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HYERS - ULAM STABILITY OF DIFFERENCE EQUATIONS OF SECOND ORDER

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ABSTRACT

In this paper, we investigate the Hyers - Ulam stability of second order difference equations of the form

$$\Delta^2 y(n) + p\Delta y(n) + qy(n) = r(n)$$

and

$$\Delta^2 y(n) - p(n)\Delta y(n) + q(n)y(n) = r(n)$$

where $p, q \in R$ *and* $\{p(n)\}, \{q(n)\}, \{r(n)\}$ *are sequences of reals. Examples are provided to illustrate the main results.*

AMS Subject Classification: *39A45, 39B42*

Keywords and Phrases: *Hyers - Ulam stability, second order, difference equation.*

1. INTRODUCTION

In recent years, there has been a great interest in investigating the Hyers - Ulam stability of various types of functional equations. This problem was first raised by Ulam [15] concerning the stability of group homomorphism, and the answer was given by Hyers [3], we refer the reader to [13] for the exact definition of Hyers - Ulam stability. Since then, the stability problems of various functional equations has been studied by many authors, see [13] and the references contained therein.

After that, Ulam stability problem for functional equations was replaced by stability of differential equations and difference equations. The differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0$$

has the Hyers - Ulam stability, if for given $\varepsilon > 0, I$ be an open interval and for any function f satisfying the differential inequality

$$| a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_1(t)y'(t) + a_0(t)y(t) + h(t) | \leq \varepsilon$$

then there exists a solution $f_0(t)$ of the above equation such that

$$| f(t) - f_0(t) | \leq K(\varepsilon) \quad and \quad \lim_{\varepsilon \to 0} K(\varepsilon) = 0, \ t \in I.$$

In [9], the authors studied the Hyers - Ulam stability of second order differential equation of the form

$$y'' + \alpha y' + \beta y = 0$$

and

$$y'' + \alpha y' + \beta y = f(t)$$

where $\alpha, \beta \in \mathbb{R}$. The Hyers - Ulam stability of differential equations have been studied in many papers, see for example [4, 5, 6, 9, 10, 14], and the references cited therein. However only few results are reported in the literature regarding the Hyers - Ulam stability of difference equations, see [1, 2, 8, 11, 12, 14] and the references cited therein.

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The aim of this paper is to study the Hyers - Ulam stability of second order difference equations of the form

$$\Delta^2 y(n) + p\Delta y(n) + qy(n) = 0, \tag{1.1}$$

$$\Delta^2 y(n) + p\Delta y(n) + qy(n) = r(n), \tag{1.2}$$

and

$$\Delta^2 y(n) - p(n)\Delta y(n) + q(n)y(n) = r(n) \tag{1.3}$$

where $p, q \in \mathbb{R}$ and $\{p(n)\}, \{q(n)\}, \{r(n)\}$ are sequences of reals.

Definition 1.1: *The difference equation*

$$a_k(n)\Delta^k y(n) + a_{k-1}(n)\Delta^{k-1} y(n) + \ldots + a_1(n)\Delta y(n) + a_0(n)y(n) + b(n) = 0$$

has the Hyers - Ulam stability, if for given $\varepsilon > 0, I$ *be an open interval and for any real function* $f(n)$ *satisfying the inequality*

$$| a_k(n)\Delta^k y(n) + a_{k-1}(n)\Delta^{k-1} y(n) + \ldots + a_1(n)\Delta y(n) + a_0(n)y(n) + b(n) | \leq \varepsilon$$

then there exists a function $f_0(n)$ *of the above difference equation such that* $| f(n) - f_0(n) | \leq K(\varepsilon)$ *and* $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$ *for* $n \in I \subset N(0) = \{0, 1, 2, \ldots\}$.

Definition 1.2: *The difference equation (1.2) has the Hyers - Ulam stability if there exists a constant* $K > 0$ *with the property: for every* $\varepsilon > 0, y(n), r(n)$ *defined for* $n \in (a, b+1), 0 < a < b < \infty,$ *if*

$$| \Delta^2 y(n) + p\Delta y(n) + qy(n) - r(n) | \leq \varepsilon \tag{1.4}$$

then there exists some $z(n), n \in (a, b+1)$ *satisfying*

$$\Delta^2 z(n) + p\Delta z(n) + qz(n) = r(n)$$

such that $| y(n) - z(n) | \leq K\varepsilon$ *. We call such* K *as a Hyers - Ulam stability constant for equation (1.2).*

The results presented in this paper are new and complement to the results reported in the literature for difference equations.

2. STABILITY RESULTS

In this section, we study the Hyers - Ulam stability of equations (1.1), (1.2) and (1.3). We begin with the following theorem.

Theorem 2.1: *If the characteristic equation* $m^2 + (p-2)m + (q - p + 1) = 0$ *have two different positive roots, then the equation (1.1) has the Hyers - Ulam stability.*

Proof: Let $\varepsilon > 0$ and $y(n), n \in (a, b+1)$ be a solution of equation (1.1) satisfying the property

$$| \Delta^2 y(n) + p\Delta y(n) + qy(n) | \leq \varepsilon.$$

Let λ and μ be the two different positive roots of the characteristic equation. For $n \in (a, b+1)$, define $g(n) = \Delta y(n) - \lambda y(n)$. Then

$$\Delta g(n) = \Delta^2 y(n) - \lambda \Delta y(n)$$

and hence

$$| \Delta g(n) - \mu g(n) | = | \Delta^2 y(n) - \lambda \Delta y(n) - \mu \Delta y(n) + \lambda \mu y(n) |$$
$$= | \Delta^2 y(n) + p\Delta y(n) + qy(n) | \leq \varepsilon.$$

Thus, $g(n)$ satisfies the relation

$$-\varepsilon \leq \Delta g(n) - \mu g(n) \leq \varepsilon. \tag{2.1}$$

From (2.1), we have

$$-\varepsilon(1+\mu)^{-(n+1)} \leq (1+\mu)^{-(n+1)}[g(n+1) - (1+\mu)g(n)] \leq \varepsilon(1+\mu)^{-(n+1)}$$

that is,

$$-\varepsilon(1+\mu)^{-(n+1)} \le \Delta\Big((1+\mu)^{-n} g(n)\Big) \le \varepsilon(1+\mu)^{-(n+1)}. \tag{2.2}$$

Summing (2.2) from n to b, we obtain

$$-\varepsilon\sum_{j=n}^{b}(1+\mu)^{-(j+1)} \le \sum_{j=n}^{b}\Delta\Big((1+\mu)^{-j} g(j)\Big) \le \varepsilon\sum_{j=n}^{b}(1+\mu)^{-(j+1)}$$

which on simplification implies that

$$-\varepsilon\frac{(1+\mu)^{-n}}{\mu} \le (1+\mu)^{-(b+1)} g(b+1) - (1+\mu)^{-n} g(b) \le \varepsilon\frac{(1+\mu)^{-n}}{\mu}.$$

Hence

$$-\varepsilon_1 \le (1+\mu)^{-(b-n+1)} g(b+1) - g(n) \le \varepsilon_1$$

where $\varepsilon_1 = \dfrac{\varepsilon}{\mu}$. Let $z(n) = (1+\mu)^{-(b-n+1)} g(b+1)$. Then $\Delta z(n) - \mu z(n) = 0$. Now $|g(n) - z(n)| \le \varepsilon_1$ implies that

$$-\varepsilon_1 \le \Delta y(n) - \lambda y(n) - z(n) \le \varepsilon_1$$

and hence

$$-\varepsilon_1(1+\lambda)^{-(n+1)} \le \Delta(1+\lambda)^{-(n+1)}\big(y(n+1) - (1+\lambda)y(n) - z(n)\big) \le \varepsilon_1(1+\lambda)^{-(n+1)}.$$

Proceeding as above, one obtains

$$-\varepsilon_1\frac{(1+\lambda)^{-n}}{\lambda} \le (1+\lambda)^{-(b+1)} y(b+1) - (1+\lambda)^{-n} y(n) - \sum_{j=n}^{b}(1+\lambda)^{-(j+1)} z(j) \le \varepsilon_1\frac{(1+\lambda)^{-n}}{\lambda}$$

or

$$-\frac{\varepsilon_1}{\lambda} \le (1+\lambda)^{-(b-n+1)} y(b+1) - y(n) - (1+\lambda)^{n}\sum_{j=n}^{b}(1+\lambda)^{-(j+1)} z(j) \le \frac{\varepsilon_1}{\lambda}.$$

Define

$$u(n) = (1+\lambda)^{-(b-n+1)} y(b+1) - \sum_{j=n}^{b}(1+\lambda)^{-(j-n+1)} z(j)$$

then $|u(n) - y(n)| \le \dfrac{\varepsilon_1}{\lambda} = \dfrac{\varepsilon}{\lambda\mu}$. It is easy to see that $\Delta u(n) = \lambda u(n) + z(n)$ and hence

$$\begin{aligned}\Delta^2 u(n) &= \lambda\Delta u(n) + \Delta z(n)\\ &= \lambda\Delta u(n) + \mu z(n)\\ &= \lambda\Delta u(n) + \mu(\Delta u(n) - \lambda u(n))\\ &= (\lambda+\mu)\Delta u(n) - \lambda\mu u(n)\end{aligned}$$

or

$$\Delta^2 u(n) + p\Delta u(n) + q u(n) = 0.$$

Consequently, the equation (1.1) has the Hyers - Ulam stability with the stability constant $K = \dfrac{1}{\lambda\mu}$. This completes the proof.

Theorem 2.2: *Assume that the characteristic equation* $m^2 + (p-2)m + (q-p+1) = 0$ *have two different positive roots. If condition (1.4) holds, then the equation (1.2) has the Hyers - Ulam stability.*

Proof: Proceeding as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}|\Delta g(n) - \mu g(n) - r(n)| &= |\Delta^2 y(n) - \lambda\Delta y(n) - \mu\Delta y(n) + \lambda\mu y(n) - r(n)|\\ &= |\Delta^2 y(n) + p\Delta y(n) + q y(n) - r(n)| \le \varepsilon.\end{aligned}$$

Hence, $g(n)$ satisfies the relation

$$-\varepsilon \le \Delta g(n) - \mu g(n) - r(n) \le \varepsilon.$$

Similar to the proof of Theorem 2.1, we have

$$-\varepsilon(1+\mu)^{-(n+1)} \le \Delta\big((1+\mu)^{-n} g(n)\big) - (1+\mu)^{-(n+1)} r(n) \le \varepsilon(1+\mu)^{-(n+1)},$$

and we let $z(n) = (1+\mu)^{-(b-n+1)} g(b+1) - \sum_{j=n}^{b} (1+\mu)^{-(j-n+1)} r(j)$, then $z(n)$ satisfies the equation

$$\Delta z(n) - \mu z(n) - r(n) = 0,$$

and $|g(n) - z(n)| \le \varepsilon_1$. Using the same type of argument as in Theorem 2.1, one can show that there exists

$$u(n) = (1+\lambda)^{-(b-n+1)} g(b+1) - \sum_{j=n}^{b} (1+\lambda)^{-(j-n+1)} z(j)$$

such that $|u(n) - y(n)| \le \dfrac{\varepsilon}{\lambda\mu}$ and $u(n)$ satisfies the equation

$$\Delta^2 u(n) + p\Delta u(n) + qu(n) - r(n) = 0.$$

This completes the proof.

Next, we study the Hyers - Ulam stability of equation (1.3). For this, we need the following result.

Lemma 2.3: *Assume that* $0 \le \alpha(n) \le \alpha < \infty$ *for every* $n \in I$. *Then for* $n \in (a, b+1)$, *the equation*

$$\Delta y(n) - \alpha(n) y(n) - r(n) = 0 \qquad (2.3)$$

has the Hyers - Ulam stability.

Proof: Let $\varepsilon > 0$ and $y(n), n \in (a, b+1)$ be a solution of equation (2.3) satisfying the property

$$|\Delta y(n) - \alpha(n) y(n) - r(n)| \le \varepsilon$$

or

$$-\varepsilon \le y(n+1) - (1+\alpha(n)) y(n) - r(n) \le \varepsilon.$$

Therefore

$$-\varepsilon\left(\prod_{i=a}^{n}(1+\alpha(i))\right)^{-1} \le \big[y(n+1) - (1+\alpha(n))y(n) - r(n)\big]\left(\prod_{i=a}^{n}(1+\alpha(i))\right)^{-1}$$
$$\le \varepsilon\left(\prod_{i=a}^{n}(1+\alpha(i))\right)^{-1}$$

implies that

$$-\varepsilon\left(\prod_{i=a}^{n}(1+\alpha(i))\right)^{-1} \le \Delta\left(y(n)\left(\prod_{i=a}^{n}(1+\alpha(i))\right)^{-1}\right)$$
$$-r(n)\left(\prod_{i=a}^{n}(1+\alpha(i))\right)^{-1} \le \varepsilon\left(\prod_{i=a}^{n}(1+\alpha(i))\right)^{-1}.$$

Summing the last inequality from a to $n-1$, we have

$$-\varepsilon\sum_{j=a}^{n-1}\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1} \le \sum_{j=a}^{n-1}\Delta\left(y(j)\left(\prod_{i=a}^{j-1}(1+\alpha(i))\right)^{-1}\right)$$
$$-\sum_{j=a}^{n-1} r(j)\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1} \le \varepsilon\sum_{j=a}^{n-1}\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1}$$

and hence

$$-\varepsilon\sum_{j=a}^{n-1}\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1} \le y(n)\left(\prod_{i=a}^{n-1}(1+\alpha(i))\right)^{-1} - y(a)$$
$$-\sum_{j=a}^{n-1} r(j)\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1} \le \varepsilon\sum_{j=a}^{n-1}\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1}$$

where we have used the convention $\prod_{i=a}^{a-1}(1+\alpha(i))=1.$ Now $1+\alpha\geq 1$, we have

$$\sum_{j=a}^{n-1}\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1}=\frac{1}{1+\alpha(a)}+\frac{1}{(1+\alpha(a))(1+\alpha(a+1))}+\ldots+\frac{1}{(1+\alpha(a))\ldots(1+\alpha(n-1))}$$
$$\leq\frac{(1+\alpha)^{n}-1}{\alpha}\left(\prod_{i=a}^{n-1}(1+\alpha(i))\right)^{-1}$$
$$\leq\frac{(1+\alpha)^{b}-1}{\alpha}\left(\prod_{i=a}^{n-1}(1+\alpha(i))\right)^{-1}.$$

Using this in (??), we obtain

$$-\varepsilon\frac{(1+\alpha)^{b}-1}{\alpha}\leq y(n)-y(a)\left(\prod_{i=a}^{n-1}(1+\alpha(i))\right)$$
$$-\prod_{i=a}^{n-1}(1+\alpha(i))\sum_{j=a}^{n-1}r(j)\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1}\leq\varepsilon\frac{(1+\alpha)^{b}-1}{\alpha}.$$

For $n\in(a,b+1)$, if we define

$$z(n)=y(a)\left(\prod_{i=a}^{n-1}(1+\alpha(i))\right)+\prod_{i=a}^{n-1}(1+\alpha(i))\sum_{j=a}^{n-1}r(j)\left(\prod_{i=a}^{j}(1+\alpha(i))\right)^{-1}$$

then it is easy to see that

$$\Delta z(n)-\alpha(n)z(n)-r(n)=0\quad and\quad |y(n)-z(n)|\leq\frac{(1+\alpha)^{b}-1}{\alpha}\varepsilon.$$

Hence equation (2.3) has the Hyers - Ulam stability with stability constant $K=\frac{(1+\alpha)^{b}-1}{\alpha}$. This completes the proof.

Theorem 2.4: *Assume that* $\{p(n)\}$ *and* $\{q(n)\}$ *are positive real sequences for every* n. *If* $\{c(n)\}$ *is a particular solution of*

$$\Delta u(n)+u(n+1)u(n)-p(n)u(n)+q(n)=0 \tag{2.4}$$

such that $0<c(n)\leq c<\infty$ *and* $d(n)=\frac{q(n)+\Delta c(n)}{c(n)}\leq d<\infty$ *for* $n\in(a,b+1)$, *then equation (1.3) has the Hyers - Ulam stability.*

Proof: Let $\varepsilon>0$ and $y(n), n\in(a,b+1)$ be a solution of equation (1.3) satisfying the property

$$|\Delta^{2}y(n)-p(n)\Delta y(n)+q(n)y(n)-r(n)|\leq\varepsilon.$$

For $n\in(a,b+1)$, define $v(n)=\Delta y(n)-c(n)y(n)$. Then

$$|\Delta v(n)-d(n)v(n)-r(n)|=|\Delta^{2}y(n)-(c(n+1)+d(n))\Delta y(n)+(c(n)d(n)-\Delta c(n))y(n)-r(n)|$$
$$=|\Delta^{2}y(n)-p(n)\Delta y(n)+q(n)y(n)-r(n)|\leq\varepsilon$$

where we have used the fact that $\{c(n)\}$ is a particular solution of equation (2.4). From Lemma 2.3, it follows that

$$\Delta v(n)-d(n)v(n)-r(n)=0$$

has the Hyers - Ulam stability with the property that $|v(n)-w(n)|\leq\frac{\varepsilon}{d}$, where

$$w(n)=v(a)\left(\prod_{i=a}^{n-1}(1+d(i))\right)+\prod_{i=a}^{n-1}(1+d(i))\sum_{i=a}^{n-1}r(j)\left(\prod_{i=a}^{j}(1+d(i))\right)^{-1}.$$

Using $v(n)$ in $|v(n)-w(n)|\leq\frac{\varepsilon}{d}$, we have

$$|\Delta y(n)-c(n)y(n)-w(n)|\leq\frac{\varepsilon}{d}$$

for $n\in(a,b+1)$. Again using Lemma 2.3, we see that

$$\Delta y(n)-c(n)y(n)-w(n)=0$$

has the Hyers - Ulam stability with the property that $|y(n)-z(n)|\leq\frac{\varepsilon}{cd}$, where

$$z(n)=y(a)\left(\prod_{i=a}^{n-1}(1+c(i))\right)+\prod_{i=a}^{n-1}(1+c(i))\sum_{j=a}^{n-1}w(j)\left(\prod_{i=a}^{j}(1+c(i))\right)^{-1}.$$

Hence, equation (1.3) has the Hyers - Ulam stability with the stability constant $K=\frac{1}{cd}$. The proof is now complete.

3. EXAMPLE

In this section we provide two examples to illustrate the main results.

Example 3.1: *Consider the second order difference equation*

$$\Delta^2 y(n)+\frac{7}{6}\Delta y(n)+\frac{1}{3}y(n)=0,\ n\geq 1. \tag{3.1}$$

The characteristic equation is $m^2-\frac{7}{6}m+\frac{1}{3}=0$ and hence the characteristic roots are $\frac{1}{2}$ and $\frac{1}{3}$. Therefore by Theorem 2.1 the equation (3.1) has Hyers-Ulam stability with stability constant $K=6$.

Example 3.2: *Consider the second order difference equation*

$$\Delta^2 y(n)-\frac{1}{(n+2)}\Delta y(n)+\frac{1}{(n+1)(n+2)}y(n)=\frac{2}{n(n+1)(n+2)} \tag{3.2}$$

on the interval $I=(1,\infty)$.

Let $c(n)=\frac{1}{n+1}, n\in I$ be a particular solution of equation

$$\Delta u(n)+u(n+1)u(n)-\frac{2}{n+2}u(n)+\frac{2}{(n+1)(n+2)}=0.$$

By Theorem 2.4, $d(n)=\frac{1}{n+2}\leq d<1, \prod_{i=1}^{n-1}(1+d(i))=\frac{n+2}{3}$ and $\prod_{i=1}^{n-1}(1+c(i))=\frac{n+1}{2}$. It is easy to verify that $w(n)\geq\frac{n+2}{3}$ and $z(n)\geq\frac{n+1}{3n}, n\in I$. Indeed, $y(n)=\frac{1}{n}$ is a solution of equation (3.2) and $|y(n)-z(n)|\leq\frac{\varepsilon}{cd}$. Hence equation (3.2) has the Hyers - Ulam stability.

We conclude this paper with the following remark.

Remark 3.3: *In this paper we investigated the Hyers - Ulam stability of different types of second order difference equations, and the results presented here are new and complement to the results reported in the literature for difference equations.*

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