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GENERALIZED DERIVATIONS ON SEMIPRIME NEAR-RINGS

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ABSTRACT

Let N be a semiprime left near-ring and α any mapping on N. A mapping $F: N \to N$ is a generalized derivation and $f: N \to N$ is a derivation. In this paper our main motive is to study the commutativity of semiprime near-rings and the nature of mappings.

Keywords: *Near-ring, Semiprime near-ring, derivation, generalized derivation, ideal.*

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1. INTRODUCTION

Throughout this paper, $Z(N)$ will denote the multiplicative center of N. Numerous outcomes in literature indicate how the worldwide structure of a near-ring N is often tightly connected to the behavior of additive mappings characterized on N. More recently several authors consider similar situation in the case the derivation d is replaced by a generalized derivation. Recently, there has been a lot of work concerning commutativity of prime and semiprime rings admitting suitably constrained derivations and generalized derivations [5]. Obviously, every derivation is a generalized derivation but the converse need not be true in general [6]. In this paper, we have proved comparable results of [6] for near-rings.

2. PRELIMINARIES

In this section, we collect all basic concepts in near-rings mostly from A. Boua and L. Oukhtite [1], H. E. Heatherly [2], G. Pilz [3], Mehsin Jabel Atteya, Dalal Jbrahee Rasen [4] and M. Samman, L. outkhtite, A. Boua [5] which are required for our study.

Definition 2.1: A **left near-ring** is a set N together with two binary operations "$+$" and "$\cdot$" such that

a) $(N, +)$ is a group (not necessarily abelian)
b) $(N, \cdot)$ is a semigroup
c) $\forall\, n_1, n_2, n_3 \in N: n_1.(n_2 + n_3) = n_1.n_2 + n_1 n_3$.

Definition 2.2: An additive endomorphism D of N is called a **derivation** on N if $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$.

Definition 2.3: An additive mapping $F: N \to N$ is said to be a right (resp., left) generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ (resp., $F(xy) = d(x)y + xF(y)$), for all $x, y \in N$, and $F: N \to N$ is said to be a **generalized derivation** with associated derivation d on F if it is both a right and left generalized derivation on N with associated derivation d.

Definition 2.4: A near-ring N is said to be **semiprime near-ring** if $xNx = \{0\}$ for $x \in N$ implies $x = 0$.

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Definition 2.5: For any $x, y \in N$, $[x, y] = xy - yx$ will denote the **commutator** and $(x \circ y) = xy + yx$ will denote the **anti-commutator**.

For any $x, y, z \in N$, the following identities hold:
i) $[x, yz] = y[x, z] + [x, y]z$
ii) $[xy, z] = x[y, z] + [x, z]y$

Definition 2.6: A normal subgroup I of $(N, +)$ is called an **ideal** of N $(I \trianglelefteq N)$ if
$\alpha)\ IN \subseteq I$
$\beta)\ \forall\, n, n' \in N\ \ \forall\, i \in I : n(n' + i) - nn' \in I.$

Normal subgroups R of $(N, +)$ with $\alpha)$ are called right ideals of N $(R \trianglelefteq_r N)$, while normal subgroups L of $(N, +)$ with $\beta)$ are said to be left ideals of N $(L \trianglelefteq_l N)$.

Definition 2.7: A **distributive near-ring** is a near-ring satisfying both distributive laws.

Definition 2.8: The symbol $Z(N)$ will represent the **multiplicative center of *N***, that is, $Z(N) = \{x \in N \,/\, xy = yx$ for all $y \in N\}$.

3. MAIN RESULTS

Theorem 3.1: Let N be an additive abelian semiprime left near-ring and I a non-zero ideal of N. Suppose that F is a left generalized derivation associated with the mapping f on N. If $F[x, y] - [x, f(y)] = 0$ for all $x, y \in I$, then $[f(y), y] = 0$ for all $y \in I$.

Proof:
Assume that
$$F[x, y] - [x, f(y)] = 0 \text{ for all } x, y \in I$$

Replacing x by yx,
$$F[yx, y] - [yx, f(y)] = 0$$
$$\Rightarrow yF[x, y] + f(y)[x, y] - yxf(y) + f(y)yx = 0$$

Adding and subtracting $y[x, f(y)]$,
$$yF[x, y] - y[x, f(y)] + f(y)[x, y] + y[x, f(y)] - yxf(y) + f(y)yx = 0$$
$$\Rightarrow f(y)xy - yf(y)x = 0 \tag{1}$$

Replace x by $xf(y)$,
$$f(y)xf(y)y - yf(y)xf(y) = 0 \tag{2}$$

Post multiply (1) by $f(y)$,
$$f(y)xyf(y) - yf(y)xf(y) = 0 \tag{3}$$

Subtracting (3) from (2),
$$f(y)x\big(f(y)y - yf(y)\big) = 0$$
$$\Rightarrow [f(y), y]x[f(y), y] = 0 \text{ for all } x, y \in I$$

Since N is semiprime near-ring, $[f(y), y] = 0$ for all $y \in I$.

Theorem 3.2: Let N be an additive abelian semiprime left near-ring and I a non-zero ideal of N. Suppose that F is a left generalized derivation associated with the mapping f on N. If $F(x \circ y) - x \circ f(y) = 0$ for all $x, y \in I$, then $[f(y), y] = 0$ for all $y \in I$.

Proof:
Assume that
$$F(x \circ y) - x \circ f(y) = 0 \text{ for all } x, y \in I$$

On replacing x by yx,
$$F(yx \circ y) - yx \circ f(y) = 0$$
$$\Rightarrow yF(x \circ y) + f(y)(x \circ y) - yxf(y) - f(y)yx = 0$$

Adding and subtracting $y\big(x \circ f(y)\big)$ *and* using hypothesis, we have
$$f(y)xy + yf(y)x = 0 \tag{4}$$

Let $x = xf(y)$,

$$f(y)xf(y)y + yf(y)xf(y) = 0 \tag{5}$$

Post multiply (4) by $f(y)$,

$$f(y)xyf(y) + yf(y)xf(y) = 0 \tag{6}$$

Subtracting (6) from (5),

$$[f(y), y]x[f(y), y] = 0 \text{ for all } x, y \in I$$

Since N is semiprime near-ring, we get $[f(y), y] = 0$ for all $y \in I$.

Theorem 3.3: Let N be an additive abelian semiprime left near-ring and I a non-zero ideal of N. Suppose that G and F are two left generalized derivations associated with the mappings g and f respectively on N.

If $G(xy) \pm [x, F(y)] \pm xy = 0$ for all $x, y \in I$, then $g(x) \in Z(N)$ for all $x \in I$.

Proof:
Case (i):
Assume that

$$G(xy) + [x, F(y)] + xy = 0 \text{ for all } x, y \in I$$

Substituting zx for x,

$$G\big((zx)y\big) + [zx, F(y)] + zxy = 0 \text{ for all } x, y, z \in I$$

$$g(z)xy + [z, F(y)]x = 0 \qquad \text{for all } x, y, z \in I \tag{7}$$

Substituting xt for x,

$$g(z)xty + [z, F(y)]xt = 0 \text{ for all } x, y, z \in I \text{ and } t \in N \tag{8}$$

Right multiply (7) by t,

$$g(z)xyt + [z, F(y)]xt = 0 \text{ for all } x, y, z \in I \text{ and } t \in N \tag{9}$$

Subtracting (8) from (9), we get

$$[y, g(z)]x[y, t] = 0 \text{ for all } x, y, z \in I \text{ and } t \in N$$

Let $t = g(z)$ and since N is a semiprime near-ring, $[y, g(z)] = 0$ for all $y, z \in I$

Substituting yr in place of y,

$$y[r, g(z)] = 0 \text{ for all } y, z \in I \text{ and } r \in N$$

Again by semiprimeness of N, $g(z) \in Z(N)$ for all $z \in I$. Hence $g(x) \in Z(N)$ for all $x \in I$.

Case (ii):
Assume that $\quad G(xy) - [x, F(y)] - xy = 0$ for all $x, y \in I$

Substituting zx for x,

$$g(z)xy - [z, F(y)]x = 0 \text{ for all } x, y, z \in I \tag{10}$$

On replacing x by xt in (10),

$$g(z)xty - [z, F(y)]xt = 0 \text{ for all } x, y, z \in I \text{ and } t \in N \tag{11}$$

Post multiply (10) by t and subtract (11)

$$g(z)x[y, t] = 0 \text{ for all } x, y, z \in I \text{ and } t \in N$$

Further, proceeding as in the proof of case (i), we have $g(x) \in Z(N)$ for all $x \in I$.

Using similar approach, the same result holds for $G(xy) \pm [x, F(y)] + xy = 0$ for all $x, y \in I$.

Theorem 3.4: Let N be an additive abelian semiprime left near-ring and I a non-zero ideal of N. Suppose that G and F are two left generalized derivations associated with the mappings g and f respectively on N. If $G(xy) \pm x \circ F(y) \pm xy = 0$ for all $x, y \in I$, then $g(x) \in Z(N)$ for all $x \in I$.

Proof:
Assume that $\quad G(xy) - x \circ F(y) - xy = 0$ for all $x, y \in I$

Substituting zx in place of x,

$$g(z)xy + [z, F(y)]x = 0 \quad (12)$$

On replacing x by xt in (12),

$$g(z)xty + [z, F(y)]xt = 0 \text{ for all } x, y, z \in I \text{ and } t \in N \quad (13)$$

Post multiply (12) by t and subtract (13)

$$[y, g(z)]x[y, t] = 0 \text{ for all } x, y, z \in I \text{ and } t \in N$$

Let $t = g(z)$ *and* since N is semiprime, $[y, g(z)] = 0$ for all $y, z \in I$.

Substituting yr in place of y, we get

$$y[r, g(z)] = 0 \text{ for all } y, z \in I \text{ and } r \in N$$

Again by semiprimeness of N, $g(z) \in Z(N)$ for all $z \in I$. Hence $g(x) \in Z(N)$ for all $x \in I$.

By using similar approach, the same result holds for $G(xy) + x \circ F(y) \pm xy = 0$ for all $x, y \in I$.

Theorem 3.5: Let N be an additive abelian semiprime left near-ring and I a non-zero ideal of N. Suppose that G and F are two left generalized derivations associated with the mappings g and f respectively on N. If one of the following holds:

i) $G(xy) \pm x \circ F(y) \pm [x, y] = 0$;
ii) $G(xy) \pm x \circ F(y) \pm x \circ y = 0$;
iii) $G(xy) \pm x \circ F(y) = 0$

for all $x, y \in I$, then $g(x) \in Z(N)$ for all $x \in I$.

Proof:

i) Assume that $G(xy) \pm x \circ F(y) \pm [x, y] = 0$ for all $x, y \in I$

Assume further that $G(xy) - x \circ F(y) - [x, y] = 0$ for all $x, y \in I$ (14)

Substituting zx instead of x in (14),

$$g(z)xy + [z, F(y)]x - [z, y]x = 0 \text{ for all } x, y, z \in I \quad (15)$$

Replace x by xt, $t \in N$ in (15)

$$g(z)xty + [z, F(y)]xt - [z, y]xt = 0 \quad (16)$$

Post multiply (15) by t and subtract (16)

$$[y, g(z)]x[y, t] = 0 \text{ for all } x, y, z \in I \text{ and } t \in N$$

Putting $t = g(z)$, then $[y, g(z)]x[y, g(z)] = 0$ for all $x, y, z \in I$

Since N is semiprime, we have $[y, g(z)] = 0$ for all $y, z \in I$

Substituting yr in place of y, we get

$$y[r, g(z)] = 0 \text{ for all } y, z \in I \text{ and } r \in N$$

Again by semiprimeness of N, $g(z) \in Z(N)$ for all $z \in I$. Hence $g(x) \in Z(N)$ for all $x \in I$.

Using similar approach, the same result holds for $G(xy) + x \circ F(y) \pm [x, y] = 0$ and $G(xy) - x \circ F(y) + [x, y] = 0$ for all $x, y \in I$.

ii) Assume that $G(xy) \pm x \circ F(y) \pm x \circ y = 0$ for all $x, y \in I$

Let $G(xy) - x \circ F(y) - x \circ y = 0$ for all $x, y \in I$

On replacing zx in place of x,

$$g(z)xy + [z, F(y)]x + [z, y]x = 0 \text{ for all } x, y, z \in I$$

Further, proceeding as in the proof of part (i), we have $g(x) \in Z(N)$ for all $x \in I$.
By using similar argument we can prove the same result for the following cases $G(xy) + x \circ F(y) \pm x \circ y = 0$ and $G(xy) - x \circ F(y) + x \circ y = 0$ for all $x, y \in I$.

iii) Assume that $G(xy) \pm x \circ F(y) = 0$ for all $x, y \in I$

Suppose that $G(xy) - x \circ F(y) = 0$ for all $x, y \in I$

Substituting zx instead of x,

$$g(z)xy + [z, F(y)]x = 0 \text{ for all } x, y, z \in I \tag{17}$$

Now replacing xt in place of x,

$$g(z)xty + [z, F(y)]xt = 0 \text{ for all } x, y, z \in I \text{ and } t \in N \tag{18}$$

Right multiply (17) by t and subtract (18)

$$[y, g(z)]x[y, t] = 0 \text{ for all } x, y, z \in I \text{ and } t \in N$$

Putting $t = g(z)$ and since N is semiprime, we have

$$[y, g(z)] = 0 \text{ for all } y, z \in I$$

Substituting yr in place of y, we get

$$y[r, g(z)] = 0 \text{ for all } y, z \in I \text{ and } r \in N$$

Again by semiprimeness of N, $g(z) \in Z(N)$ for all $z \in I$. Hence $g(x) \in Z(N)$ for all $x \in I$.

By using similar approach, the same result holds for $G(xy) + x \circ F(y) = 0$ for all $x, y \in I$.

Theorem 3.6: Let N be an additive abelian semiprime distributive near-ring and I a non-zero ideal and α any mapping on N. Suppose that G and F are two left generalized derivations associated with the mappings g and f respectively on N. If $G(xy) + F(x)F(y) \pm [x, \alpha(y)] = 0$ for all $x, y \in I$, then $[f(x), x] = 0$ and $[g(x), x] = 0$ for all $x \in I$. Moreover, if α is an automorphism, then N is commutative.

Proof:

Assume that $G(xy) + F(x)F(y) - [x, \alpha(y)] = 0$ for all $x, y \in I$

Substituting zx for x,

$$g(z)xy + f(z)xF(y) - [z, \alpha(y)]x = 0 \qquad \text{for all } x, y, z \in I \tag{19}$$

Substituting xr in place of x, we obtain

$$g(z)xry + f(z)xrF(y) - [z, \alpha(y)]xr = 0 \tag{20}$$

Substituting ry in place of y in (20) and subtract from (20)

$$-f(z)xf(r)y + [z, \alpha(ry)]x - [z, \alpha(y)]xr = 0 \tag{21}$$

On replacing x by xr in (21),

$$-f(z)xrf(r)y + [z, \alpha(ry)]xr - [z, \alpha(y)]xr^2 = 0 \tag{22}$$

Post multiply (21) by r and subtract from (22)

$$f(z)x[f(r)y, r] = 0 \tag{23}$$

Replace x by xzy in (23),

$$f(z)xzy[f(r)y, r] = 0 \tag{24}$$

Replace x by xy in (23),

$$f(z)xy[f(r)y, r] = 0 \tag{25}$$

Pre multiply (25) by z and subtract from (24)

$$[f(z)x, z]y[f(r)y, r] = 0$$

Let $r = z$ and $y = x$ and since N is semiprime,

$$f(z)[x, z] + [f(z), z]x = 0 \tag{26}$$

Substituting xr for x in (26),

$$f(z)x[r, z] + f(z)[x, z]r + [f(z), z]xr = 0 \tag{27}$$

Post multiply (26) by r and subtract from (27)

$$[f(z),z]x[f(z),z]=0$$

Since N is semiprime, $[f(z),z]=0$ for all $z\in I$

$$\Rightarrow [f(x),x]=0 \text{ for all } x\in I.$$

Again replacing x by zx in (19),

$$g(z)zxy+f(z)zxF(y)-[z,\alpha(y)]zx=0 \tag{28}$$

Pre multiply (19) by z and subtract from (28)

$$[g(z),z]xy+\big[z,[z,\alpha(y)]\big]x=0 \tag{29}$$

Putting $x=xr$,

$$[g(z),z]xry+\big[z,[z,\alpha(y)]\big]xr=0 \tag{30}$$

Post multiply (29) by r and subtract (30)

$$[g(z),z]x[y,r]=0 \text{ for all } x,y,z\in I \text{ and } r\in N$$

Let $r=g(z)$ and $y=z$,

$$[g(z),z]=0 \text{ for all } z\in I \tag{31}$$

Hence $[f(x),x]=0$ and $[g(x),x]=0$ for all $x\in I$.

Next, we assume the case, when α is an automorphism.

Applying (31) in (29), we have

$$\big[z,[z,\alpha(y)]\big]=0 \tag{32}$$

Linearizing this expression,

$$\big[x,[z,\alpha(y)]\big]+\big[z,[x,\alpha(y)]\big]=0 \tag{33}$$

On replacing xz for x,

$$\big(\big[x,[z,\alpha(y)]\big]+\big[z,[x,\alpha(y)]\big]\big)z+[z,x][z,\alpha(y)]=0$$

Using (33), we obtain

$$[z,x][z,\alpha(y)]=0 \text{ for all } x,y,z\in I$$

Substituting $\alpha(y)x$ for x,

$$[z,\alpha(y)]=0 \text{ for all } y,z\in I$$

Put $y=y\alpha^{-1}(r)$

$$\big[z,\alpha\big(y\alpha^{-1}(r)\big)\big]=0 \;\Rightarrow [z,r]=0 \text{ for all } z\in I \text{ and } r\in N$$

i.e. $I\subset Z(N)$.

Hence N is commutative.Using similar approach, the same result holds for $G(xy)+F(x)F(y)+[x,\alpha(y)]=0$ for all $x,y\in I$.

Using similar techniques with some necessary variations we can prove the following Theorem:

Theorem 3.7: Let N be an additive abelian semiprime distributive near-ring and I a non-zero ideal and α any mapping on N. Suppose that G and F are two left generalized derivations associated with the mappings g and f respectively on N. If $G(xy)-F(x)F(y)\pm[x,\alpha(y)]=0$ for all $x,y\in I$, then $[f(x),x]=0$ and $[g(x),x]=0$ for all $x\in I$. Moreover, if α is an automorphism, then N is commutative.

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