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CHARACTERIZATION OF NORM-ATTAINABLE OPERATORS

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ABSTRACT

In this paper we characterize norm-attainable elementary operators, we show that $\delta_{P,Q}$ is norm-attainable if both P and Q are norm-attainable and $\delta_{P,Q}$ is norm-attainable $\delta_{P,Q}$ if is normally represented.

Classification: *46C15.*

Keywords: *norm-attainability, normally represented, nonseparable Hilbert spaces.*

1. INTRODUCTION

The norm of elementary operators has been a subject of many papers in operator theory but still it remains interesting to many mathematicians because calculating these norms requires finding a formula that involves their coefficients. Up-to- date, there is no known formular for calculating the norm of an arbitrary elementary operator acting on a general Banach algebra. About the discussion of the norms of elementary operators one can trace back the work of Stampfli [15]. Properties of elementary operators have also been investigated under variety of aspects. Some interesting results about the Spectra, numerical ranges, boundedness, orthogonality and norms have been obtained. On our paper we have given properties of norm-attainable operators. Let H be an infinite dimensional complex nonseparable Hilbert space and ENA(H) be the set of all norm-attainable elementary operators. Let $T: NA(H) \rightarrow NA(H)$ be defined by $T(X) = \sum_{i=1}^{n} P_i X Q_i$ for all $X \in NA(H)$ where$P_i\ Q_i$ are fixed in $NA(H)$. We have the following examples of elementary operators.

(i) the inner derivations $\delta_P = PX - XP$
(ii) the generalized derivation $\delta_{P,Q} = PX - XQ$
(iii) the basic elementary operator $M_{P,Q} = PXQ$
(iv) the Jordan elementary operator $U_{P,Q} = PXQ + QXP$. An operator $A \in B(H)$ is said to be norm-attainable if there exists a unit vector $x \in H$ such that $\|Ax\| = \|A\|$.

Definition 1.1 A normed space in which every cauchy sequence is convergent is called a complete normed space or Banach space.

Definition 1.2 A norm or length function on a vector space X is a nonnegative real valued functions $\|\|: X \rightarrow \mathbf{R}$ (real number) satisfying the following axioms:

(i) $\|x\| \geq 0\ \forall x \in X$.
(ii) $\|x\| = 0$ if and only if $x = 0\ \forall\ x \in X$.
(iii) $\|\lambda x\| = |\lambda|\|x\|\ \forall x \in X$ and λ is scalar.
(iv) $\|x + y\| \leq \|x\| + \|y\| \forall\ x, y \in X$.

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Definition 1.3 Let X and Y be two normed spaces. A mapping T: X → Y is called a linear operator if:
(i) $T(x + y) = T(x) + T(y)$ ∀ x, y ∈ X.
(ii) $T(\alpha X) = \alpha T(x)$ ∀ x ∈ X and complex number α.
(iii) $T(\alpha X + \beta y) = \alpha T(x) + \beta T(y)$ ∀ x,y ∈ X and complex numbers α and β.
(iv) T is bounded if there exists a constant K > 0 such that $\|Tx\| \leq K\|x\|$ ∀ x ∈ X.

Definition 1.4 For any two elements x and y belonging to an inner product space, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Definition 1.5 A normed space in which every cauchy sequence is convergent is called a complete normed space or Banach space.

Theorem 1.6 A Banach space is a Hilbert space if and only if its norm satisfies the parallelogram law.

Definition 1.7: A Hilbert space H is a complete inner product space.

Definition 1.8: An operator A∈B(H) is said to be norm-attainable if there exists a unit vector x ∈ H such that $\|Ax\| = \|A\|$.

Lemma 1.9: For an operator T∈B(H), the operator T is norm-attainable if and only if its adjoint T* is norm-attainable.

Definition 1.10: Let T: $H \rightarrow H$ the adjoint of T is T*: $H \rightarrow H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all x, y∈ H.

Definition 1.11: An operator T∈B(H) is said to be a projection if T^2 = T and self adjoint if T = T*.

Definition 1.12: A necessary and sufficient condition for an operator T to be normal is that $\|Tx\| = \|T^*x\|$ for any vector x ∈ H .

Definition 1.13: An operator T∈B(H) is said to be normal if it commutes with its adjoint, that is TT* = T*T and unitary if

Example 1.14: Let T : $X \rightarrow X$ be given by T= 2iI where I is the identity operator. Then T is normal since TT* = T*T= I.

$$\begin{aligned} TT^* &= (2iI)(2iI)^* \\ &= (2Ii)(-2iI) \\ &= -4i^2 I \\ &= 4I \end{aligned}$$

and

$$\begin{aligned} T^*T &= (2iI)^*(2iI) \\ &= (-2Ii)(2iI) \\ &= -4i^2 I \\ &= 4I \end{aligned}$$

This implies that TT* = T*T.

Example 1.15: Let T: X −→ X be given by T = $\begin{pmatrix} -3 & 1 \\ 5 & -2 \end{pmatrix}$

To show it is unitary we show that TT* = T*T = I where $T^* = T^{-1}$ then then we have

$$\begin{aligned} TT^* &= \begin{pmatrix} -3 & 1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -5 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

Also we have

$$\begin{aligned} T^*T &= \begin{pmatrix} -2 & -1 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 5 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

This means that TT* = T*T = I which implies that T is unitary.

Definition 1.16: A bouned operator U is called isometry if $\|Ux\| = \|x\|$ for all $x \in H$. Uis called partial isometry when restricted to $(Ker\ U)^{\perp}$.

Definition 1.17: An operator T is normaloid if $\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$

Definition 1.18: An operator T is hyponormal if $\|T^*x\| \leq \|Tx\|$ for every $x \in H$.

Definition 1.19: An operator T is quasinormal if T^*T commutes with T. That is, $(T^*T)T = T(T^*T)$

Definition 1.20: An operator T is posinormal if there exists a positive operator $P \in B(H)$ such that $TT^* = T^*PT$. Here P is an interrupter of T. The associated interrupter P must satisfy the following condition $\|P\| \geq 1$

Since $\|T^2\| = \|TT^*\| = \|T^*PT\| \leq \|T^*\|\|P\|\|T\| = \|P\|\ \|T\|^2$

Definition 1.21: A generalized derivation$\delta_{P,Q}$ on a C*-algebra Ω, is said to be norm-attainable if there exists a function $\varphi \in H^*$ such that $\|\delta_{P,Q}\varphi\| = \|\delta_{P,Q}\|$

Lemma 1.22: Lex X be an inner product space. Then for all x, y ∈ X we have $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$. Equality holds if and only if x and y are linearly dependent. Thus the Cauchy-Schwarz Inequality can be written as

$$\langle x, y\rangle \leq \|x\|\|y\| \tag{1}$$

2. RESULTS AND DISCUSSION

In this section we start by stating some Lemmas (whose proofs are easy to show) that will apply to prove other lemmas.

Lemma 2.1: Let H be an infinite dimensional complex nonseparable Hilbert space and T : H →H. T is norm-attainable if it is self adjoint.

Lemma 2.2: Let H be an infinite dimensional complex nonseparable Hilbert space and T : H →H. T is norm-attainable if it is normal.

Lemma 2.3: Let H be an infinite dimensional complex nonseparable Hilbert space and T : H →H. T is norm-attainable if it is unitary.

Lemma 2.4: Let H be an infinite dimensional complex nonseparable Hilbert space and T : H →H. T is norm-attainable if it is an isometry.

Lemma 2.5: Let H be an infinite dimensional complex nonseparable Hilbert space and T :. H →H T is norm-attainable if it is a paranormal.

Theorem 2.6 If T ∈ NA(H) then the following are equivalent:
(i) T is unitary.
(ii) T is isometry.
(iii) T is paranormal.

Proof:
(i)⇒ (*ii*)
Suppose *T* is unitary, then $T^*T = TT^* = I$ and from $T^*T = I$ we have for each $\zeta \in H$,

$$\|\zeta\|^2 = \langle\zeta,\zeta\rangle = \langle I\zeta,\zeta\rangle = \langle T^*T\zeta,\zeta\rangle = \langle T\zeta,T\zeta\rangle = \|T\zeta\|^2$$

Getting the positive square on both sides we get $\|\zeta\| = \|T\zeta\|$ implying that T is isometry. Similary for $TT^* = I$ we have,

$$\|\zeta\|^2 = \langle\zeta,\zeta\rangle = \langle I\zeta,\zeta\rangle = \langle T^*T\zeta,\zeta\rangle = \langle T^*\zeta,T^*\zeta\rangle = \|T^*\zeta\|^2$$

Getting the positive square on both sides we get $\|\zeta\| = \|T^*\zeta\|$ implying that T* is isometry.

(*ii*)⇒ (*iii*)
Suppose *T* is isometry, then $\|T\zeta\| = \|\zeta\|$ which implies that

$$\|T\zeta\|^2 = \|\zeta\|^2 = \langle\zeta,\zeta\rangle = \langle I\zeta,\zeta\rangle = \langle T^*T\zeta,\zeta\rangle \leq \|T^*T\zeta\|\|\zeta\| \leq \|T^*T\zeta\| \leq \|T^*T\zeta\| \leq \|T^2\zeta\|$$

which implies that *T* is paranormal.

$(iii) \Rightarrow (i)$
Suppose T is paranormal then $\| T\zeta \|^2 \geq \| T^2 \zeta \| \geq \langle T^*T\zeta, \zeta\rangle$

This implies that $\| T\zeta \|^2 = \langle T^*T\zeta, \zeta\rangle \leq \|T^*T\zeta\| \| \zeta \| \leq \|T^*\| \|T\zeta\| \| \zeta \|$

This implies $\| T\zeta \| \leq \|T^*\| \| \zeta \|$ which implies that $\| T \| \leq \|T^*\|$

Applying this relation to T^* we have $\| T^*\| \leq \|(T^*)^*\| \leq \|T\|$ which implies that $\| T \| = \|T^*\|$ and therefore by left multiplication with T and by right multiplication of T we get $\| TT \| = \|T^*T\| = \| T T^*\| = I$. Thus unitary.

Lemma 2.7: Let H be an infinite dimensional complex nonseparable Hilbert space and B(H) the algebra of all bounded linear operators on H. Let δ: B(H) →B(H) defined by $\delta_P(X) = PX - XP$. Then δ_P is norm-attainable if P is norm-attainable.

Proof: Let P be norm-attainable, we need to show that δ_P is norm-attainable. By supremum norm we have:

$$\| \delta_P (X) \| = \sup\{\|PX - XP\| \text{ and } \|PX\| = \|P\| , \|X\| = 1\}$$

Since P is norm-attainable, there exists a unit sequence $\{s\}_n$ such that $\|s_n\| = 1, \|Ps_n\| \to \|P\|$. Set $Ps_n = \lambda_n s_n + \mu_n t_n$, where $\langle s_n, t_n\rangle = 0$ and $\|t_n\| = 1$.

Set $U_n s_n = s_n$ and $U_n t_n = -t_n$. Then

$$\begin{aligned}\|(P U_n - U_n P)s_n\|^2 &= \|P U_n s_n - U_n P s_n \|^2 = \|P s_n - U_n(\lambda_n s_n + \mu_n t_n) \|^2\\ &= \| P s_n - U_n\lambda_n s_n - U_n\mu_n t_n) \|^2 = \|P s_n - \lambda_n s_n + \mu_n t_n) \|^2\\ &= \| \mu_n s_n + \mu_n s_n \|^2 = \| 2 \mu_n t_n \|^2 = 4\| \mu_n t_n \|^2\\ &= 4 \|P s_n - \lambda_n s_n \|^2 \geq 4 (\|P s_n \|^2 - \| \lambda_n s_n \|^2)\\ &\geq 4 (\|P \|^2 - \| \lambda_n \|^2)\end{aligned}$$

Allowing $\lambda_n \to 0$ as $n \to \infty$ we get, $\|(P U_n - U_n P)s_n\|^2 \geq 4 \|P \|^2$

Taking the square root on both sides we get $\|(P U_n s_n - U_n P)s_n\| \geq 2 \|P \|$

which implies that $\| \delta_P (X) \| \geq 2 \|P \|$

Since $\| \delta_P (X) \| \leq 2\|P\|$ for any P, sufficiency is proved and this implies that

$$\| \delta_P (X) \| = 2\|P\|.$$

Since $\langle Ps, s\rangle = 0$, then via results in [[15], Theorem 1] we have

$\| \delta_P (X) \| = \|PX - XP\| = 2\|P\| = \| \delta_P \|$ which means that δ_P is norm-attainable.

Lemma 2.8: *Let H be an infinite dimensional complex nonseparable Hilbert space and B(H) the algebra of all bounded linear operators on H. Let δ: B(H) → B(H) defined by $\delta_P (X) = PX - XP$. Then δ_P is norm-attainable if and only if δ_P is normal.*

Proof: Suppose that δ_P is norm-attainable, then by Lemma 2.2, δ_P is normal.

Conversely suppose that δ_P is normal then for any an operator $X \in H$, with $\|X\| = 1$ and by Lemma 2.1, we have

$$\| \delta_P^* \delta_P (X) \| = \|\delta^2{}_P \delta_P (X) \|.$$

Let $Y = \frac{\delta_P (X)}{\| \delta_P \|}$ then Y is an operator such that $\|Y \| = 1$ and hence

$$\| \delta_P^* Y\| = \| \delta_P^* \frac{\delta_P (X)}{\| \delta_P \|} \| = \frac{1}{\| \delta_P \|} \| \delta_P^* \delta_P (X)\| = \frac{1}{\| \delta_P \|} \|\delta^2{}_P (X) \| = \| \delta_P (X) \| = \| \delta_P\|$$

But by Lemma 2.1 we have $\| \delta_P^* Y\| = \| \delta_P\|$ this implying that $\| \delta_P (X) \| = \| \delta_P\|$ this means that is δ_P norm-attainable.

Lemma 2.9: Let H be an infinite dimensional complex nonseparable Hilbert space and B(H) the algebra of all bounded linear operators on H. Let δ : B(H) → B(H) defined by $\delta_P (X) = PX - XP$. Then δ_P is norm-attainable if δ_P is normally represented.

Proof: To show that δ_P is normally represented it is equivalent in showing that for $\delta_P \neq 0$ is norm-attainable if its adjoint δ_P^* is norm-attainable. Let $\delta_P \in E[NA(H)]$ be norm-attainable, then there exists an operator $X \in H$ with $\|X\| = 1$ such that $\| \delta_P X\| = \|\delta_P \|$. That is, That is , $\delta_P^* \delta_P (X) = \|\delta_P \|^2 X$

Let, Y = $\frac{\delta_P(X)}{\|\delta_P\|}$ then Y is an operator such that $\|Y\| = 1$ and hence $\|\delta_P^* Y\| = \|\delta_P\|$. but δ_P is self adjoint, then

$$\|\delta_P^* Y\| = \|\delta_P\| = \|\delta_P^*\|$$

This implies that $\|\delta_P^* Y\| = \|\delta_P\| = \|\delta_P^*\|$

Thus δ_P^* is norm-attainable implying that is δ_P normally represented.

Lemma 2.10: Let H be an infinite dimensional complex nonseparable Hilbert space and B(H) the algebra of all bounded linear operators on H. Let δ: B(H) → B(H) defined by $\delta_{P,Q}$ (X) = PX – XQ. Then $\delta_{P,Q}$ is norm-attainable if both P and Q are norm-attainable.

Proof: By Cauchy Schwarz inequality in Lemma 1.22, $|\langle Q\zeta,\zeta\rangle| \le \|Q\zeta\|\|\zeta\|$.

But if Qζ and ζ are linearly dependent the equality holds, that is $|\langle Q\zeta,\zeta\rangle| = \|\zeta Q\|\|\zeta\|$.

Suppose Qζ and ζ are linearly independent, that is, $Q\zeta = \varphi\|Q\|\zeta$, then it is true that $|\varphi| = 1$ and $|\langle P\zeta,\zeta\rangle| = \|P\|$.
It follows that $|\langle P\eta,\eta\rangle| = \|P\|$,

which implies that $P\eta = \psi \|Q\| \eta$ and $\|\psi\| = 1$ and by [[11], Theorem 3.4] we have, $\langle\frac{P\eta}{\|A\|},\eta\rangle = \psi = -\langle\frac{Q\zeta}{\|Q\|},\zeta\rangle$

Defining X as $X : \zeta \to \eta$ we have $\|X\| = 1$ and $(PX - XQ)\eta = \varphi(\|P\| + \|Q\|)\zeta$

which implies that $\|PX- XQ\| = \|(PX- XQ)\eta\| = \|P\| + \|Q\|$

By [[15], Theorem 1] we have

$$\|\delta_{P,Q}(X)\| = \|(PX - XQ)\| = \|P\| + \|Q\| = \|\delta_{P,Q}\|,$$

which implies that $\delta_{P,Q}$ is norm-attainable.

Lemma 2.11: Let H be an infinite dimensional complex nonseparable Hilbert space and B(H) the algebra of all bounded linear operators on H. Let Let δ : B(H) → B(H) defined by $\delta_{P,Q}$ (X) = PX – XQ. Then $\delta_{P,Q}$ is norm-attainable if is $\delta_{P,Q}$ normally represented.

Proof: To show that $\delta_{P,Q}$ is normally represented it is in enough to show that for $\delta_{PQ} \neq 0$ is norm-attainable if its adjoint $\delta^*_{P,Q}$ is norm-attainable. Let $\delta_{PQ} \in E[NA(H)]$ be norm-attainable, then there exists an operator $S \in NA(H)$ with $\|S\| = 1$, $\|\delta_{PQ}S\| = \|\delta_{PQ}\|$.

That is, $\delta_{P,Q}^* \delta_{PQ}S = \|\delta_{PQ}\|S$.

.

Let T = $\frac{\delta_{PQ}S}{\|\delta_{PQ}\|}$, then T is a vector such that $\|T\| = 1$ and hence

$$\|\delta_{P,Q}^* T\| = \|\delta_{PQ}\|.$$

but δ_{PQ} is self adjoint, then

$$\|\delta_{P,Q}^* T\| = \|\delta_{PQ}\|.$$
$$= \|\delta_{P,Q}^*\|.$$

This implies that,

$$\|\delta_{P,Q}^* T\| = \|\delta_{PQ}\|.$$

Thus $\delta_{P,Q}^*$ is norm-attainable implying that δ_{PQ} is normally represented.

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