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OSCILLATION CRITERIA FOR SECOND ORDER DIFFERENCE EQUATIONS WITH NONLINEAR NEUTRAL TERM

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ABSTRACT

In this paper, the authors obtain some new sufficient conditions for the oscillation of all solutions of certain class of second order difference equations with nonlinear neutral term. Examples are included to illustrate the main results.

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INTRODUCTION

Consider the nonlinear neutral difference equation of the form

$$\Delta\left(a_n\Delta\left(x_n + p_n x_{n-k}^{\alpha}\right)\right) + q_n x_{n+1-l}^{\beta} = 0, n \geq n_0 \tag{1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H1) $0 < \alpha \leq 1$ and β are ratios of odd positive integers;
(H2) $\{a_n\}$ is a positive real sequence for all $n \geq n_0$;
(H3) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences for all $n \geq n_0$;
(H4) k is a positive integer and l is a nonnegative integer.

Let $\theta = max\,\{k, l\}$. By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, that satisfies equation (1) for all $n \geq n_0$. A solution of equation (1) is called oscillatory if its terms are neither eventually positive nor eventually negative, and nonoscillatory otherwise. If all solutions of the equation are oscillatory then the equation itself called oscillatory.

In the past few years, there has been a great interest in studying the oscillatory and asymptotic behavior of neutral type difference equations, see [1-3, 9] and the references cited therein.

In [5], Thandapani et.al investigated the oscillation of all solutions of the equation

$$\Delta\left(a_n\Delta\left(x_n - p_n x_{n-k}^{\alpha}\right)\right) + q_n x_{n+1-l}^{\beta} = 0, n \geq n_0 \tag{2}$$

where $p > 0$ is a real number, k and l are positive integers, $0 < \alpha \leq 1$ and β are ratios of odd positive integers, and $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$.

A special case of the equation studied by Yildiz and Ogunmez [10] has the form

$$\Delta^2\left(x_n + p_n x_{n-k}^{\beta}\right) + q_n x_{n-l}^{\beta} = 0, n \geq n_0 \tag{3}$$

where $\{p_n\}$ is a real sequence, $\{q_n\}$ is a nonnegative real sequence, and $\alpha > 1$ and β are again ratios of odd positive integers. They too discussed the oscillatory behavior of solutions of equation [3].

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In [6] and [8], Thandapani *et.al* considered equation (1) and obtained criteria for the oscillation of solutions provided either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty. \tag{4}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty. \tag{5}$$

Using some inequalities and Riccati type transformation.

Motivated by this observation, in this paper we obtain sufficient conditions for the oscillation of all solutions of equation (1) in the two cases (4) and (5). Our method of proof is different from that of in [6, 8, 10, 11], and hence our results are new and complement to those reported in [5, 6, 8, 10, 11]. Example are presented to illustrate the main results.

OSCILLATION RESULTS

In this section, we obtain sufficient conditions for the oscillation of all solutions of the equation (1). We set

$$z_n = x_n + p_n x_{n-k}^{\alpha}.$$

Due to the form of our equation, we only need to give proofs for the case of eventually positive solutions since the proofs for the eventually negative solutions would be similar.

We begin with the following theorem.

Theorem 1: Assume that $(H_1) - (H_4)$ and (4) hold. If

$$\sum_{n=n_1}^{\infty} q_n (M^{1-\alpha} - p_n)^{\beta} = \infty \tag{6}$$

holds for all constants $M > 0$, then every solution of equation (1) is oscillatory.

Proof: Assume to the contrary that equation (1) has an eventually positive solution $\{x_n\}$, say $x_n > 0, x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. From equation (1), we have

$$\Delta(a_n \Delta z_n) = -q_n x_{n+1-l}^{\beta} \leq 0, n \geq n_1. \tag{7}$$

In view of condition (4), it is easy to see that $\Delta z_n > 0$ for all $n \geq n_1$. Now, it follows from the definition of z_n, and using $z_n \geq M > 0$ for all $n \geq n_1$, we have

$$x_n = z_n - p_n x_{n-k}^{\alpha} \geq (z_n^{1-\alpha} - p_n) z_n^{\alpha} \geq (M^{1-\alpha} - p_n) z_n^{\alpha}, n \geq n_1. \tag{8}$$

From (7) and (8), we have

$$\Delta(a_n \Delta z_n) + q_n (M^{1-\alpha} - p_{n+1-l})^{\beta} z_{n+1-l}^{\alpha\beta} \leq 0, n \geq n_1. \tag{9}$$

Summing equation (9) from n_1 to n and using $z_n \geq M$, the last inequality yields

$$M^{\alpha\beta} \sum_{s=n_1}^{n} q_s (M^{1-\alpha} - p_{s+1-l})^{\beta} < \infty$$

which contradicts (6) as $n \to \infty$. This completes the proof of the theorem.

In the next theorem, we reduce the oscillation of equation (1) to that of a first order delay difference equation. Define

$$R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}.$$

Theorem 2: Assume that $(H_1) - (H_4)$ and (4) hold. If the first order delay difference equation

$$\Delta w_n + q_n (M^{1-\alpha} - p_{n+1-l})^{\beta} R_{n+1-l}^{\alpha\beta} w_{n+1-l}^{\alpha\beta} = 0 \tag{10}$$

is oscillatory, then every solution of equation (1) is oscillatory.

Proof: Assume to the contrary that equation (1) has an eventually positive solution $\{x_n\}$, say $x_n > 0, x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. Proceeding as in proof of Theorem 1 we obtain (9).

Now

$$z_n = z_{n_1} + \sum_{s=n_1}^{n-1} \frac{a_s \Delta z_s}{a_s} \geq a_n \Delta z_n R_n, \ \ n \geq n_1 \tag{11}$$

where we have used $\{a_n \Delta z_n\}$ is positive and decreasing for all $n \geq n_1$. Combining (9) and (11), we obtain

$$\Delta(a_n \Delta z_n) + q_n (M^{1-\alpha} - p_{n+1-l})^{\beta} R_{n+1-l}^{\alpha\beta} (a_{n+1-l} \Delta z_{n+1-l})^{\alpha\beta} \leq 0, n \geq n_1.$$

Set $w_n = a_n \Delta z_n$. Then $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + q_n (M^{1-\alpha} - p_{n+1-l})^{\beta} R_{n+1-l}^{\alpha\beta} w_{n+1-l}^{\alpha\beta} \leq 0,.$$

It follows from Lemma 2.7 of [7], the corresponding difference equation (10) also has a positive solution, which is a contradiction. The proof is now completed.

Remark 1: Employing sufficient conditions for oscillation of equation (10), one can obtain easily verifiable criteria for the oscillation of all solutions of equation (1).

Corollary 3: Assume that $(H_1) - (H_4)$ and (4) hold. If $\alpha\beta = 1$, $l \geq 2$, and

$$lim_{n\to\infty} \, inf \sum_{s=n-l+1}^{n-1} q_s (M^{1-\alpha} - p_{s+1-l})^{\beta} \, R_{s+1-l} > \left(\frac{l-1}{l}\right)^{l} \tag{12}$$

then every solution of equation (1) is oscillatory.

Proof: Condition (12) and Theorem 7.5.1 of [3] implies oscillation of equation (10). The assertion now follows from Theorem 2. This completes the proof.

Corollary 4: Assume that $(H_1) - (H_4)$ and (4) hold. If $\alpha\beta < 1$, and

$$\sum_{n=n_1}^{\infty} q_n (M^{1-\alpha} - p_{n+1-l})^{\alpha} R_{n+1-l}^{\alpha\beta} = \infty$$

then every solution of equation (1) is oscillatory.

Proof: Condition (13) and Theorem 1 of [4] implies oscillation of equation (10). The assertion now follows from Theorem 2. The proof is now completed.

Corollary 5: Assume that $(H_1) - (H_4)$ and [4] hold. If $\alpha\beta > 1$, $l \geq 2$, and there exists a $\lambda > \frac{1}{l-1} \log \alpha\beta$ such that

$$\lim_{n\to\infty} \left[q_n (M^{1-\alpha} - p_{n+1-l})^{\beta} R_{n+1-l}^{\alpha\beta} \exp(-e^{\lambda_n}) \right] > 0$$

then every solution of equation (1) is oscillatory.

Proof: Condition (14) and Theorem 2 of [4] implies oscillation of equation (10). The conclusion now follows from Theorem 2, and the proof is completed.

Our next results are for the case where (5) holds in place of (4). We let

$$A_n = \sum_{s=n}^{\infty} \frac{1}{a_s}.$$

We will also need the condition

$$E_n = \frac{K^{1-\alpha}}{A_n^{1-\alpha}} \left(1 - \frac{p_n A_{n-k}^{\alpha}}{K^{1-\alpha} A_n}\right) > 0$$

for all constant K>0 and all $n \geq n_1 \geq n_0$.

Theorem 6: Let $\alpha\beta \geq 1$ and $(H_1) - (H_4)$, (5) and (15) hold. If there exists a positive nondecreasing real sequence $\{\rho_n\}$ such that

$$\lim_{n\to\infty} \sup \sum_{s=n_1}^{n} \left[\rho_s q_s (M^{1-\alpha} - p_{s+1-l})^{\beta} - \frac{a_{s-l} (\Delta\rho_s)^2}{4\alpha\beta\rho_s} \right] = \infty, \tag{16}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n_1}^{n} \left[\rho_s q_s (M^{1-\alpha} - p_{s+1-l})^{\beta} - \frac{a_{s-l} (\Delta\rho_s)^2}{4\alpha\beta\rho_s} \right] = \infty, \tag{17}$$

hold for all constants $M > 0$ and $K > 0$, then every solution of equation (1) is oscillatory.

Proof: Assume to the contrary that equation (1) has an eventually positive solution such that $x_n > 0, x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1 \geq n_0$. From equation (1) that (7) holds, we then have either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ holds, then we can proceed as in the proof of Theorem 1, we obtain (9). Define

$$w_n = \rho_n \frac{a_n \Delta z_n}{z_{n-l}^{\alpha\beta}}, n \geq n_1. \tag{18}$$

Then $w_n > 0$ for $n \geq n_1$, and from (18) and (19), we obtain

$$\Delta w_n \leq -\rho_n q_n (M^{1-\alpha} - p_{n+1-l})^{\beta} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_{n-l}^{\alpha\beta}}{z_{n-l}^{\alpha\beta}}. \tag{19}$$

By Mean value theorem, we have

$$\Delta z_{n-l}^{\alpha\beta} \leq \alpha\beta\, z_{n+1-l}^{\alpha\beta-1} \Delta z_{n-l}$$

and from (19), we obtain

$$\Delta w_n \leq -\rho_n q_n (M^{1-\alpha} - p_{n+1-l})^{\beta} + \frac{\Delta\rho_{n+1}}{\rho_{n+1}} w_{n+1} - \alpha\beta \frac{\rho_n}{\rho_{n+1}^2 a_{n-l}} w_{n+1}^2 \tag{20}$$

for $n \geq n_1$, where we have used $a_n \Delta z_n$ is decreasing and z_n is increasing. Using the completing square in (20), and then summing the resulting inequality from n_1 to n, we obtain

$$\sum_{s=n_1}^{n} \left[\rho_s q_s (M^{1-\alpha} - p_{s+1-l})^{\beta} - \frac{a_{s-l}(\Delta\rho_s)^2}{4\alpha\beta\rho_s}\right] \leq w_{n_1}$$

which contradicts (16) as $n \to \infty$.

Next assume that $\Delta z_n < 0$ for all $n \geq n_1$. Define

$$u_n = \frac{a_n \Delta z_n}{z_n^{\alpha\beta}}, n \geq n_1. \tag{21}$$

Thus $u_n < 0$ for all $n \geq n_1$ and from (7) we have

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_s}, s \geq n.$$

Summing the last inequality from n to j, we obtain

$$z_{j+1} - z_n \leq a_n \Delta z_n \sum_{s=n}^{j} \frac{1}{a_s}$$

and then letting $j \to \infty$ gives

$$\frac{a_n \Delta z_n A_n}{z_n} \geq -1, n \geq n_1. \tag{22}$$

Thus $-\frac{a_n \Delta z_n (-a_n \Delta z_n)^{\alpha\beta-1} A_n^{\alpha\beta}}{z_n^{\alpha\beta}} \leq 1$ for $n \geq n_1$. Since $-a_n \Delta z_n > 0$ and (7) and (21) hold, we have

$$-\frac{1}{L^{\alpha\beta-1}} \leq u_n A_n^{\alpha\beta} \leq 0, \tag{23}$$

where $L = -a_{n_1} \Delta z_{n_1}$. On the other hand from (22),

$$\Delta\left(\frac{z_n}{A_n}\right) \geq 0, n \geq n_1. \tag{24}$$

From the definition of z_n and (24), we have

$$x_n = z_n - p_n x_{n-k}^{\alpha} \geq z_n - p_n z_{n-k}^{\alpha} \geq z_n - p_n \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}} z_n^{\alpha}$$

$$\geq \left(K^{1-\alpha} A_n^{1-\alpha} - \frac{p_n A_{n-k}^{\alpha}}{A_n^{\alpha}}\right) z_n^{\alpha}, n \geq n_1, \tag{25}$$

where we have used $\frac{z_n}{A_n} \geq K > 0$ for all $n \geq n_1$. From (7) and (25), we obtain

$$\Delta(a_n \Delta z_n) \leq -q_n E_{n+1-l}^{\beta} z_{n+1-l}^{\alpha\beta}, n \geq n_1.$$

From (18), we have

$$\Delta u_n = \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\alpha\beta}} - \frac{a_n \Delta z_n}{z_n^{\alpha\beta} z_{n+1}^{\alpha\beta}} \Delta z_n^{\alpha\beta}, n \geq n_1.$$

By Mean value theorem

$$\Delta z_n^{\alpha\beta} = \begin{cases} \alpha\beta z_{n+1}^{\alpha\beta-1}\Delta z_n, & \text{if } \alpha\beta > 1; \\ \alpha\beta\, z_n^{\alpha\beta-1}\Delta z_n, & \text{if } \alpha\beta < 1, \end{cases} \tag{28}$$

So combining (28) and (27) and then using the fact that $\Delta z_n < 0$ gives

$$\begin{aligned} \Delta u_n &\le \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\alpha\beta}} - \alpha\beta \frac{u_n^2}{a_n} z_n^{\alpha\beta-1} \\ &\le -q_n E_{n+1-l}^{\alpha} - \alpha\beta K^{\alpha\beta-1} A_n^{\alpha\beta-1} \frac{u_n^2}{a_n}, n \ge n_1, \end{aligned} \tag{29}$$

where we have used $\frac{z_{n+1-l}}{z_{n+1}} \ge 1$ for all $n \ge n_1$. Multiplying (29) by $A_{n+1}^{\alpha\beta}$, and then summing it from n_1 to $n-1$, we have

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta} \Delta u_s + \sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta} q_s E_{s+1-l}^{\beta} + \sum_{s=n_1}^{n-1} \alpha\beta K^{\alpha\beta-1} A_s^{\alpha\beta-1} A_{s+1}^{\alpha\beta} \frac{u_n^2}{a_n} \le 0. \tag{30}$$

By summation by parts formula and Mean value theorem, we obtain

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta} \Delta u_s \ge A_n^{\alpha\beta} u_n - A_{n_1}^{\alpha\beta} u_{n_1} + \sum_{s=n_1}^{n-1} \frac{\alpha\beta u_s A_s^{\alpha\beta-1}}{a_s}. \tag{31}$$

From (31) and (30), we obtain

$$\begin{aligned} A_n^{\alpha\beta} u_n - A_{n_1}^{\alpha\beta} u_{n_1} + \sum_{s=n_1}^{n-1} \frac{\alpha\beta u_s A_s^{\alpha\beta-1}}{a_s} &+ \sum_{s=n_1}^{n-1} \alpha\beta K^{\alpha\beta-1} A_s^{\alpha\beta-1} A_{s+1}^{\alpha\beta} \frac{u_n^2}{a_n} \\ &+ \sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta} q_s E_{s+1-l}^{\beta} \le 0 \end{aligned}$$

$$\sum_{s=n_1}^{n-1} \left[A_{s+1}^{\alpha\beta} q_s E_{s+1-l}^{\beta} - \frac{\alpha\beta K^{1-\alpha\beta} A_s^{\alpha\beta-1}}{4 a_s A_{s+1}^{\alpha\beta}} \right] \le \frac{1}{L^{\beta-1}} + A_{n_1}^{\alpha\beta} u_{n_1}$$

when using (23). This contradicts (17) as $n \to \infty$, and the proof is now completed.

Corollary 7: Let $\alpha\beta > 1$ and $(H_1) - (H_4)$, (5) and (15) hold. If condition (17) and $l \ge 2$ hold, and there exists a $\lambda > \frac{1}{l-1} \log \alpha\beta$ such that (14) holds then every solution of equation (1) is oscillatory.

Proof: The proof follows from Corollary 5 and Theorem 6. This completes the proof.

Theorem 8: Let $0 < \alpha\beta < 1$ and $(H_1) - (H_4)$, (5) and (15) hold. If condition (13) holds, and

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[K^{\alpha\beta-1} A_{s+1} q_s E_{s+1-l}^{\beta} - \frac{1}{4 a_s A_{s+1}} \right] = \infty \tag{32}$$

for all constant $K > 0$, then every solution of equation (1) is oscillatory.

Proof: Proceeding as in the proof of Theorem 6, we see that $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ for all $n \ge n_1$ then by Corollary 4, we obtain a contradiction to (13). Next we assume that $\Delta z_n < 0$ for all $n \ge n_1$. Proceeding as in the proof of Theorem 6 we obtain (26). Define

$$u_n = \frac{a_n \Delta z_n}{z_n}, n \ge n_1. \tag{33}$$

Thus $u_n < 0$ for all $n \ge n_1$.

From (33) and (26), we see that

$$\Delta u_n \le -q_n E_{n+1-l}^{\beta} \frac{z_{n+1-l}^{\alpha\beta}}{z_{n+1}} - \frac{u_n^2}{a_n}, n \ge n_1. \tag{34}$$

Since $\{z_n\}$ is positive and decreasing there exists a constant $K > 0$ such that $z_n \le K$ for all $n \ge n \ge 1$. Using the last inequality in (34), we obtain

$$\Delta u_n \le -q_n \frac{E_{n+1-l}^{\beta}}{K^{1-\alpha\beta}} - \frac{u_n^2}{a_n}, n \ge n_1.$$

Multiplying the last inequality A_{n+1} and then summing it from n_1 to $n-1$, we have

$$A_n u_n - A_{n_1} u_{n_1} + \sum_{s=n_1}^{n-1} K^{\alpha\beta-1} A_{s+1} q_s E_{s+1-l}^{\beta} + \sum_{s=n_1}^{n-1} \left(\frac{u_s}{a_s} + A_{s+1} \frac{u_s^2}{a_s} \right) \leq 0$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1} \left[K^{\alpha\beta-1} A_{s+1} q_s E_{s+1-l}^{\beta} - \frac{1}{4 a_s A_{s+1}} \right] \leq 1 + A_{n_1} u_{n_1}$$

when using (22). This contradicts (32) as $n \to \infty$, and the proof is now complete.

EXAMPLES

In this section, we present some examples to illustrate our main results.

Example 1: Consider the neutral difference equation

$$\Delta\left(\frac{1}{2n+1} \Delta\left(x_n + \frac{1}{n} x_{n-1}^{\frac{1}{3}} \right) \right) + \left(\frac{4(n+1)}{(2n+1)(2n+3)} + \frac{2}{n(n+2)} \right) x_{n-1}^{3} = 0, n \geq 1. \tag{35}$$

Here $a_n = \frac{1}{2n+1}$, $p_n = \frac{1}{n}$, $q_n = \frac{4(n+1)}{(2n+1)(2n+3)} + \frac{2}{n(n+2)}$, $\alpha = \frac{1}{3}$, $\beta = 3$, $k = 2$ and $l = 2$. Simple calculation shows that $R_n = n^2 - 1$ and we see that all conditions of Corollary 3 are satisfied. Hence every solution of equation (35) is oscillatory, and in fact $\{x_n\} = \{(-1)^{3n}\}$ is are such oscillatory solution of equation (35).

Example 2:.Consider the neutral difference equation

$$\Delta\left(n \Delta\left(x_n + \frac{1}{n} x_{n-2}^{\frac{1}{3}} \right) \right) + \frac{1}{n} x_{n-1}^{\frac{1}{5}} = 0, n \geq 1. \tag{36}$$

Here $a_n = n$, $p_n = \frac{1}{n}$, $q_n = \frac{1}{n}$, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{5}$, $k = 1$ and $l = 2$. It is easy to see that all conditions of Corollary 4 are satisfied, and hence every solution of equation (36) is oscillatory.

Example 3: Consider the neutral difference equation

$$\Delta\left((n+1)(n+2) \Delta\left(x_n + \frac{1}{n(n+1)} x_{n-1}^{\frac{1}{3}} \right) \right) + 4(n+2)^2 x_{n-1}^{3} = 0, n \geq 1. \tag{37}$$

Here $a_n = (n+1)(n+2)$, $p_n = \frac{1}{n(n+1)}$, $q_n = 4(n+2)^2$, $\alpha = \frac{1}{3}$, $\beta = 3$, $k = 1$ and $l = 2$.

Simple calculation shows that $A_n = \frac{1}{n} + 1$, $\alpha\beta = 1$ and $K^{\frac{2}{3}}(n+1)^{\frac{2}{3}}\left(1 - \frac{K^{-\frac{2}{3}}}{n^{\frac{4}{3}}}\right) > 0$. The conditions (16) and (17) are also satisfied with $\rho_n = 1$. Therefore by Theorem 8, every solution of equation (37) is oscillatory.

We conclude this paper with the following remark.

Remark 3.4: In this paper, we obtain oscillation criteria for equation (1) by employing comparison method and Riccati type transformation involving both α and β. Therefore the results presented in this paper are new and complement to the existing results reported in the literature. Further condition (15) is some what restrictive and it implies that we must have $\{p_n\} \to 0$ as $n \to \infty$. It would be good to obtain a result that did not require this added condition.

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