIfcl(A) \cup FIfcl(B)$.
Proof of e) is easy.

4. FINEINTUITIONISTIC FUZZY TOPOLOGICAL RING

Definition 4.1: Let R be a ring. A fineintuitionistic fuzzy set $A = \left\langle x, \tau_{\lambda A}, \tau_{\nu A} \right\rangle$ in R is called a***fine intuitionistic fuzzy topological ring***on R if it satisfies the following conditions

i. $\tau_{\mu_A}(x + y) \ge \tau_{\mu_A}(x) \wedge \tau_{\mu_A}(y)$.
ii. $\tau_{\mu_A}(xy) \ge \tau_{\mu_A}(x) \wedge \tau_{\mu_A}(y)$.

iii. $\tau_{\gamma_A}(x+y) \le \tau_{\gamma_A}(x) \vee \tau_{\gamma_A}(y)$.

iv. $\tau_{\gamma_A}(xy) \le \tau_{\gamma_A}(x) \vee \tau_{\gamma_A}(y) \ \ \forall\, x, y \in R$.

Definition 4.2: Let $(R, +, \bullet)$ be a ring. A family σ of a fineintuitionistic fuzzy topological ring on R if it satisfies the following axioms.

1. $0_{X\sim}, 1_{X\sim} \in \sigma_f$.
2. If for any $G_1, G_2 \in \tau_f$ then $\mu_1 \cap \mu_2 \in \sigma_f$.
3. $\bigcup \mu_i \in \sigma_f$ for any arbitrary family $\{G_i : i \in J\} \subseteq \sigma_f$.

The triplet (R, σ, σ_f) is called a ***fine intuitionistic fuzzy topological ring*** (FIfTR)or ***fine intuitionistic fuzzy ring structure space*** (FIfRSS)for short and any FIfTR in σ_f is known as a fine intuitionistic fuzzy open ring (FIfOR) in (R, σ, σ_f) and its complement is a fine intuitionistic fuzzy closed ring(FIfCR).

Example 4.1: Let $R = \{0, 1\}$ be a set of integers of modulo 2 with two binary operations as follows

+	0	1
0	0	1
1	1	0

•	0	1
0	0	0
1	0	1

Then $(R, +, \bullet)$ is a ring.
Define a fine intuitionistic fuzzy rings B and C as follows

$\tau_{\mu_B}(0) = 0.35, \tau_{\mu_B}(1) = 0.65 \; and \; \tau_{\gamma_B}(0) = 0.49, \tau_{\gamma_B}(1) = 0.29$

$\tau_{\mu_C}(0) = 0.40, \tau_{\mu_C}(1) = 0.89 \; and \; \tau_{\gamma_C}(0) = 0.39, \tau_{\gamma_C}(1) = 0.10$

Then $\sigma_f = \{0_{X_\sim}, 1_{X_\sim}, B, C\}$ is a fine intuitionistic fuzzy topological ring structure on R. Thus (R, σ, σ_f) is called a ***fine intuitionistic fuzzy ring structure space*** (FIfRSS).

Notation: Let (R, σ, σ_f) be anyfine intuitionistic fuzzy ring structure space. Then FIfO(R) denotes the family of all fine intuitionistic fuzzy open ring in (R, σ, σ_f). Then FIfC(R) denotes the family of all fine intuitionistic fuzzy closed ring in (R, σ, σ_f).

Definition 4.3: Let (R, σ, σ_f) be a fine intuitionistic fuzzy ringstructure space and $A = \left\langle x, \tau_{\mu_A}, \tau_{\gamma_A} \right\rangle$ be a fine intuitionistic fuzzy ring in R. A ***fine intuitionistic fuzzyring interior*** **and** ***fine intuitionistic fuzzy ring closure*** of A is defined by

$FIfRcl(A) = \bigcap \{B = \left\langle x, \tau_{\mu_B}, \tau_{\gamma_B} \right\rangle : B \; is \; a \; FIfC(R) \; in \; X \; and \; B \subseteq A\}$

$FIfRint(A) = \bigcup \{B = \left\langle x, \tau_{\mu_B}, \tau_{\gamma_B} \right\rangle : B \; is \; a \; FIfO(R) \; in \; X \; and \; A \subseteq B\}$.

Corollary 4.1: Let (R, σ, σ_f) be a fine intuitionistic fuzzy ring structure space. Then the following statements are hold for anyfine intuitionistic fuzzy ring structure in *R*.

i. *FIfRint*(A)=*A* iff A is afine intuitionistic fuzzy topological open ring.

ii. *FIfRcl*(A)=*A* iff A is a fine intuitionistic fuzzy topological closed ring.

iii. $FIfRint\,(A) \subseteq A \subseteq FIfRcl\,(A)$.

iv. $FIfRint(1_{X_\sim}) = 1_{X_\sim}$ and $FIfRint(0_{X_\sim}) = 0_{X_\sim}$.

v. $FIfRcl(1_{X_\sim}) = 1_{X_\sim}$ and $FIfRcl(0_{X_\sim}) = 0_{X_\sim}$.

vi. $FIfRcl(\overline{A}) = \overline{FIfRint(A)} \; FIfRint(\overline{A}) = \overline{FIfRcl(A)}$.

vii. $\bigcup_{i=1}^{\infty} FIfRcl(A_i) \subseteq FIfRcl(\bigcup_{i=1}^{\infty} A_i)$.

viii. $\bigcap_{i=1}^{n} FIfRcl(A_i) = FIfRcl(\bigcup_{i=1}^{n} A_i)$.

ix. $\bigcap_{i=1}^{\infty} FIfRcl(A_i) \subseteq FIfRcl(\bigcap_{i=1}^{\infty} A_i)$

x. $\bigcup_{i=1}^{\infty} FIfRint(A_i) \subseteq FIfRint(\bigcup_{i=1}^{\infty} A_i)$.

Proof: The Proof is simple

Definition 4.4: Let (R, σ, σ_f) be a fine intuitionistic fuzzy ring structure space and if $A = \left\langle x, \tau_{\mu_A}, \tau_{\gamma_A} \right\rangle$ is any fineintuitionistic fuzzyring in R. Then $FIfRint(\overline{A})$ is called a ***fine intuitionistic fuzzy ring exterior of*** **A** is denoted by FIfRExt(A).

Proposition 4.1: Let (R, σ, σ_f) be a fineintuitionistic fuzzy ring structure space. Then the following statements are hold for any two fine intuitionistic fuzzy rings A and B.

i. $FIfRExt(A) \subseteq \overline{A}$.

ii. $FIfRExt(A) = \overline{FIfRcl(A)}$.

iii. $FIfRExt(FIfRExt(A)) = FIfint(FIfRcl(A))$.

iv. If A $\subseteq$ B then $FIfRExt(A) \supseteq FIfRExt(B)$.

v. $FIfRExt(1_{X_\sim}) = 0_{X_\sim}$, $FIfRExt(0_{X_\sim}) = 1_{X_\sim}$.

vi. $FIfRExt(A \bigcup B) = FIfRExt(A) \bigcap FIfRExt(B)$.

Proof It is easy by using above definition.

Definition 4.5: Let (R, σ, σ_f) be any fine intuitionistic fuzzyring structure space. Let $A = \left\langle x, \tau_{\mu_A}, \tau_{\gamma_A} \right\rangle$ be a fine intuitionistic fuzzy ring in R. Then A is said to be a ***fine intuitionistic fuzzy G_δ ring*** in (R, σ, σ_f) if $A = \bigcap_{i=1}^{\infty} A_i$, where $A = \left\langle x, \tau_{\mu_A}, \tau_{\gamma_A} \right\rangle$ is a fine intuitionistic fuzzy open ring in (R, σ, σ_f). The complement of a fine intuitionistic fuzzy G_δ ring in (R, σ, σ_f) is a fine intuitionisticfuzzy F_σ ring in (R, σ, σ_f).

Definition 4.6: Let (R, σ, σ_f) be any fine intuitionistic fuzzy ring structure space. Let $A = \left\langle x, \tau_{\mu_A}, \tau_{\gamma_A} \right\rangle$ be a fine intuitionistic fuzzy ring in R. Then A is said to be a ***fine intuitionistic fuzzy dense ring*** in (R, σ, σ_f) if there exists no fine intuitionistic fuzzy closed ring B in (R, σ, σ_f) such that $A \subset B \subset 1_{X\sim}$ and ***fine intuitionistic fuzzy nowhere dense ring*** in (R, σ, σ_f) if there exists no fine intuitionistic fuzzy open ring B in (R, σ, σ_f) such that $B \subset FIfRcl(A)$ (i.e) $FIfRint(FIfRcl(A)) = 0_{X\sim}$.

Definition 4.7: Let (R, σ, σ_f) be any fine intuitionistic fuzzy structure ring space. Let $A = \left\langle x, \tau_{\mu_A}, \tau_{\gamma_A} \right\rangle$ be a fine intuitionistic fuzzy ring in R. Then A is said to be a ***fine intuitionistic fuzzy first category ring*** in (R, σ, σ_f) if $A = \bigcup_{i=1}^{\infty} A_i$ where A_i's are fine intuitionistic fuzzy nowhere dense rings in (R, σ, σ_f) and its complement is a ***fine intuitionistic fuzzy residual ring*** in (R, σ, σ_f).

Proposition 4.2: Let (R, σ, σ_f) be a fine intuitionistic fuzzyring structure space. If A is a fine intuitionistic fuzzy G_δ ring and the fine intuitionistic fuzzy ring exterior of $\overline{A}$ is a fine intuitionistic fuzzy dense ring in (R, σ, σ_f) then A is a fine intuitionistic fuzzy first category ring in (R, σ, σ_f).

Proof: Since A is fine intuitionistic fuzzy G_δ ring in (R,σ,σ_f), $A=\bigcap_{i=1}^{\infty} A_i$ where A_i's are fine intuitionistic fuzzy open rings. Since the fine intuitionistic fuzzy ring exterior of $\overline{A}$ is afine intuitionistic fuzzy dense ringin (R,σ,σ_f), $FIfRcl(FIfRExt(\overline{A}))=1_{X\sim}$.Since $FIfRExt(\overline{A})\subseteq A\subseteq FIfRcl(A)$, $FIfRExt(\overline{A})\subseteq FIfRcl(A)$ from this we have $FIfRcl(FIfRExt(\overline{A}))\subseteq FIfRcl(A), 1_{X\sim}\subseteq FIfRcl(A)$. Therefore, $FIfRcl(A)=1_{X\sim}$.
(i.e.) $FIfRcl(A)=FIfRcl(\bigcap_{i=1}^{\infty} A_i)=1_{X\sim}$ .But $FIfRcl(\bigcap_{i=1}^{\infty} A_i)\subseteq\bigcap_{i=1}^{\infty} FIfRcl(A_i)$.
Hence, $1_{X\sim}\subseteq\bigcap_{i=1}^{\infty} FIfRcl(A_i)\Rightarrow\bigcap_{i=1}^{\infty} FIfRcl(A_i)=1_{X\sim}$. This implies that $FIfRcl(A_i)=1_{X\sim}$
Foreach $A_i\in\tau_f$. Thus $FIfRcl(FIfRint(\overline{A}))=1_{X\sim}$.

Consider, $FIfRint(FIfRcl(\overline{A_i}))=FIfRint(\overline{FIfRint(A_i)})$

$$=\overline{FIfRcl(FIfRint(A_i))}=0_{X\sim}.$$

Therefore, $\overline{A_i}$ is the fine intuitionistic fuzzy nowhere dense ring in (R,σ,σ_f). Now,
$\overline{A}=\overline{\bigcap_{i=1}^{\infty} A_i}=\bigcup_{i=1}^{\infty}\overline{A_i}$. Hence, $\overline{A}=\bigcup_{i=1}^{\infty}\overline{A_i}$ where $\overline{A_i}$'s are the fineintuitionistic fuzzy nowhere dense rings in (R,σ,σ_f). Therefore, $\overline{A}$ is a fine intuitionistic fuzzy first category ring in (R,σ,σ_f).

Proposition 4.3: If A is a fine intuitionistic fuzzy first category ring in a fine intuitionistic fuzzy ring structure space (R,σ,σ_f) such that $B\subseteq\overline{A}$ where B is non-zero fine intuitionistic fuzzy G_δ ring and the fine intuitionistic fuzzy ring exterior of $\overline{B}$ is a fine intuitionistic fuzzy dense ring in (R,σ,σ_f) then A is a fine intuitionistic fuzzy nowhere dense ring in (R,σ,σ_f).

Proof: Let A be a fine intuitionistic fuzzy first category ring in (R,σ,σ_f). Then, $A=\bigcup_{i=1}^{\infty} A_i$ where A_i's are the fine intuitionistic fuzzy nowhere dense rings in (R,σ,σ_f). Now, $\overline{FIfRcl(A_i)}$ is a fine intuitionistic fuzzy open ring in (R,σ,σ_f). Let $B=\bigcap_{i=1}^{\infty}\overline{FIfRcl(A_i)}$.
Then, B is non-zero fine intuitionistic fuzzy G_δ ring in (R,σ,σ_f).

Consider $B=\bigcap_{i=1}^{\infty}\overline{FIfRcl(A_i)}=\overline{\bigcup_{i=1}^{\infty} FIfRcl(A_i)}\subseteq\overline{\bigcup_{i=1}^{\infty} A_i}=\overline{A}$
Hence, $B\subseteq\overline{A}$. Then, $A\subseteq\overline{B}$.

Let $FIfRint(FIfRcl(A)\subseteq FIfRint(FIfRcl(\overline{(B)}))$

$$=FIfRint(\overline{FIfRint(B)})$$
$$=\overline{FIfRcl(FIfRint(B))}$$
$$=\overline{FIfRcl(FIfRExt(\overline{B}))}$$

We know that $FIfRExt(\overline{B})$ is a fine intuitionistic fuzzy dense ring in (R,σ,σ_f), $FIfRcl(FIfRExt(\overline{B}))=1_{X\sim}$. Therefore $FIfint(FIfRcl(A))\subseteq 0_{X\sim}$. Then $FIfint(FIfRcl(A))=0_{X\sim}$. Hence, A is a fine intuitionistic fuzzy nowhere dense ring in (R,σ,σ_f).

Definition 4.8: Let (R,σ,σ_f) be any fine intuitionistic fuzzy ring structure space. Let A be a fine intuitionistic fuzzy ring in (R,σ,σ_f). Then A is said to be a ***fine intuitionistic fuzzy regular closed ring*** in (R,σ,σ_f)

if $FIfRcl(FIfint(A)) = A$. The complement of a fine intuitionistic fuzzy regular closed ring in (R, σ, σ_f) is a fine intuitionistic fuzzy regular open ring in (R, σ, σ_f).

Remark 4.1: Every fine intuitionistic fuzzy regular closed ring is a fine intuitionistic fuzzy closed ring.

Definition 4.9: Let (R, σ, σ_f) be any fine intuitionistic fuzzy ring structure space. Then (R, σ, σ_f) is called a ***fine intuitionistic fuzzy ring*** $G_\delta T_{1/2}$ ***space*** if every nonzero fine intuitionistic fuzzy G_δ ring in (R, σ, σ_f) is a fine intuitionistic fuzzy open ring in (R, σ, σ_f).

Proposition 4.4: If the fine intuitionistic fuzzy ring structure space (R, σ, σ_f) is a fine intuitionistic fuzzy ring $G_\delta T_{1/2}$ space and if A is a fine intuitionistic fuzzy first category ring in (R, σ, σ_f), then A is not a fine intuitionistic fuzzy dense ring in (R, σ, σ_f).

Proof: Suppose that A is a fine intuitionistic fuzzy first categoryring in (R, σ, σ_f) such that A is a fine intuitionistic fuzzy dense ring in (R, σ, σ_f),that is, $FIfRcl(A) = 1_{X\sim}$.Then, $FIfRcl(FIfRExt(\overline{B})) = 1_{X\sim}$, $A = \bigcup_{i=1}^{\infty} A_i$ where A_i's are fineintuitionistic fuzzy nowhere dense rings in (R, σ, σ_f).

Now, $\overline{FIfRcl(A_i)}$ is a fine intuitionistic fuzzy open ring in (R, σ, σ_f).Let $B = \bigcap_{1=1}^{\infty} \overline{FIfRcl(A_i)}$.Then, B is non-zero, fine intuitionistic fuzzy G_δ ring in (R, σ, σ_f).

Consider $B = \bigcap_{i=1}^{\infty} \overline{FIfRcl(A_i)} = B = \overline{\bigcup_{i=1}^{\infty} FIfRcl(A_i)} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i} = \overline{A}$. Hence $B \subseteq \overline{A}$. Then $FIfR\text{int}(B) \subseteq FIfR\text{int}(\overline{A}) \subseteq \overline{FIfRcl(A)} = 0_{X\sim}$. That is, $FIfR\text{int}(B) = 0_{X\sim}$.Since (R, σ, σ_f) is a fine intuitionistic fuzzy ring $G_\delta T_{1/2}$ space, $FIfR\text{int}(B) = B$, which implies that $B = 0_{X\sim}$. This is a contradiction. Therefore, A is not a fine intuitionistic fuzzy dense ring in (R, σ, σ_f).

5. FINEINTUITIONISTIC FUZZY NORMALSUBRING

Definition 5.1: A fine intuitionistic fuzzy subset δ of a ring $(R, +, .)$ is said to be a **fineintuitionistic fuzzy subring** ofa ring $(R, +, .)$ is denoted by (FIfSR) if it satisfies the following conditions

a. $\tau_{\mu_\delta}(x - y) \geq Min\{\tau_{\mu_\delta}(x), \tau_{\mu_\delta}(y)\}$
b. $\tau_{\mu_\delta}(xy) \geq Min\{\tau_{\mu_\delta}(x), \tau_{\mu_\delta}(y)\}$
c. $\tau_{\gamma_\delta}(x - y) \leq Max\{\tau_{\gamma_\delta}(x), \tau_{\gamma_\delta}(y)\}$
d. $\tau_{\gamma_\delta}(xy) \leq Max\{\tau_{\gamma_\delta}(x), \tau_{\gamma_\delta}(y)\} \ \forall\, x, y \in R.$

Definition 5.2: Let $(R, +, .)$ be a ring. A fine intuitionistic fuzzy subset δ of R is said to be **a fine intuitionistic fuzzy normal subring** of R is denoted by FIfNSR if it satisfies

i. $\tau_{\mu_\delta}(xy) = \tau_{\mu_\delta}(yx)$
ii. $\tau_{\gamma_\delta}(xy) = \tau_{\gamma_\delta}(yx) \ \forall\, x, y \in R.$

Definition 5.3: Let R_1 and R_2 be two rings with identity. Let δ and θ be the two fine intuitionistic fuzzy subsets of the fine intuitionistic fuzzy rings and $\delta \times \theta$ is a fine intuitionistic fuzzy subring of $R_1 \times R_2$ if the following condition holds

Let δ and θ be two fine intuitionistic fuzzy subsets of the fine intuitionistic fuzzy rings with an identity R_1 and R_2 and $\delta\times\theta$ is a fine intuitionistic fuzzy subring of $R_1\times R_2$.thefollowingholds

i. If $\tau_{\mu_\delta}(x)\le\tau_{\mu_\theta}(e^1)$ and $\tau_{\gamma_\delta}(x)\ge\tau_{\gamma_\theta}(e^1)$ then δ is a fine intuitionistic fuzzy subring of R_1.

ii. If $\tau_{\mu_\theta}(x)\le\tau_{\mu_\delta}(e)$ and $\tau_{\gamma_\theta}(x)\ge\tau_{\gamma_\delta}(e)$ then θ is a fine intuitionistic fuzzy subring of R_2.

iii. Either δ is a fine intuitionistic fuzzy subring of R_1(or) fine intuitionistic fuzzy subring of R_2.

Definition 5.4: Let A and B be two fine intuitionistic fuzzy subring of rings of rings R_1 and R_2respectively. The product of A and B denoted by $A\times B$ is defined as
$A\times B=\{\langle(x,y),\tau_{\mu_{A\times B}}(x,y),\tau_{\gamma_{A\times B}}(x,y)\rangle:\text{for all } x\in R_1 \text{ and } y\in R_2\}$ where
$\tau_{\mu_{A\times B}}(x,y)=\text{Min}\{\tau_{\mu_A}(x),\tau_{\mu_B}(y)\}$ and $\tau_{\gamma_{A\times B}}(x,y)=\text{Max}\{\tau_{\gamma_A}(x),\tau_{\gamma_B}(y)\}$.

Theorem 5.1: Let $(R,+,.)$ be a ring. If Aand B are two fine intuitionistic fuzzy normal subrings of R. Then their intersection $A\bigcap B$ is a fine intuitionistic fuzzy normal subring of R.
The intersection $A\bigcap B$of any two fine intuitionistic fuzzy normal subrings of a ring$(R,+,.)$ is afine intuitionistic fuzzy normal subring of R.

Proof: Let x, y$\in$R. Let $A=\{\langle x,\tau_{\mu_A}(x),\tau_{\gamma_A}(x)\rangle: x\in R\}andB=\{\langle x,\tau_{\mu_B}(x),\tau_{\gamma_B}(x)\rangle: x\in R\}$be a fine intuitionistic fuzzy subring of a ring R. Let C = $A\bigcap B$ and $C=\{\langle x,\tau_{\mu_C}(x),\tau_{\gamma_C}(x)\rangle: x\in R\}$ where min $\{\tau_{\mu_A}(x),\tau_{\mu_B}(x)\}=\tau_{\mu_C}(x)$ and max$\{\tau_{\gamma_A}(x),\tau_{\gamma_B}(x)\}=\tau_{\gamma_C}(x)$. Hence, C is a fine intuitionistic fuzzy subring of a ring R.
Since A and B are two intuitionistic fuzzy subrings of a ring R.
$\tau_{\mu_C}(xy)=\min\{\tau_{\mu_A}(x\,y),\tau_{\mu_B}(x\,y)\}=\min\{\tau_{\mu_A}(yx),\tau_{\mu_B}(yx)\}=\tau_{\mu_C}(yx)$. $\tau_{\mu_C}(xy)=\tau_{\mu_C}(yx)\ \forall\ x,y\in R$, also $\tau_{\gamma_C}(xy)=\max\{\tau_{\gamma_A}(xy),\tau_{\gamma_B}(xy)\}=\max\{\tau_{\gamma_A}(yx),\tau_{\gamma_B}(yx)\}=\tau_{\gamma_C}(yx)$. $\tau_{\gamma_C}(xy)=\tau_{\gamma_C}(yx)\ \forall\ x,y\in R$.
Hence, intersection of any two fine intuitionistic fuzzy normal subring is a fine intuitionistic fuzzy normal subring of a ring R.

Theorem 5.2: Let A and B be fine intuitionistic fuzzy subsets of the rings with an identity R_1 and R_2 and $A\times B$ is a fine intuitionistic fuzzy normal subring of $R_1\times R_2$the following holds

i. If $\tau_{\mu_A}(x)\le\tau_{\mu_B}(e')$ and $\tau_{\gamma_A}(x)\ge\tau_{\gamma_B}(e')$ then A is a fine intuitionistic fuzzy normal subring of R_1.

ii. If $\tau_{\mu_B}(x)\le\tau_{\mu_A}(e)$ and $\tau_{\gamma_B}(x)\ge\tau_{\gamma_A}(e)$ then B is a fine intuitionistic fuzzy normal subring of R_2.

iii. either A is afine intuitionistic fuzzy normal subring of R_1 or B is afine intuitionistic fuzzy normal subring of R_2.

Proof: Let $A\times B$ be a fine intuitionistic fuzzy normal subring of $R_1\times R_2$ andx, y in R_1 and$e'\in R_2$. Then (x,e') and (y,e') are in $R_1\times R_2$and Clearly, $A\times B$ is a fine intuitionistic fuzzy subring of R_1. $\tau_{\mu_A}(xy)=\min\{\tau_{\mu_A}(xy),\tau_{\mu_B}(e'e')\}=\tau_{\mu_{A\times B}}((xy),(e'e'))=\tau_{\mu_{A\times B}}((x,e'),(y,e'))=\tau_{\mu_{A\times B}}((y,e'),(x,e'))=\min\{\tau_{\mu_A}(yx),\tau_{\mu_B}(e'e')\}=\tau_{\mu_A}(yx)$. Thus, $\tau_{\mu_A}(xy)=\tau_{\mu_A}(yx)\ \forall\ x,y\in R$.
$\tau_{\gamma_A}(xy)=\max\{\tau_{\gamma_A}(xy),\tau_{\gamma_B}(e'e')\}=\tau_{\gamma_{A\times B}}((xy),(e'e'))=\tau_{\gamma_{A\times B}}((x,e'),(y,e'))=\tau_{\gamma_{A\times B}}((y,e'),(x,e'))=\tau_{\gamma_{A\times B}}((yx),(e'e'))=\max\{\tau_{\gamma_A}(yx),\tau_{\gamma_B}(e'e')\}=\tau_{\gamma_A}(yx)$.

Thus, $\tau_{\gamma_A}(xy)=\tau_{\gamma_A}(yx)\ \forall\ x,y\in R$. Hence, A is a fine intuitionistic fuzzy subring of R_1. $\tau_{\mu_B}(x)\le\tau_{\mu_A}(e)$, $\tau_{\gamma_B}(x)\ge\tau_{\gamma_A}(e)$, $\forall\ x\in R_2$ and let $x,y\in R_2$ and e $\in R_1$.Then$(e,x)$and(e,y) are in $R_1\times R_2$. Thus B is a fine intuitionistic fuzzy subring of R_2. $\tau_{\mu_B}(xy)=\min\{\tau_{\mu_B}(xy),\tau_{\mu_A}(ee)\}=\min\{\tau_{\mu_A}(ee),\tau_{\mu_B}(xy)\}=\tau_{\mu_{A\times B}}((ee),(xy))=\tau_{\mu_{A\times B}}((e,x),(e,y))=\tau_{\mu_{A\times B}}((e,y),(e,x))=\tau_{\mu_{A\times B}}((ee),(yx))=\min\{\tau_{\mu_A}((ee),\tau_{\mu_B}(yx))\}=\tau_{\mu_B}(yx)$.
$\tau_{\mu_B}(xy)=\tau_{\mu_B}(yx)\ \forall\ x,y\in R$.

$\tau_{\gamma_B}(xy) = \max\{\tau_{\gamma_B}(xy), \tau_{\gamma_A}(e'e')\} = \max\{\tau_{\gamma_A}(e'e'), \tau_{\gamma_B}(xy)\} = \tau_{\gamma_{A\times B}}((ee),(xy)) = \tau_{\gamma_{A\times B}}((e,x)(e,y)) = \tau_{\gamma_{A\times B}}((ee),(yx)) = \max\{\tau_{\gamma_A}(ee), \tau_{\gamma_B}(yx)\} = \tau_{\gamma_B}(yx)$

$\tau_{\gamma_B}(xy) = \tau_{\gamma_B}(yx)\ \forall\ x, y \in R$. Hence, B is a fine intuitionistic fuzzy normal subring of R_2.

Theorem 5.3: If A is a fine intuitionistic fuzzy normal subring of R, then □ A is a fine intuitionistic fuzzy normal subring of a ring R.

Proof: Let □A = B = $\{\langle x, \tau_{\mu_B}(x), \tau_{\gamma_B}(x)\rangle\}$. Hence, □A is a fine intuitionistic fuzzy subring of a ring R. Since A is a fine intuitionistic fuzzy subring of a ring R. Let x, y$\in R$. Then $\tau_{\mu_B}(x+y) = \tau_{\mu_B}(y+x)$ and $\tau_{\mu_B}(xy) = \tau_{\mu_B}(yx)$, $\tau_{\mu_A}(x+y) = \tau_{\mu_A}(y+x) . 1_{X\sim} - \tau_{\gamma_B}(x+y) = 1_{X\sim} - \tau_{\gamma_B}(y+x)$

(i.e) $\tau_{\gamma_B}(x+y) = \tau_{\gamma_B}(y+x)$ and $\tau_{\mu_A}(xy) = \tau_{\mu_A}(yx)$. This implies $1_{X\sim} - \tau_{\gamma_B}(xy) = 1_{X\sim} - \tau_{\gamma_B}(yx)$.

(i.e) $\tau_{\gamma_B}(xy) = \tau_{\gamma_B}(yx)$. Hence, B = □A is a fine intuitionistic fuzzy normal subring of a ring R.

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Source of support: National Conference on "New Trends in Mathematical Modelling" (NTMM - 2018), Organized by Sri Sarada College for Women, Salem, Tamil Nadu, India.