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COINCIDENCE POINTS AND COMMON FIXED POINT THEOREM IN CONE METRIC SPACES

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ABSTRACT

In this paper, we prove a unique common fixed point theorem in cone metric spaces which generalize and extend metric space into cone metric spaces without appealing to commutativity. These results generalize and extend some recent results.

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Keywords: *Common fixed point, cone metric space, coincidence points, asymptotically regular.*

1. INTRODUCTION AND PRELIMINARIES

In 2007 Huang and Zhang [6] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [6], [15] and the references mentioned therein).Recently S.L. Singh, Apichai Hematulin and Rajendra Pant [21] have obtained coincidence points and fixed points results for three mappings in metric spaces. In this paper, we extend the fixed point theorem of S.L. Singh *et.al*. [21] in metric space into cone metric space without appealing to commutativity.

In all that follows B is a real Banach Space. For the mapping f, g: $X \to X$, let C (f, g) denote the set of coincidence points of f and g, that is C(f, g) = {z $\in$ X: fz = gz}.

We recall some definitions of cone metric spaces and some of their properties [6].

Definition 1.1: Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:
(a). P is closed, non-empty and $P \neq \{0\}$;
(b). a, b $\in R$, a, b ≥ 0, x, y $\in P$ implies ax+by $\in P$;
(c). x $\in$ P and -x $\in P$ implies x =0.

Definition 1.2: Let P be a cone in a Banach space B, define partial ordering '$\leq$' with respect to P by x $\leq$ y if and only if y-x $\in P$ .We shall write x<y to indicate $x \leq y$ but $x \neq y$ while x<<y will stand for y-x $\in$ Int P, where Int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3: Let B be a Banach space and $P \subset B$ be an order cone. The order cone P is called normal if there exists L>0 such that for all x, y $\in$ B.
$0 \leq x \leq y$ implies $\|x\| \leq L\|y\|$.

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4: Let X be a nonempty set of B .Suppose that the map d: $X \times X \to B$ satisfies:
(d1). $0 \leq d(x, y)$ for all x, y $\in X$ and d(x, y) = 0 if and only if x = y ;
(d2). d(x, y) = d(y, x) for all x, y $\in X$;
(d3). d(x, y) $\leq$ d(x, z) +d(y, z) for all x, y, z $\in X$.

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Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.5: ([6]). Let $E = R^2$, $P = \{(x, y) \in E$ such that: $x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$ such that

$d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant Then (X, d) is a cone metric space.

Definition 1.6: Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with c >>0, there is N such that for all n, m>N, $d(x_n, x_m) << c$;

(ii) a convergent sequence if for any c >>0, there is an N such that for all n > N, $d(x_n, x) << c$, for some fixed x in X. We denote this $x_n \to x$ (as $n \to \infty$) .

Lemma 1.7: Let (X, d) be a cone metric space, and let P be a normal cone with normal constant L. Let $\{x_n\}$ be a sequence in X .Then
(i) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.
(ii) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

Definition 1.8: Let f, g and h be maps on X with values in a cone metric space (X, d). The pair (f, g) is asymptotically regular with respect to h at $x_0 \in X$ if there exists a sequence $\{x_n\}$ in X such that

$hx_{2n+1} = fx_{2n},$

$hx_{2n+2} = gx_{2n+1}$, n = 0, 1, 2,…, and $\lim\limits_{n \to \infty} d(fx_n, fx_{n+1}) = 0$.

Definition 1.9[21]: Let φ denote the class of all functions $\varphi: R_+ \to R_+$ satisfying: for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \leq \varepsilon$.

2. MAIN RESULT

In this section we obtain, a common fixed point theorem for self-mappings without appealing to commutativity condition, defined on a cone metric space, which is an extension of metric space into cone metric space.

The following Theorem generalizes the Theorem 2.7 of [21].

Theorem 2.1: Let (X, d) be a cone metric space, P be a normal cone with normal constant L and f, g, h: $X \to X$ be self maps. Let (f, g) be asymptotically regular with respect to h at $x_0 \in X$ and the following conditions are satisfied.

(E_1): $f(X) \cup g(X) \subseteq h(X)$;

(E_2): $d(fx, gy) \leq \varphi(s(x, y))$ for all $x, y \in X$, where $s(x, y) = d(hx, hy) + \gamma\,[d(fx, hx) + d(gy, hy)]$, $0 \leq \gamma \leq 1$, and $\varphi: R_+ \to R_+$ continuous.

If f(X) or g(X) or h(X) is a complete subspace of X, then the maps f, g and h have a coincidence points in X. Then f, g, and h have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X.Since (f, g) is asymptotically regular with respect to h, then there exist a sequence $\{x_n\}$in X such that

$hx_{2n+1} = fx_{2n}$

$hx_{2n+2} = gx_{2n+1}$, n = 0, 1, 2 … and $d(hx_n, hx_{n+1}) = 0$.

Now we shall show that $\{hx_n\}$ is Cauchy sequence.

Suppose $\{hx_n\}$ is not Cauchy.

Then there exists μ > 0 and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers, Such that for all $n \leq m_k < n_k$,

$d(hx_{m_k}, hx_{n_k}) \geq \mu$ and $d(hx_{m_k}, hx_{n_{k-1}}) < \mu$.

By the triangle inequality,

$d(hx_{m_k}, hx_{n_k}) \leq d(hx_{m_k}, hx_{n_{k-1}}) + d(hx_{n_{k-1}}, hx_{n_k})$.

Letting $k \to \infty$, we get

$d(hx_{m_k}, hx_{n_k}) < \mu$.

Thus $d(hx_{m_k}, hx_{n_k}) \to \mu$ as $k \to \infty$.

By (E_2) we have

$$d(hx_{m_{k+1}}, hx_{n_{k+1}}) = d(fx_{m_k}, gx_{n_k})$$
$$\leq \varphi(s(x_{m_k}, x_{n_k}))$$
$$= \varphi(d(fx_{m_k}, gx_{n_k}) + \gamma[d(fx_{m_k}, hx_{m_k}) + d(gx_{n_k}, hx_{n_k})]).$$

Letting $k \to \infty$,

$\mu \leq \varphi(\mu) < \mu$, a contradiction.

Thus $\{hx_n\}$ is Cauchy Sequence.

Then $\{hx_n\}$ being contained in h(X) has limit in h(X), let it be z.

Let $u = h^{-1} z$.

Thus hu = z for some $u \in X$.

Note that the subsequences $\{hx_{2n+1}\}$ and $\{hx_{2n+2}\}$ also converge to z. Now by (E_2)

$d(fu, gx_{2n+1}) \leq \varphi(d(hu, hx_{2n+1}) + \gamma[d(fu, hu) + d(gx_{2n+1}, hx_{2n+1})])$.

Letting $n \to \infty$.

$d(fu, hu) \leq \varphi(\gamma\, d(fu, hu)) < d(fu, hu)$, which is a contradiction.

Therefore, fu = hu = z, u is a coincidence point of f and h. (1)

Since, $f(X) \cup g(X) \subseteq h(X)$.

Therefore, there exists $v \in X$ such that

fu = hv. We claim that hv = gv using (E_2)

$$d(hv, gv) = d(fu, gv)$$
$$\leq \varphi(d(hu, hv) + \gamma[d(fu, hu) + d(gv, hv)])$$
$$= \varphi(d(hv, gv))$$
$$< d(hv, gv), \text{ which is a contradiction.}$$

Therefore, hv = gv = z, v is a coincidence point of g and h. … (2)

From (1) and (2), fu = hu = gv = hv = z. … (3)

Now using (E_2),

$d(hu,hhu) = d(hu,y_{n+1}) + d(y_{n+1},hhu)$ (By the triangle inequality)

$= d(hu,y_{n+1}) + d(fx_{2n},ghu)$ (since hu = gu)

$\leq d(hu,y_{n+1}) + \varphi\ (d(hx_{2n}, hhu) + \gamma\,[d(fx_{2n}, hx_{2n}) + d(ghu, hhu)]\,).$

From (1.3)

$\|d(hu,hhu)\| \leq L(\|d(hu,y_{n+1}) + \varphi\ (d(hx_{2n}, hhu) + \gamma\,[d(fx_{2n}, hx_{2n}) + d(ghu, hgu)]\,)\|)$

$\leq L(\|d(hu,y_{n+1})\| + \|\varphi\,(d(hx_{2n}, hhu)\| + \gamma\,[\|d(fx_{2n}, hx_{2n})\| + \|d(ghu, ghu)\|\,])).$

Letting $n \to \infty$,

$\|d(hu,hhu)\| \leq L(\|d(z, z)\| + \varphi\ (\|d(z, hhu)\| + \gamma\,[\|d(z, z)\| + \|d(ghu, ghu)\|\,]))$

$\leq L(\varphi\,\|d(hu, hhu)\|) < \|d(hu, hhu)\|,$ which is a contradiction.

$\Rightarrow \|d(hu, hhu)\| = 0$

$\Rightarrow d(hu, hhu) = 0$

$\Rightarrow hhu = hu(= z).$ … (4)

Now,
$d(gu,ggu) = d(fu,ggu)$ (since gu=fu)

$\leq \varphi\ (d(hu, hgu) + \gamma\,[d(fu,hu) + d(ggu, hgu)])$ (by E_2)

From **(1.3)**

$\|d(gu,ggu)\| \leq L(\|\varphi\ (d(gu, ggu) + \gamma\,[d(z,z) + d(ggu, ggu)])\|)$ (since hu = gu = z)

$\leq L\,(\varphi\,\|d(gu, ggu)\|\,) < \|d(gu, ggu)\|,$

which is a contradiction

$\Rightarrow \|d(gu,ggu)\| = 0$

$\Rightarrow d(gu,ggu) = 0$

$\Rightarrow ggu = gu(=z)$ … (5)

And $d(fu,ffu) = d(fu,gfu)$ (since fu = gu)

$\leq \varphi\ (d(hu, hfu) + \gamma\,[d(fu,hu) + d(gfu, hfu)]\,)$ (by E_2)
From **(1.3)**

$\|d(fu,ffu)\| \leq L(\|\varphi\ (d(fu, ffu) + \gamma\,[d(z,z) + d(gfu, gfu)]\,)\|)$ (since hu = fu = gu = z)

$\leq L(\varphi\,\|d(fu, ffu)\|) < \|d(fu, ffu)\|$

Which is a contradiction.

$\Rightarrow \quad \| d(fu, ffu) \| = 0$

$\Rightarrow \quad d(fu, ffu) = 0$

$\Rightarrow \quad ffu = fu\ (=z).$ … (6)

From (4), (5) and (6) it follows that the mappings f, g, and h have a common fixed point.

Uniqueness

Let z and z_1 be two distinct common fixed points of mappings f, g and h. From (E_2)

$$d(z, z_1) = d(fz,gz_1) \leq \varphi\ (d(hz,hz_1) + \gamma\,[d(fz,hz) + d(\ gz_1,h\ z_1)])$$

$$\leq \varphi\ (d(z,z_1) + \gamma\,[d(z,z) + d(\ z_1, z_1)])$$

$$\leq \varphi\ d(z,z_1)$$

$$< d(z,z_1),\ \text{a contradiction.}$$

$\Rightarrow \quad d(z,z_1) = 0.$

$\Rightarrow \quad z = z_1.$

Therefore f, g and h have a unique common fixed point.

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