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OSCILLATIONS OF THIRD ORDER LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper we study the sufficient conditions for oscillations of third order linear neutral delay differential equations of the form

$$\frac{d}{dt}\left\{r_1(t)\frac{d^2}{dt^2}\left(m(t)y(t)+\frac{r(t)}{r(t-\tau)}y^{\alpha}(t-\tau)\right)\right\}+f(t)y(t-\sigma)=0\ ; \qquad t\geq t_0$$

where $r_1(t), r(t), m(t)$ *are positive real valued continuous functions and* $f(t)\geq 0$ *and* α *is the ratio of odd positive integers such that* $0<\alpha\leq 1$.

Key Words: *Oscillations, Third order, Neutral Differential Equation.*

1. INTRODUCTION

This paper is concerned with the oscillatory behavior of third order linear neutral delay differential equation

$$\frac{d}{dt}\left\{r_1(t)\frac{d^2}{dt^2}\left\{m(t)y(t)+\frac{r(t)}{r(t-\tau)}y^{\alpha}(t-\tau)\right\}\right\}+f(t)y(t-\sigma)=0 \tag{1}$$

where $r_1(t)\in C([t_0,\infty),(0,\infty))$, $r(t)$, $f(t)\in C([t_0,\infty),[0,\infty))$.

Recently oscillation of solutions of neutral delay differential equations attracted many researchers [1, 2, 10 and 12]. Hanan[16], Erbe;[10] studied the special case of the equation (1) given by

$$\frac{d^3y}{dt^3}+f(t)y(t))=0 \tag{2}$$

.

We can find theoscillatory behavior of solutions of third order delay equations in [3,4,6,7,8,9,11,20] and [14-19, 21-27].

We present sufficient conditions for the solutions to be oscillatory or asymptotically tend to zero.

By a solution of equation (1) we mean a function $y(t)\in C([T_y,\infty))$ where $T_y\geq t_0$ which satisfies(1) on $[T_y,\infty)$. We consider only those solutions of *y(t)* of (1) which satisfy $Sup\left\{|y(t)|: t\geq T\right\}>0$ for all $T\geq T_y$ and assume that (1) possesses such solutions.

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A solution of equation (1) is called oscillatory if it has arbitrary large zeros on $[T_y,\infty)$; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory it all its solutions oscillate .Unless otherwise stated, when we write a functional inequality, it will be assumed to hold for sufficiently large t in our subsequent discussion.

II. MAIN RESULTS

We need the following in our discussion:

(H_1): $r_1(t), m(t), \in C^{'}([t_0,\infty),(0,\infty))$; $r_1(t), m(t) > 0$

(H_2): $r(t), f(t) \in C(t_0,\infty),[0,\infty))$, $f(t) > 0.$

(H_3): $0 < \alpha \le 1$, and α is the ratio of odd positive integers.

Set $$z(t) = m(t)y(t) + \frac{r(t)}{r(t-\tau)} y^{\alpha}(t-\tau) \tag{3}$$

and $$R(t) = \int_{t_0}^{t} \frac{1}{r_1(s)} ds = \infty. \tag{4}$$

We have the following Lemma

Lemma 2.1: If $a > 0, b > 0$ and $0 < \alpha \le 1$, then

$$a^{\alpha} b^{1-\alpha} \le \alpha a + (1-\alpha) b. \tag{5}$$

Proof: The proof of this Lemma be found in (13)

Lemma 2.2 (See [5 Lemma 3]): Assume that $u(t) > 0, u^{'}(t) > 0, and\ u^{''}(t) \le 0\ for\ t \ge t_0$. Then for every $\alpha \in (0,1),\ there\ exists\ a\ \alpha T_{\alpha} \ge T_0\ such\ that\ \dfrac{u(t-\sigma)}{t-\sigma} \ge \dfrac{\alpha u(t)}{t}\ for\ t \ge T_0.$

Lemma 2.3: If u satisfies conditions of Lemma **2.2,** then $\dfrac{u(t-\sigma)}{u(t)} \ge \dfrac{\alpha(t-\sigma)}{t}$ for $i = 1,2,.....m$.

Lemma 2.4: (See [5, Lemma 4]) Assume that $u(t) > 0, u^{'}(t) > 0, and\ u^{''}(t) \le 0\ for\ t \ge t_o$. Then for each $\beta \in (0,1)$, there exists a $T\beta \ge t_0\ such\ that\ u(t) \ge \dfrac{\beta t u^{'}(t)}{2}\ for\ t \ge t_{\beta}$.

Lemma 2.5: If u satisfies conditions of Lemma **2.4** then $\dfrac{u(t-\sigma)}{u^{'}(t-\sigma)} \ge \dfrac{\beta(t-\sigma)}{2}$ for $i = 1,2,.......m$

We use the Integral averaging technique to establish a Philos-Type criteria for (1)

Let

$$D = \{(t,s) : t \ge s \ge t_0\} \text{ and } D_0 = \{(t,s) : t > s \ge t_0\}.$$

(i) $H(t,t) = 0,\quad t \ge t_0;\quad H(t,s) > 0, (t,s) \in D_0;$

(ii) H has a non positive continuous partial derivative $\partial H/\partial S$ on D_0 w.r.t the second variable, and there exists a function $\rho \in C^{'}([t,\infty),(0,\infty))$, $\delta \in C^{'}([t_0,\infty),R)$ and $h \in C(D_0,R)$ such that

$$\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)} H(t,s) = -h(t,s)\sqrt{H(t,s).} \tag{6}$$

Now we state our main Theorem

Theorem 2.4: Assuming that conditions $(H_1)-(H_3)$ are satisfied. If

$$Lim_{t\to\infty}\ Sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)G(s)-\frac{1}{4}\rho(s)r(s)h^2(t,s)\right]ds=\infty \tag{7}$$

for some $\alpha\in(0,1)$, $\beta\in(0,1)$ and for some $H\in X$, where

$$G(s)=-\rho(s)\left(f(t)\frac{1}{m(s-\sigma)}\left\{1\text{-}\alpha\frac{r(s-\sigma)}{r(s-\sigma-\tau)}-\frac{r(s-\sigma)}{r(s-\sigma-\tau)}(1-\alpha)\frac{1}{M}\right\}\frac{\alpha\beta(s-\sigma)^2}{2\text{s}}\right)$$

Then every solution of y of (1) is either oscillatory or satisfies $\underset{t\to\infty}{Lim}\,y(t)=0.$

Set

$$z(t)=m(t)y(t)+\frac{r(t)}{r(t-\tau)}y^{\alpha}(t-\tau)$$

$$m(t)y(t)=z(t)-\frac{r(t)}{r(t-\tau)}y^{\alpha}(t-\tau)$$

$$y(t)=\frac{1}{m(t)}\left\{z(t)-\frac{r(t)}{r(t-\tau)}y^{\alpha}(t-\tau)\right\}$$

$$y(t)=\frac{1}{m(t)}\left\{z(t)-\frac{r(t)}{r(t-\tau)}z^{\alpha}(t-\tau)\right\}$$

$$y(t)\ge\frac{1}{m(t)}\left\{z(t)-\frac{r(t)}{r(t-\tau)}z^{\alpha}(t)\right\} \tag{8}$$

Using the above Lemma with $b=1$

$$y(t)\ge\frac{1}{m(t)}\left\{z(t)-\frac{r(t)}{r(t-\tau)}\big(\alpha\,z(t)+(1-\alpha)\big)\right\}$$

$$y(t)\ge\frac{1}{m(t)}\left\{1-\alpha\frac{r(t)}{r(t-\tau)}-\frac{r(t)}{r(t-\tau)}(1-\alpha)\frac{1}{M}\right\}z(t) \tag{9}$$

we have used $z(t)\ge M>0$ for all $t\ge t_1$.

Using (1) and (9)

$$\frac{d}{dt}\left\{r_1(t)\frac{d^2}{dt^2}\left(m(t)y(t)+\frac{r(t)}{r(t-\tau)}\right)\right\}\le$$

$$-f(t).\frac{1}{m(t-\sigma)}\left\{1-\alpha\frac{r(t-\sigma)}{r(t-\sigma-\tau)}-\frac{r(t-\sigma)}{r(t-\sigma-\tau)}(1-\alpha)\frac{1}{M}\right\}z(t-\sigma) \tag{10}$$

Define

$$\omega(t)=\rho(t).\frac{r_1(t)\,z''(t)}{z'(t)}\qquad\text{for}\quad t\ge t_1 \tag{11}$$

Then $\omega(t)>0$

$$\omega'(t)=\rho'(t)\frac{r_1(t)z''(t)}{z'(t)}+\rho(t)\left\{\frac{r_1(t)z''(t)}{z'(t)}\right\}'$$

$$=\rho'(t)\frac{r_1(t)z''(t)}{z'(t)}+\rho(t)\left[\frac{z'(t)\{r_1(t)z''(t)\}'-r_1(t)z''(t)\{z''(t)\}}{\{z^{'}(t)\}^2}\right]$$

$$= \rho'(t)\frac{r_1(t)z''(t)}{z'(t)} + \rho(t)\frac{\{r_1(t)z''(t)\}'}{z'(t)} - \left[\rho(t)\frac{r_1(t)z''(t)\{z''(t\}}{\{z'(t)\}^2}\right]$$

$$= \rho'(t)\frac{r_1(t)z''(t)}{z'(t)} + \rho(t)\frac{\{r_1(t)z''(t)\}'}{z'(t)} - \left[\rho(t)\frac{r_1(t)\{z''(t\}^2}{\{z'(t)\}^2}\right]$$

$$= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\{r_1(t)z''(t)\}'}{z'(t)} - \left[\rho(t)\frac{r_1(t)\{z''(t\}^2}{\{z'(t)\}^2}\right] \tag{12}$$

By (11) we have

$$\frac{\omega(t)}{\rho(t)} = \frac{r_1(t)z''(t)}{z'(t)};$$

$$\frac{z''(t)}{z'(t)} = \frac{\omega(t)}{r_1(t)\rho(t)}$$

Squaring both sides $\left\{\frac{z''(t)}{z'(t)}\right\}^2 = \frac{\omega^2(t)}{r^2{}_1(t)\rho^2(t)}$ (13)

Substituting (10) and (13) into (12) we obtain

$$\omega'(t) \le$$

$$\frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\left(-f(t)\frac{1}{m(t-\sigma)}\left\{1-\alpha\frac{r(t-\sigma)}{r(t-\sigma-\tau)} - \frac{r(t-\sigma)}{r(t-\sigma-\tau)}(1-\alpha)\frac{1}{M}\right\}\frac{z(t-\sigma)}{z^{'}(t)}\right)$$

$$-\left[\frac{\omega^2(t)}{\rho(t)r_1(t)}\right]$$

$$\omega'(t) \le$$

$$-\rho(t)\left(f(t)\frac{1}{m(t-\sigma)}\left\{1-\alpha\frac{r(t-\sigma)}{r(t-\sigma-\tau)} - \frac{r(t-\sigma)}{r(t-\sigma-\tau)}(1-\alpha)\frac{1}{M}\right\}\frac{z(t-\sigma)}{z^{'}(t)}\right)$$

$$+\frac{\rho'(t)}{\rho(t)}\omega(t) - \left[\frac{\omega^2(t)}{\rho(t)r_1(t)}\right] \tag{14}$$

It follows from Lemma 2.3 and Lemma 2.5 that for any $\alpha \in (0,1)$, and $\beta \in (0,1)$

$$\frac{z(t-\sigma)}{z'(t)} = \frac{z(t-\sigma)}{z'(t-\sigma)}.\frac{z^{'}(t-\sigma)}{z^{'}(t)} \ge \frac{\alpha\beta}{2}\frac{(t-\sigma)^2}{t} \tag{15}$$

Again substituting (15) into (14) we get

$$\omega'(t) \le$$

$$-\rho(t)\left(f(t)\frac{1}{m(t-\sigma)}\left\{1-\alpha\frac{r(t-\sigma)}{r(t-\sigma-\tau)} - \frac{r(t-\sigma)}{r(t-\sigma-\tau)}(1-\alpha)\frac{1}{M}\right\}\frac{\alpha\beta(t-\sigma)^2}{2t}\right)$$

$$+\frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{1}{\rho(t)r_1(t)}\omega^2(t)$$

$$\omega^{'}(t) \le -G(t) + A(t)\omega(t) - B(t)\omega^2(t)$$

where $A(t) = \frac{\rho^{'}(t)}{\rho(t)}$; $B(t) = \frac{1}{\rho(t)r_1(t)}$ and

$$G(t) = -\rho(t)\left(f(t)\frac{1}{m(t-\sigma)}\left\{1\text{-}\alpha\frac{r(t-\sigma)}{r(t-\sigma-\tau)} - \frac{r(t-\sigma)}{r(t-\sigma-\tau)}(1-\alpha)\frac{1}{M}\right\}\frac{\alpha\beta(t-\sigma)^2}{2\mathrm{t}}\right)$$

Multiplying both sides by *H(t,s)* and integrating w.r.t s from $T_1\,(T_1 \geq T)$ to *t* we derive from *H(t, t)=0* and (1) that

$$\int_{t_1}^{t} H(t,s)G(s)\,ds \leq \int_{t_1}^{t} H(t,s)\left[-\omega'(s)+A(s)\omega(s)-B(s)\omega^2(s)\right]ds$$

$$= H(t,T_1)\omega(T_1)+\int_{t_1}^{t}\left[\left(\frac{\partial H(t,s)}{\partial s}+A(s)H(t,s)\right)\omega(s)-H(t,s)B(s)\omega^2(s)\right]ds$$

$$= H(t,T_1)\omega(T_1)-\int_{t_1}^{t}\left[h(t,s)\sqrt{H(t,s)}\omega(s)+H(t,s)B(s)\omega^2(s)\right]ds$$

$$= H(t,T_1)\omega(T_1)-\int_{t_1}^{t}\left(\sqrt{H(t,s)B(s)}\omega(s)+\frac{h(t,s)}{2\sqrt{B(s)}}\right)^2 ds+\int_{T_1}^{t}\frac{h^2(t,s)}{4B(s)}ds$$

Thus

$$\int_{T_1}^{t} H(t,s)G(s)\,ds \leq H(t,T_1)\omega(T_1)+\int_{t_1}^{t}\frac{h^2(t,s)}{4B(s)}ds$$

and hence

$$\lim_{t\to\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t}\left[H(t,s)G(s)-\frac{1}{4}\rho(s)r(s)h^2(t,s)\right]ds \leq \omega(T_1)$$ which contradicts (7).

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