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FIXED POINT THEOREMS THROUGH CONTRACTIVETYPE MAPPINGS

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ABSTRACT

In this research article, some fixed point theorems in 2- metric spaces are established. It also introduces contraction type mappings in 2-metric spaces. The theorems are generalizations of some fixed point theorems of Pal and Maiti [2].

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INTRODUCTION

The notion of 2-metric space was introduced by *Gahler* [1] in 1963 as a generalizationof area function for Euclidean triangles. Fixed point theory was first studied by *Poincare* and developed by many mathematicians *Brouwer, Banach, Schauder, Rhoades* [4], etc. Let X be a non-empty set. A function $f: X \rightarrow X$ from X into itself is called a self-map on X. A point $z \in X$ is called a fixed point of self-map $f: X \rightarrow X$ if $f(z)=z$, Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on [0,1] simply serves the counter example.

In this present work, some of the fixed point theoremsof *Pal* and *Maiti* are extended to a more generalized 2-metric space setting.

In what follows X stands for a 2-metric space.

1. PRELIMINARIES

This section is devoted to some basic definitions which are needed for the further study of this Article.

Definition 1.1: Let X be a non-empty set and $\lambda: X \times X \times X \rightarrow R$. For all x, y, z and u in X, if λ satisfies the following conditions

(a) $\lambda(x, y, z)=0$ if atleast two of x, y, z are equal

(b) $\lambda(x, y, z)=\lambda(x, z, y)=\lambda(y, z, x)=\ldots$

(c) $\lambda(x, y, z) \leq \lambda(x, y, u)+\lambda(x, u, z)+\lambda(u, y, z)$

Then d is called a 2-*metric*on X and the pair (X, λ) is called a *2-metric space.*

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Definition 1.2: Let (X,λ) be a 2-metric space. A mapping $T:X\to X$ is said to be a *Contractive* if

$$\lambda(Tx,Ty,a)<\lambda(x,y,a) \text{ for all } x,y,a \text{ in } X.$$

Remark 1.3: It is well known that a contractive mapping on a complete metric space has a unique fixed point in X. But a contractive mapping on a complete 2-metric space (X,λ) neednot have a fixed point. It can be seen from the following example.

Example- 1.4: Let $X=\{x\in R/x\geq 2\}=[2,\infty)$ with 2-metric defined by

$$\lambda(x,y,z)=\min\{|x-y|,|y-z|,|z-x|\}.$$

Define $f:X\to X$ by $f(x)=\frac{1}{x}\ \forall\ x\in X$.

$$\begin{aligned}\lambda(fx,fy,a)&=\lambda\left(\frac{1}{x},\frac{1}{y},a\right)\\&=\min\left\{\left|\frac{1}{x}-\frac{1}{y}\right|,\left|\frac{1}{y}-a\right|,\left|\frac{1}{x}-a\right|\right\}\\&<\min\{|x-y|,|y-a|,|a-x|\}\\&=\lambda(x,y,a)\end{aligned}$$

$\Rightarrow$ f is a contractive mapping on X ,but it has no fixed point in X.

Definition 1.5: Let (X,λ) be a 2-metric space. A mapping $T:X\to X$ is said to be a *Generated Contractive* if

$$\lambda(Tx,Ty,a)<\max\left\{\lambda(x,y,a),\lambda(x,Tx,a),\lambda(y,Ty,a),\frac{1}{2}[\lambda(x,Ty,a)+\lambda(y,Tx,a)]\right\} \text{ for all } x,y,a \text{ in } X.$$

Remark 1.6: A Generated contractive mapping is need not be contractive mapping. Itcan be seen from the following example.

Example 1.7: Let $X=[0,6]$ with 2-metric defined as $\lambda(x,y,z)=\min\{|x-y|,|y-z|,|z-x|\}$ for all x,y,z in X.

We define a map $T:[0,6]\to[0,6]$ by $T(x)=\begin{cases}3x & if \quad x\in[0,3]\\ 3x-1 & if \quad x\in(3,6]\end{cases}$

Clearly T is generated contractive mapping.

But $\lambda(T(3),T(5),0)=\lambda(9,14,0)=5>\lambda(3,5,0)$

$\Rightarrow$ T is not a contractive mapping.

Definition 1.8: A 2-metric space (X,λ) is said to be *2-compact*if every sequence in X has a convergent subsequence.

2. FIXED POINT THEOREMS FOR GENERATED CONTRACTIVE MAPPINGS

In this section we proved fixed point theorems for Generated Contractive type mappings.

The following Theorem-2.1 is a generalization of *Pal* and *Maiti* [2] in 2-metric space setting.

Theorem 2.1: Let T be a continuous generated contractive self-mapping on a compact 2-metric space X such that $\lambda(x,Ty,a)+\lambda(y,Tx,a)<2\lambda(x,y,a)$ for all x,y,a in X. Then T has a unique fixed point in X.

Proof: Let $s_0 \in X$..

Define a sequence $\{s_n\}$ in X such that $s_n = T^n s_0$ when $n = 1, 2, 3, \ldots$

Consider $\lambda(x, Ty, a) + \lambda(y, Tx, a) < 2\lambda(x, y, a)$

Put $x = s_{n+1}$, $y = s_n$

$$\lambda\left(s_{n+1}, Ts_n, a\right) + \lambda\left(s_n, Ts_{n+1}, a\right) < 2\lambda\left(s_{n+1}, s_n, a\right)$$

$$\Rightarrow \lambda\left(s_{n+1}, s_{n+1}, a\right) + \lambda\left(s_n, s_{n+2}, a\right) < 2\lambda\left(s_{n+1}, s_n, a\right)$$

$$\Rightarrow \lambda\left(s_n, s_{n+2}, a\right) < 2\lambda\left(s_{n+1}, s_n, a\right)$$

$$\Rightarrow \frac{1}{2}\lambda\left(s_n, s_{n+2}, a\right) < \lambda\left(s_{n+1}, s_n, a\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

.Put $c_n = d\left(s_n, s_{n+1}, a\right)$

Since T is generated contractive,

$$\lambda(Tx, Ty, a) < \max\left\{\lambda(x, y, a), \lambda(x, Tx, a), \lambda(y, Ty, a), \frac{1}{2}\left[\lambda(x, Ty, a) + \lambda(y, Tx, a)\right]\right\}$$

Put $x = s_{n+1}$, $y = s_n$

$$\lambda\left(Ts_{n+1}, Ts_n, a\right)$$

$$< \max\left\{\lambda\left(s_{n+1}, s_n, a\right), \lambda\left(s_{n+1}, Ts_{n+1}, a\right), \lambda\left(s_n, Ts_n, a\right), \frac{1}{2}\left[\lambda\left(s_{n+1}, Ts_n, a\right) + \lambda\left(s_n, Ts_{n+1}, a\right)\right]\right\}$$

$$= \max\left\{\lambda\left(s_{n+1}, s_n, a\right), \lambda\left(s_{n+1}, s_{n+2}, a\right), \lambda\left(s_n, s_{n+1}, a\right), \frac{1}{2}\left[\lambda\left(s_{n+1}, s_{n+1}, a\right) + \lambda\left(s_n, s_{n+2}, a\right)\right]\right\}$$

$$\Rightarrow \lambda\left(s_{n+2}, s_{n+1}, a\right) < \max\left\{\lambda\left(s_{n+1}, s_n, a\right), \lambda\left(s_{n+1}, s_{n+2}, a\right), \frac{1}{2}\lambda\left(s_n, s_{n+2}, a\right)\right\}$$

Let $M = \lambda\left(s_{n+1}, s_{n+2}, a\right)$

Then $\lambda\left(s_{n+1}, s_{n+2}, a\right) < \lambda\left(s_{n+1}, s_{n+2}, a\right)$

This is a contradiction.

So $M \neq \lambda\left(s_{n+1}, s_{n+2}, a\right)$

By (1), $M = \lambda\left(s_n, s_{n+1}, a\right)$

So we get $\lambda\left(s_{n+1}, s_{n+2}, a\right) < \lambda\left(s_n, s_{n+1}, a\right)$

$$\Rightarrow c_{n+1} < c_n$$

Similarly $c_{n+1} < c_n < c_{n-1} < \ldots < c_0$

$\Rightarrow \{c_n\}$ is a monotonically decreasing and bounded sequence of nonnegative real numbers.

$\Rightarrow \{c_n\}$ Converges.

Suppose that $\lim_{n\to\infty} c_n = l$

Since X is compact, $\{s_n\}$ has a convergent subsequence, say, $\{p_m\}$.

Let $\lim_{m\to\infty} p_m = u,$ where $u \in X$.

Since T is continuous and $\lim_{m\to\infty} p_m = u,$ we have $\lim_{m\to\infty} Tp_m = Tu$

$$\Rightarrow \lim_{m\to\infty} Tp_{m+1} = Tu$$

Now we show that u is a fixed point of T Assume that $u \neq Tu$

Then $\lambda(u,Tu,a) \leq \lambda(u,Tu,p_m) + \lambda(u,p_m,a) + \lambda(p_m,Tu,a)$

$$= \lambda(u,Tu,p_m) + \lambda(u,p_m,a) + \lambda(Tp_{m-1},Tu,a)$$

$$< \lambda(u,Tu,p_m) + \lambda(u,p_m,a) + \max\left\{\lambda(p_{m-1},u,a), \lambda(p_{m-1},Tp_{m-1},a), \lambda(u,Tu,a), \frac{1}{2}\left[\lambda(p_{m-1},Tu,a) + \lambda(u,Tp_{m-1},a)\right]\right\}$$

$$= \lambda(u,Tu,p_m) + \lambda(u,p_m,a) + \max\left\{\lambda(p_{m-1},u,a), \lambda(p_{m-1},p_m,a), \lambda(u,Tu,a), \frac{1}{2}\left[\lambda(p_{m-1},Tu,a) + \lambda(u,p_m,a)\right]\right\}$$

Letting $m \to \infty$

$$\lambda(u,Tu,a) < \max\left\{\lambda(u,Tu,a), \frac{1}{2}\lambda(u,Tu,a)\right\}$$

$$\lambda(u,Tu,a) < \lambda(u,Tu,a)$$

This is a contradiction.

Therefore our assumption is false.

Hence $Tu = u$.

$\Rightarrow u$ is a fixed point of T in X.

Now we show that it is unique.

Let v be another fixed point of T

i.e, $Tv = v$

Assume that $u \neq v$

Then $\lambda(u,v,a) = \lambda(Tu,Tv,a)$

$$< \max\left\{\lambda(u,v,a), \lambda(u,Tu,a), \lambda(v,Tv,a), \frac{1}{2}\left[\lambda(u,Tv,a) + \lambda(v,Tu,a)\right]\right\}$$

$$= \max\left\{\lambda(u,v,a), \lambda(u,u,a), \lambda(v,v,a), \frac{1}{2}\left[\lambda(u,v,a) + \lambda(v,u,a)\right]\right\}$$

$$= \lambda(u,v,a)$$

$$\Rightarrow \lambda(u,v,a) < \lambda(u,v,a)$$

This is a contradiction.

Hence $u = v$.

$\Rightarrow T$ has a unique fixed point in X.

Theorem- 2.2: Let (X,λ) be a compact 2-metric space. Suppose that S and T are two continuous self-maps on a X such that

$$(1)\ \lambda(x,Ty,a)+\lambda(y,Sx,a)<2\lambda(x,y,a)$$

$$(2)\ \lambda(Sx,Ty,a)<\max\left\{\lambda(x,y,a),\lambda(x,Sx,a),\lambda(y,Ty,a),\frac{1}{2}\left[\lambda(x,Ty,a)+\lambda(y,Sx,a)\right]\right\}$$

for all x,y,a in X.

Then S and T have a unique common fixed point in X.

Proof: Let $p_0 \in X$.

Define a sequence $\{p_n\}$ in X such that $p_{2n+1}=Sp_{2n}$, where $n=0,1,2,3,\ldots$

And $p_{2m}=Tp_{2m-1}$, where $m=1,2,3,\ldots$

Suppose that $\lambda_{2n}=\lambda(p_{2n},p_{2n+1},a)$, where $n=0,1,2,3,\ldots$

Consider $\lambda(x,Ty,a)+\lambda(y,Sx,a)<2\lambda(x,y,a)$

Put $x=p_{2n}$ and $y=p_{2n-1}$

$$\lambda(p_{2n},Tp_{2n-1},a)+\lambda(p_{2n-1},Sp_{2n},a)<2\lambda(p_{2n},p_{2n-1},a)$$

$$\Rightarrow \lambda(p_{2n},p_{2n},a)+\lambda(p_{2n-1},p_{2n+1},a)<2\lambda(p_{2n},p_{2n-1},a)$$

$$\Rightarrow \lambda(p_{2n-1},p_{2n+1},a)<2\lambda(p_{2n},p_{2n-1},a)$$

$$\Rightarrow \frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)<\lambda(p_{2n-1},p_{2n},a)\ (1)$$

Now, consider $\lambda(Sx,Ty,a)<\max\left\{\lambda(x,y,a),\lambda(x,Sx,a),\lambda(y,Ty,a),\frac{1}{2}\left[\lambda(x,Ty,a)+\lambda(y,Sx,a)\right]\right\}$

Put $x=p_{2n}$ and $y=p_{2n-1}$

$$\lambda(Sp_{2n},Tp_{2n-1},a)$$

$$<\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},Sp_{2n},a),\lambda(p_{2n-1},Tp_{2n-1},a),\frac{1}{2}\left[\lambda(p_{2n},Tp_{2n-1},a)+\lambda(p_{2n-1},Sp_{2n},a)\right]\right\}$$

$$=\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\lambda(p_{2n-1},p_{2n},a),\frac{1}{2}\left[\lambda(p_{2n},p_{2n},a)+\lambda(p_{2n-1},p_{2n+1},a)\right]\right\}$$

$$\Rightarrow \lambda(p_{2n+1},p_{2n},a)<\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)\right\}$$

If $\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)\right\}=\lambda(p_{2n},p_{2n+1},a)$

Then $\lambda(p_{2n},p_{2n+1},a)<\lambda(p_{2n},p_{2n+1},a)$

This is a contradiction.

So $\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)\right\}\neq\lambda(p_{2n},p_{2n+1},a)$

By (1), we get

$$\max\left\{\lambda\left(p_{2n}, p_{2n-1}, a\right), \lambda\left(p_{2n}, p_{2n+1}, a\right), \frac{1}{2}\lambda\left(p_{2n-1}, p_{2n+1}, a\right)\right\} = \lambda\left(p_{2n-1}, p_{2n}, a\right)$$

$$\Rightarrow d_{2n} < d_{2n-1}$$

Similarly $d_{2n} < d_{2n-1} < d_{2n-2} < ... < d_0$

$\Rightarrow \{d_{2n}\}$ is a monotonically decreasing and bounded sequence of nonnegative real numbers.

$\Rightarrow \{d_{2n}\}$ Converges.

Suppose that $\lim\limits_{n\to\infty} d_{2n} = l$

Since X is compact, every sequence in X has a convergent subsequence

So $\{p_n\}$ has a convergent subsequencein X.

Suppose that $\{p_{2n}\}$ is a convergent subsequence of $\{p_n\}$ in X and let $\lim\limits_{n\to\infty} p_{2n} = u,$ where $u \in X$.

Now we prove that u is acommon fixed point of S and T.

Assume that $u \neq Su$

Since S is continuous on X and $\lim\limits_{n\to\infty} p_{2n} = u,$ we have $\lim\limits_{n\to\infty} Sp_{2n} = Su$

$$\Rightarrow \lim_{n\to\infty} p_{2n+1} = Su$$

Since T is continuous on X and $\lim\limits_{n\to\infty} p_{2n+1} = Su,$ we have $\lim\limits_{n\to\infty} Tp_{2n+1} = TSu$

$$\Rightarrow \lim_{n\to\infty} Tp_{2n+1} = TSu$$

$\Rightarrow \lim\limits_{n\to\infty} p_{2n+2} = TSu$ because $Tp_{2n+1} = p_{2n+2}$

Then
$$\lambda(u, Su, a) \le \lambda(u, Su, p_{2n}) + \lambda(u, p_{2n}, a) + \lambda(p_{2n}, Su, a)$$
$$= \lambda(u, Su, p_{2n}) + \lambda(u, p_{2n}, a) + \lambda(Tp_{2n-1}, Su, a)$$
$$< \lambda(u, S\ u p_{2n}) + \lambda(u, p_{2n}, a)$$
$$+ \max\left\{\lambda(u, p_{2n-1}, a), \lambda(u, Su, a), \lambda\left(p_{2n-1}, Tp_{2n-1}, a\right), \frac{1}{2}\left[\lambda(u, Tp_{2n-1}, a) + \lambda\left(p_{2n-1}, Su, a\right)\right]\right\}$$
$$= \lambda(u, S\ u p_{2n}) + \lambda(u, p_{2n}, a)$$
$$+ \max\left\{\lambda(u, p_{2n-1}, a), \lambda(u, Su, a), \lambda\left(p_{2n-1}, p_{2n}, a\right), \frac{1}{2}\left[\lambda(u, p_{2n}, a) + \lambda\left(p_{2n-1}, Su, a\right)\right]\right\}$$

Letting $n \to \infty$

$$\lambda(u, Su, a) < \max\left\{\lambda(u, Su, a), \frac{1}{2}\lambda(u, Su, a)\right\}$$

$$\Rightarrow \lambda(u, Su, a) < \lambda(u, Su, a)$$

This is a contradiction.Soour assumption is false.

Hence $Su = u$.

$\Rightarrow u$ is a fixed point of S in X.

Consider $\lambda(x,Ty,a)+\lambda(y,Sx,a)<2\lambda(x,y,a)$

Put $x=y=u$

$$\lambda(u,Tu,a)+\lambda(u,Su,a)<2\lambda(u,u,a)$$

Since $Su=u$, $\lambda(u,Tu,a)<0$.

But $\lambda(u,Tu,a)\geq 0$.

$$\Rightarrow \lambda(u,Tu,a)=0$$

$$\Rightarrow Tu=u$$

Hence u is a common fixed point of S and T in X.

Now we show that it is unique.

Let v be another fixed point of T

i.e, $Sv=Tv=v$

Assume that $u\neq v$

Then $\lambda(u,v,a)=\lambda(Su,Tv,a)$

$$<\max\left\{\lambda(u,v,a),\lambda(u,Su,a),\lambda(v,Tv,a),\frac{1}{2}\left[\lambda(u,Tv,a)+\lambda(v,Su,a)\right]\right\}$$

$$=\max\left\{\lambda(u,v,a),\lambda(u,u,a),\lambda(v,v,a),\frac{1}{2}\left[\lambda(u,v,a)+\lambda(v,u,a)\right]\right\}$$

$$=\lambda(u,v,a)$$

$$\Rightarrow \lambda(u,v,a)<\lambda(u,v,a)$$

This is a contradiction.

Hence $u=v$.

Thus S and T have a unique common fixed point in X.

Remark-2.3: If we take $S=T$ in Theorem-2.2 then we obtain Theorem-2.1. So 2.2 is a further generalization of Theorem-2.1. A point is a unique fixed point of $T:X\to X$ iff it is unique fixed point of any positive power of T. This fact leads us to the following theorem and proof of the following is similar to the proof of previous theorem.

Theorem- 2.4: Let (X,λ) be a compact 2-metric space. Suppose that S and T are two continuous self-maps on X such that

(1) $\lambda(x,T^q y,a)+\lambda(y,S^p x,a)<2\lambda(x,y,a)$

(2) $\lambda(S^p x,T^q y,a)<\max\left\{\lambda(x,y,a),\lambda(x,S^p x,a),\lambda(y,T^q y,a),\frac{1}{2}\left[\lambda(x,T^q y,a)+\lambda(y,S^p x,a)\right]\right\}$

For all x,y,a in X and for all $p,q\in\mathbb{N}$.

Then S and T have a unique common fixed point in X.

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