International Journal of Mathematical Archive-9(6), 2018, 179-189
Available online through www.ijma.info ISSN 2229 – 5046

KERNEL IDEALS AND ∗-IDEALS IN PSEUDO-COMPLEMENTED ALMOST SEMILATTICES

G. NANAJI RAO[1] AND S. SUJATHA KUMARI*[2]

Department of Mathematics,
Andhra University, Visakhapatnam - 530003, Andhra Pradesh, India.

E-mail: nani6us@yahoo.com, sskmaths9@gmail.com.

(Received On: 11-05-18; Revised & Accepted On: 20-06-18)

ABSTRACT

The concepts of kernel ideal, ∗-ideal and ∗-congruence are introduced and derive smallest ∗-congruence with kernel ideal in ∗-commutative Pseudo-complemented Almost Semilattice (∗-commutative PCASL) and the largest ∗-congruence with kernel ideal in ∗-commutative PCASL. Also, characterize the ∗-congruence ψ on ∗-commutative PCASL L defined by $(x, y) \in \psi$ if and only if $x^{**} = y^{**}$ in terms of largest ∗-congruences.

Key Words: *Almost Semilattice, ∗-commutative Pseudo-complemented Almost Semilattice (∗-commutative PCASL), Kernel ideal, ∗-ideal, Smallest ∗-congruence, Largest ∗-congruence.*

AMS Mathematics Subject classification (2000): *06D99, 06D15.*

1. INTRODUCTION

The concept of Almost semilattice (ASL) was introduced by Nanaji Rao,G. and Trefe Getachew Beyene [3]. They proved several properties of ASL and derived necessary and sufficient conditions for an ASL to become a semilattice. They introduced the concept of maximal set and amicable set in ASL and established relation between these two sets. Also, they introduced the concepts like ideals, filters, principal ideals, principal filters, prime ideals, prime filters, anhilator ideals, anhilator preserving homomorphisms and proved several results on these concepts. Next, the concept of pseudo-complementation on ASL L was introduced by Nanaji Rao,G. and Sujatha Kumari,S. [2]. and observed that pseudo-complementation on an ASL is not unique and proved several basic properties of pseudo-complementation ∗ on L. They proved that pseudo-complementation ∗ on L is equationally definable and established a one-to-one correspondence between set of all maximal elements in an ASL L and set of all pseudo-complementations on L and proved that the set of all ∗-elements in a ∗-commutative pseudo-complemented ASL form a Boolean algebra which is independent (up to isomorphism) of the pseudo-complementation ∗. In this paper we introduced the concepts of kernel ideal, ∗-ideal and ∗-congruence in ∗-commutative $PCASL$ and several examples are obtained for these concepts. We derived a necessary and sufficient conditions for an ASL congruence to become a ∗-congruence. Also, derived necessary and sufficient conditions for an ideal in ∗-commutative $PCASL$ to become a kernel ideal. Next, we established the smallest ∗-congruence with given kernel ideal and largest ∗-congruence with given kernel ideal and charectarized the largest ∗-congruence in terms of smallest ∗-congruence and the ∗-congruence ψ on ∗-commutative $PCASL$ defined by $(x, y) \in \psi$ if and only if $x^{**} = y^{**}$.

2. PRELIMINARIES

In this section we collect few important definitions and results which are already known and which will be used more frequently in the paper.

Corresponding Author: S. Sujatha Kumari[2], Department of Mathematics, Andhra University, Visakhapatnam - 530003, Andhra Pradesh, India.*

Definition 2.1 [3]**:** An almost semilattice (ASL) is an algebra $(L,\circ)$ where L is a non-empty set and $\circ$ is a binary operation on L, satisfies the following conditions:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)
(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)
(3) $x \circ x = x$, for all $x, y, z \in L$ (Idempotent)

Definition 2.2 [3]**:** An ASL with 0 is an algebra $(L,\circ,0)$ of type $(2,0)$ satisfies the following axioms:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)
(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)
(3) $x \circ x = x$ (Idempotent)
(4) $0 \circ x = 0$, for all $x, y, z \in L$.

Theorem 2.3 [3]**:** Let L be an ASL. Define a relation $\leq$ on L by $a \leq b$ if and only if $a \circ b = a$. Then $\leq$ is a partial ordering on L.

Theorem 2.4 [3]**:** Let L be an ASL. Then for any $a, b \in L$, we have the following:

(1) $a \circ b \leq b$.
(2) $a \circ b = b \circ a$ whenever $a \leq b$.

Theorem 2.5 [3]**:** Let L be an ASL with 0. Then for any $a, b \in L$, we have the following:

(1) $a \circ 0 = 0$.
(2) $a \circ b = 0$ if and only if $b \circ a = 0$.
(3) $a \circ b = b \circ a$ whenever $a \circ b = 0$.

Definition 2.6 [3]**:** Let L be an ASL. Then an element $m \in L$ is said to be unimaximal if $m \circ x = x$, for all $x \in L$.

Corollary 2.7 [3]**:** Let L be an ASL and I be an ideal of L. Then, for any $a, b \in L, a \circ b \in I$ if and only if $b \circ a \in I$.

Definition 2.8 [3]**:** Let L be an ASL and $a \in L$. Then $(a] = \{a \circ x \mid x \in L\}$ is an ideal of L.

Lemma 2.9 [3]**:** Let L be an ASL and for any $a, b \in L, a \in (b]$ if and only if $a = b \circ a$.

Definition 2.10[3]: For any non-empty subset A of an ASL L with 0, define $A^* = \{x \in L : x \circ a = 0,\ \textit{for all } a \in A\}$. Then A^* is called the annihilator of A. Note that, if $A = \{a\},$ then we denote $A^* = \{a\}^*$ by $[a]^*$.

Lemma 2.11 [3]**:** Let L be an ASL with 0. Then for any $a, b \in L,$ $[a \circ b]^{**} = [a]^{**} \cap [b]^{**}$.

Definition 2.12 [2]**:** Let L be an ASL with zero. Then a unary operation $a \mapsto a^*$ on L is said to be pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:

1. $a \circ b = 0 \Rightarrow a^* \circ b = b$
2. $a \circ a^* = 0$.

Here onwords by a $PCASL$, we mean a Pseudo-complemented Almost Semilattice and $*$-commutative in PCASL we mean all $*$-elements are commute.

Lemma 2.13 [2]: Let L be a $PCASL$. Then for any $a,b \in L$, we have the following:

(1) $0^* \circ a = a$
(2) 0^* is unimaximal
(3) 0^* is maximal
(4) $a^{**} \circ a = a$
(5) a is unimaximal $\Rightarrow$ $a^* = 0$
(6) $0^{**} = 0$.

Theorem 2.14 [2]: Let L be an ASL with 0. Then a unary operation $*: L \to L$ is a pseudo-complementation on L if and only if it satisfies the following conditions:

(1) $a^* \circ b = (a \circ b)^* \circ b$
(2) $0^* \circ a = a$
(3) $0^{**} = 0$.

Theorem 2.15 [2]: Let L be a $*$-commutative $PCASL$. Then for any $a,b \in L$, we have the following:

(1) $a \le b \Rightarrow b^* \le a^*$
(2) $a^{***} = a^*$
(3) $a^* \le b^* \Leftrightarrow b^{**} \le a^{**}$.

Theorem 2.16 [2]: Let L be a $*$-commutative $PCASL$. Then for any $a,b \in L$, we have the following:

(1) $(a \circ b)^{**} = a^{**} \circ b^{**}$
(2) $(a \circ b)^* = (b \circ a)^*$
(3) $a^*, b^* \le (a \circ b)^*$.

Theorem 2.17 [4] : The set $\varepsilon(M)$ of all equivalence relations of a set M is a relatively complemented semimodular complete lattice with respect to the ordering relation defined by $\phi \le \psi (\phi, \psi \in \Re(M))$ if and only if, for any pair of elements $x, y \in M, (x,y) \in \phi$ implies $(x,y) \in \psi$. Accordingly, $\varepsilon(M)$ is called the equivalence lattice of the set M. Let $\{\Theta_\gamma\}_{\gamma \in \Gamma}$ be any subset of $\varepsilon(M)$. The infimum and supremum of the set $\{\Theta\}_{\gamma \in \Gamma}$ in $\varepsilon(M)$ is the relation Π and Σ defined as follows: For a pair of elements x, y of M, $x \equiv y(\Pi)$ if and only if $x \equiv y(\Theta_\gamma)$ for every γ, $x \equiv y(\Sigma)$ if and only if in the set M a finite sequence of elements, $x = t_0, t_1, ..., t_r = y$ exists such that to every one of the indices $j = 1,2,...,r$ there can be found a Θ_{γ_j} $(\gamma_j \in \Gamma)$ such that $t_{j-1} \equiv t_j (\Theta_{\gamma_j})$.

Theorem 2.18 [4]: The set $\mathcal{K}$(A) of the congruence relation of an algebra $A = A(\{f_\gamma\}_{\gamma \in \Gamma})$ is a complete sublattice of the lattice $\varepsilon(A)$ of all equivalence relations on A.

3. KERNEL IDEALS AND $*$-IDEALS

In this section, we introduce the concept of $*$-congruence on $PCASL$ L and give several examples of $*$-congruence. We derive necessary and sufficient condition for an almost semilattice congruence to become a $*$-congruence in $*$-commutative $PCASL$. Next, we introduce the concept of kernel ideal in $PCASL$ and obtain a necessary and sufficient condition for ideal to become a kernel ideal in $*$-commutative $PCASL$. Finally, in this section we introduce the concept of $*$-ideal in $PCASL$ and obtain a necessary and sufficient condition for a $*$-ideal to become a kernel ideal. First, we begin this section with the following definition.

Definition 3.1: Let $(L,\circ)$ be a semilattice. Then an equivalence relation R on L is said to be an almost semilattice congruence if for any $(a,b),(c,d)\in R,(a\circ c,b\circ d)\in R$.
It can be easily seen that if θ is an equivalence relation on an ASL L then θ is a congruence relation on L if and only if for any $(a,b)\in\theta$, $x\in L,(a\circ x,b\circ x),(x\circ a,x\circ b)\in\theta$.

Definition 3.2: Let $(L,\circ,*,0,)$ be a $PCASL$. Then a congruence relation θ on L is said to be a $*$-congruence if for any $(x,y)\in\theta,(x^*,y^*)\in\theta$.

In the following we give certain examples of $*$-congruences in $PCASL$ L.

Example 3.3: Let L be a $*$-commutative PCASL and I be an ideal of L. Define a relation R_I on L by $(x,y)\in R_I$ if and only if $x\circ i^* = y\circ i^*\in I$, for some $i\in I$. Then clearly, R_I is a $*$-congruence on L.

Example 3.4: Let L be a $*$-commutative $PCASL$ and I be an ideal of L. Define a relation R_I on L by $(x,y)\in R_I$ if and only if $i^*\circ x = i^*\circ y\in I$, for some $i\in I$. Then clearly, R_I is a $*$-congruence on L.

Example 3.5: Let L be a $*$-commutative $PCASL$ and I be an ideal of L. Define a relation R^I on L by $(x,y)\in R^I$ if and only if $a\circ x, a\circ y\in I$, for any $a\in L$. Then clearly, R^I is a $*$-congruence on L.

Example 3.6: Let L be a $*$-commutative $PCASL$ and I be an ideal of L. Define a relation R^I on L by $(x,y)\in R^I$ if and only if $a^{**}\circ x = a^{**}\circ y\in I$, for some $a\in I$. Then clearly, R^I is a $*$-congruence on L.

Example 3.7: Let L be a $*$-commutative $PCASL$ and I be an ideal of L. Define a relation R^I on L by $(x,y)\in R^I$ if and only if $a\circ x = a\circ y\in I$, for some $a\in I$. Then clearly, R^I is a $*$-congruence on L.

Example 3.8: Let L be a $*$-commutative $PCASL$ and I be an ideal of L. Define a relation R^I on L by $(x,y)\in R^I$ if and only if $(a\circ x\in I\Leftrightarrow a\circ y\in I$, for any $a\in L$). Then clearly, R^I is a $*$-congruence on L.
In the following we give a necessary and sufficient condition for almost semilattice congruence to become a $*$-congruence. Recall that in a $PCASL$, for any $a,b\in L, a^*\circ b=(a\circ b)^*\circ b$.

Theorem 3.9: Let L be a $*$-commutative $PCASL$ and let θ be a congruence on L. Then θ is a $*$-congruence if and only if for any $(x,0)\in\theta$ implies $(x^*,0^*)\in\theta$.

Proof: Suppose θ is a $*$-congruence on L. If $(x,0)\in\theta,$ then by the definition of $*$-congruence, $(x^*,0^*)\in\theta.$
Conversely, assume the condition. Suppose $(x,y)\in\theta$. Since θ is reflexive, $(x^*,x^*)\in\theta$. Therefore $(x\circ x^*, y\circ x^*)\in\theta$ and hence $(0,y\circ x^*)\in\theta$. This implies $(y\circ x^*,0)\in\theta$.
It follows that $((y\circ x^*)^*,0^*)\in\theta$. Therefore $(x^*\circ(y\circ x^*)^*, x^*\circ 0^*))\in\theta.$
Since $x^*\circ(y\circ x^*)^* = y^*\circ x^* = x^*\circ y^*$, $(x^*\circ y^*, x^*)\in\theta$. Therefore $(x^*, x^*\circ y^*)\in\theta$. Similarly, we can prove that $(y^*\circ x^*, y^*)\in\theta$. Therefore $(x^*\circ y^*, y^*)\in\theta$. Hence by transitive $(x^*,y^*)\in\theta$. Thus θ is a $*$-congruence on L.
First, observe that in a $PCASL$ L, if I is an ideal then not necessarily I is closed under $**$ a pseudo-complementation on L; that is, if $a\in I$ then $a^{**}\notin I$. For, consider the following example.

Example 3.10: Let L = { a, b, c, 0 }. Now, define a binary operation $\circ$ on L as follows:

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	a	b	c

It can be easily seen that the ASL L is a $PCASL$ under the unary operation $\circ$ on L defined by $0^* = b, x^* = 0$ for all $x \neq 0 \in L$. Now, put $I = \{0, a\}$. Then clearly I is an ideal of L. But, $a^{**} = 0^* = b \notin I$.

It can be easily observed that, if θ is a congruence relation on an ASL L with 0, then the congruence class of 0 with respect to θ (denoted by $\theta_0 = \{x \in L : (x,0) \in \theta\}$) is an ideal of L and is called the kernel of θ. Next, we introduce the concept of kernel ideal in PCASL and give necessary and sufficient condition for ideal in PCASL to become a kernel ideal.

Definition 3.11: An ideal I of $PCASL$ L is said to be a kernel ideal if I is the kernel of a $*$-congruence on L.

Theorem 3.12: An ideal I of a $*$-commutative $PCASL$ L is a kernel ideal of L if and only if for any $i, j \in I$ implies $(i^* \circ j^*)^* \in I$.

Proof: Suppose I is a kernel ideal of L. Then I is a kernel of a $*$-congruence θ on L. Therefore $I = \theta_0 = \{x \in L : (x,0) \in \theta\}$. Now, we shall prove that for any $i, j \in I, (i^* \circ j^*)^* \in I$. Let $i, j \in I = \theta_0$. Then $(i,0),(j,0) \in \theta$. Since θ is a $*$-congruence, it follows that $(i^*,0^*),(j^*,0^*) \in \theta$. Hence $(i^* \circ j^*,0^*) \in \theta$. This implies $((i^* \circ j^*)^*,0^{**}) \in \theta$. Therefore $((i^* \circ j^*)^*,0) \in \theta$. Thus $(i^* \circ j^*)^* \in \theta_0 = I$. Conversely, assume the condition. Now, we shall prove that I is kernel ideal of L. Consider the relation R_I on L defined by $(x, y) \in R_I \Leftrightarrow x \circ i^* = y \circ i^*$, for some $i \in I$. Then clearly, R_I is both reflexive and symmetric. Suppose $(x, y),(y, z) \in R_I$. Then there exists $i, j \in I$ such that $x \circ i^* = y \circ i^*$ and $y \circ j^* = z \circ j^*$. Since $i, j \in I$, we have $(i^* \circ j^*)^* \in I$. Now, put $k = (i^* \circ j^*)^*$. Then we have $k \in I$. Now,consider $x \circ k^* = x \circ (i^* \circ j^*)^{**} =$
$= x \circ (i^{***} \circ j^{***}) = x \circ (i^* \circ j^*) = ((x \circ i^*) \circ j^*) = ((y \circ i^*) \circ j^*) = ((i^* \circ y) \circ j^*) = (i^* \circ (y \circ j^*)) =$
$(i^* \circ (z \circ j^*)) = ((i^* \circ z) \circ j^*) = ((z \circ i^*) \circ j^*) = z \circ (i^* \circ j^*) = z \circ (i^{***} \circ j^{***}) = z \circ (i^* \circ j^*)^{**} = z \circ k^*.$
Therefore $(x, z) \in R_I$. Thus R_I is transitive. Hence R_I is an equivalence relation on L.

Now, we shall prove that R_I is a congruence on L. Suppose $(x, y),(z,t) \in R_I$. Then there exists $i, j \in I$ such that $x \circ i^* = y \circ i^*$ and $z \circ j^* = t \circ j^*$. Since $i, j \in I$, $(i^* \circ j^*)^* \in I$. Now, put $k = (i^* \circ j^*)^*$. Then $k \in I$. Now, consider $(x \circ z) \circ k^* = (x \circ z) \circ (i^* \circ j^*)^{**} = (x \circ z) \circ (i^{***} \circ j^{***}) =$
$= (x \circ z) \circ (i^* \circ j^*) = x \circ (z \circ (i^* \circ j^*)) = x \circ ((z \circ i^*) \circ j^*) = x \circ ((i^* \circ z) \circ j^*) = (x \circ (i^* \circ z)) \circ j^* =$
$((x \circ i^*) \circ z) \circ j^* == (x \circ i^*) \circ (z \circ j^*) = (y \circ i^*) \circ (t \circ j^*) = ((y \circ i^*) \circ t) \circ j^*) = (y \circ (i^* \circ t)) \circ j^* =$
$(y \circ (t \circ i^*)) \circ j^* = ((y \circ t) \circ i^*) \circ j^*) = (y \circ t) \circ (i^* \circ j^*) = (y \circ t) \circ (i^{***} \circ j^{***}) = (y \circ t) \circ (i^* \circ j^*)^{**}$
$= (y \circ t) \circ k^*.$ Therefore $(x \circ z, y \circ t) \in R_I$. Therefore R_I is a congruence on L. Now, we shall prove that R_I is a $*$-congruence on L. That is, enough to prove that if $(x,0) \in R_I$. Then $(x^*, 0^*) \in R_I$. Suppose $(x, 0) \in R_I$. Then $x \circ i^* = 0 \circ i^*$, for some $i \in I$. This implies $x \circ i^* = 0$. It follows that $x^* \circ i^* = i^*$. Since 0^* is unimaximal, $x^* \circ i^* = 0^* \circ i^*$. Hence $(x^*,0^*) \in R_I$. It follows by theorem 3.9, R_I is a $*$- congruence on L.

Finally, we shall prove that the congruence class of the zero element is I. We have for any $i, j \in I, (i^* \circ j^*)^* \in I$. Then $i^{**} \in I$. Let $x \in (R_I)_0$. Then $(x,0) \in R_I$. This implies $x \circ i^* = 0 \circ i^*$ for some $i \in I$. It follows that $x \circ i^* = 0$ and hence $i^* \circ x = 0$. Therefore $i^{**} \circ x = x$. Since $i \in I$, $i^{**} \in I$. Therefore $i^{**} \circ x \in I$. Hence $x \in I$. Therefore $(R_I)_0 \subseteq I$. Conversely, suppose $x \in I$. Then we have $x \circ x^* = 0 = 0 \circ x^*$ and $x \in I$. It follows that $(x,0) \in R_I$ and hence $x \in (R_I)_0$. Therefore $I \subseteq (R_I)_0$. Hence $(R_I)_0 = I$. Thus I is a kernel ideal of L.

Corollary 3.13: An ideal I of a $*$-commutative $PCASL$ L is a kernel ideal if and only if

(i) $i \in I \Rightarrow i^{**} \in I$

(ii) $i, j \in I \Rightarrow \exists\, k \in I$ such that $i^* \circ j^* = k^*$.

Proof: Suppose I is a kernel ideal of L and suppose $i \in I$. Then by theorem 3.12, we get $(i^* \circ i^*)^* \in I$. Hence $i^{**} \in I$. Again, let $i, j \in I$. Then by theorem 3.12, $(i^* \circ j^*)^* \in I$. Now, put $k = (i^* \circ j^*)^*$. Then clearly, $k \in I$ and $k^* = (i^* \circ j^*)^{**} = i^{***} \circ j^{***} = i^* \circ j^*$. Convesely, assume the conditions. Let $i, j \in I$. Then by (2), there exist $k \in I$ such that $i^* \circ j^* = k^*$. Since $k \in I$, by(1), $k^{**} \in I$. It follows that $(i^* \circ j^*)^* \in I$. Thus by theorem 3.12, I is a kernel ideal of L.

Recall that, if L is a $*$-commutative $PCASL$. Then $S(L) = \{x^* : x \in L\}$ is a Boolean algebra.

Corollary 3.14: If $x \in S(L)$, then $(x]$ is a kernel ideal.

Proof: Suppose $x \in S(L)$. Then we have $x = x^{**}$. Now, let $i, j \in (x]$. Then $i = x \circ i$ and $j = x \circ j$.

Now, $i^{**} = (x \circ i)^{**} = x^{**} \circ i^{**} = i^{**} \circ x^{**} = i^{**} \circ x$. Similarly, we get $j^{**} = j^{**} \circ x$. Hence $i^{**} \leq x$ and $j^{**} \leq x$. This implies $x^* \leq i^{***} = i^*$ and $x^* \leq j^{***} = j^*$. Therefore $x^* \leq i^* \circ j^*$. It follows that $(i^* \circ j^*)^* \leq x^{**} = x$. Hence $(i^* \circ j^*)^* = (i^* \circ j^*)^* \circ x = x \circ (i^* \circ j^*)^* \in (x]$, since $x \in S(L)$. Thus $(x]$ is a kernel ideal.

In the following, (in view of example 3.10), we introduce the concept of $*$-ideal in $PCASL$ and we give an example of $*$-ideal.

Definition 3.15: An ideal I of a $PCASL$ L is said to be a $*$-ideal if , $i \in I$ then $i^{**} \in I$.

Example 3.16: Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ are two discrete ASLs.

Let $L = A \times B = \{(0,0), (0,b_1), (0,b_2), (a,0), (a,b_1), (a,b_2)\}$. Define a binary operation $\circ$ on L under point-wise operations as follows:

$\circ$	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(0, b_1)	(0, 0)	(0, b_1)	(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)
(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)	(0, 0)	(0, b_1)	(0, b_2)
(a, 0)	(0, 0)	(0, 0)	(0, 0)	(a, 0)	(a, 0)	(a, 0)
(a, b_1)	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)
(a, b_2)	(0, 0)	(0, b_1)	(0, b_2)	(a, 0)	(a, b_1)	(a, b_2)

Then clearly, $(L,\circ)$ is an ASL. Now, define a unary operation $*$ on L, by $(0,0)^* = (a,b_1)$, $(0,b_1)^* = (0,b_2)^* = (a,0)$, $(a,0)^* = (0,b_1)$ and $(a,b_1)^* = (a,b_2)^* = (0,0)$. Then clearly, $*$ is a pseudo-complementation on L. Now, put $I = \{(0,0),(0,b_1),(0,b_2)\}$. Then clearly, I is an ideal of L which is a $*$-ideal.
It can be easily observed that every kernel ideal is $*$-ideal. But, converse is not true. For, in the example 3.16, put $I = \{(0,0),(0,b_1),(0,b_2),(a,0)\}$. Then clearly, I is a $*$-ideal. But I is not a kernel ideal. For, we have $(0,\mathrm{b}_1),(\mathrm{a},0) \in \mathrm{I}$.
Now, $((0,b_1)^* \circ (a,0)^*)^* = ((a,0)\circ(0,b_1))^* = (0,0)^* = (a,b_1) \notin I.$ Now, we derive a necessary and sufficient condition for a $*$-ideal to become a kernel ideal.

Theorem 3.17: Let I be $*$-ideal of a $*$-commutative $PCASL$ L. Then I is a kernel ideal if and only if $\sup_{\mathrm{S(L)}}\{\mathrm{i}^{**},\mathrm{j}^{**}\}$ belongs to I, for all $i,j \in I$.

Proof: Proof followes by theorem 3.12, and by the definition of supremum of any two elements in $\mathrm{S(L)}$.

4. SMALLEST AND LARGEST $*$-CONGRUENCES

In this section, we describe the smallest $*$-congruence with kernel ideal and largest $*$-congruence with kernel ideal and largest $*$-congruence with kernel ideal and characterize the largest $*$-congruence with kernel ideal in terms of smallest $*$-congruence and the $*$-congruence ψ. Recall that not every ideal in $PCASL$ is a kernel ideal(example 3.16). In the following we describe the smallest $*$-congruence with kernel ideal on $*$-commutative PCASL.

Theorem 4.1: Let L be a $*$-commutative $PCASL$ L and let I be a kernel ideal of L. Then the smallest $*$-congruence with kernel I is given by $(x,y)\in R_I$ if and only if $i^*\circ x = i^*\circ y$, for some $i \in I$.

Proof: Suppose I is a kernel ideal. Now, we shall prove that R_I is the smallest $*$-congruence on L with kernel I. Clearly, $\mathrm{R_I}$ is reflexive and symmetric. Now, let $(x,y),(y,z)\in R_I$. Then $i^*\circ x = i^*\circ y$ and $j^*\circ y = j^*\circ z$ for some $i,j\in I$. Now, put $k=(i^*\circ j^*)^*$. Then clearly, $k\in I$. Now, consider
$k^*\circ x = (i^*\circ j^*)^{**}\circ x = (i^{***}\circ j^{***})\circ x = ((i^*\circ j^*)\circ x) = ((j^*\circ i^*)\circ x) =$
$(j^*\circ(i^*\circ x)) = (j^*\circ(i^*\circ y)) = ((j^*\circ i^*)\circ y) = ((i^*\circ j^*)\circ y) = (i^*\circ(j^*\circ y)) = (i^*\circ(j^*\circ z)) =$
$((i^*\circ j^*)\circ z) = (i^{***}\circ j^{***})\circ z = ((i^*\circ j^*)^{**})\circ z = k^*\circ z.$ Therefore $(x,z)\in R_I$. Hence $\mathrm{R_I}$ ia an equivalence relation on L. Suppose $(x,y),(z,t)\in R_I$. Then $i^*\circ x = i^*\circ y$ and $j^*\circ z = j^*\circ t$ for some $i,j\in I$. Now, put $k=(i^*\circ j^*)^*$. Then clearly, $k\in I$.
Now, consider $k^*\circ(x\circ z) =$
$(i^*\circ j^*)^{**}\circ(x\circ z) = (i^{***}\circ j^{***})\circ(x\circ z) = ((i^*\circ j^*)\circ(x\circ z) = i^*\circ(j^*\circ(x\circ z)) = i^*\circ((j^*\circ x)\circ z)$
$i^*\circ((x\circ j^*)\circ z) = (i^*\circ(x\circ j^*))\circ z = ((i^*\circ x)\circ j^*)\circ z = (i^*\circ x)\circ(j^*\circ z) = (i^*\circ y)\circ(j^*\circ t) =$
$i^*\circ(y\circ(j^*\circ t)) = i^*\circ((y\circ j^*)\circ t)) = i^*\circ((j^*\circ y)\circ t)) = (i^*\circ(j^*\circ y))\circ t = ((i^*\circ j^*)\circ y)\circ t =$
$(i^*\circ j^*)\circ(y\circ t) = (i^{***}\circ j^{***})\circ(y\circ t) = (i^*\circ j^*)^{**}\circ(y\circ t) = k^*\circ(y\circ t).$ Therefore $(x\circ z,\ y\circ t)\in R_I$. Hence R_I is a congruence relation on L. Now, we shall prove that R_I is a $*$-congruence on L. That is, enough to prove that if $(x,0)\in R_I$ then $(x^*,0^*)\in R_I$. Suppose $(x,0)\in R_I$. Then $i^*\circ x = i^*\circ 0$ for some $i\in I$. Therefore $i^*\circ x = 0$. This implies $x\circ i^* = 0$. It follows that $x^*\circ i^* = i^*$. Since 0^* is unimaximal, $i^*\circ x^* = i^* = 0^*\circ i^* = i^*\circ 0^*$. Hence $(x^*,0^*)\in R_I$. Hence R_I is a $*$-congruence on L. Next, we prove that $(R_I)_0 = I$. Let $x\in(R_I)_0$. Then $(x,0)\in R_I$. This implies $i^*\circ x = i^*\circ 0$ for some $i\in I$. Hence $i^*\circ x = 0$.

Therefore $i^{**}\circ x = x$. Since $i \in I, i^{**} \in I$. It follows that $x = i^{**}\circ x \in I$. Therefore $(R_I)_0 \subseteq I$. Conversely, suppose $x \in I$. Then we have $x^*\circ x = 0 = x^*\circ 0$ and $x \in I$. Hence $(x,0)\in R_I$. It follows that $x \in (R_I)_0$. Therefore $I \subseteq (R_I)_0$. Thus we get $(R_I)_0 = I$.

Now, we shall prove that R_I is the smallest $*$-congruence on L with kernel I. Suppose θ is a $*$-congruence on L with kernel I. Let $(x, y)\in R_I$. Then $i^*\circ x = i^*\circ y$ for some $\mathrm{i} \in \mathrm{I}$. Since $i \in I = \theta_0, (i,0)\in\theta$. This implies $(i^*,0^*)\in\theta$. It follows that $(i^*\circ x, 0^*\circ x)\in\theta$ and hence $(i^*\circ x, x)\in\theta$. This implies $(x, i^*\circ x)\in\theta$. Similarly, we can prove that $(i^*\circ y, y)\in\theta$. Therefore by transitive $(x, y)\in\theta$. Thus R_I is the smallest $*$-congruence with kernel I.

In the following, we describe the largest $*$-congruence with kernel ideal on $*$-commutative $PCASL$. It can be easily seen that, if L is $*$-commutative $PCASL$ then a relation ψ on L defined by $(x, y)\in\psi$ if and only if $x^{**} = y^{**}$ is a $*$-congruence on L.

First we prove the following

Lemma 4.2: Let L be a $*$-commutative $PCASL$ L and let I be a kernel ideal of L. Define a relation R^I on L by $(x, y)\in R^I$ if and only if $(a\circ x\in I \Leftrightarrow a\circ y \in I,$ for any $a \in L).$ Then R^I is a $*$-congruence with kernel I.

Proof: Suppose I is a kernel ideal. Now, we shall prove that R^I is a $*$-congruence with kernel I. Then clearly, R^{I} is an equivalence relation on L. Suppose $(x, y),(w, z)\in R^I$. Then for any $a \in L, a\circ x \in I \Leftrightarrow a\circ y \in I$ and $a\circ w \in I \Leftrightarrow a\circ z \in I$. Suppose $t \in L$ and suppose $t\circ(x\circ w)\in I$.

Now, consider $t\circ(x\circ w)\in I \Leftrightarrow (t\circ x)\circ w \in I \Leftrightarrow w\circ(t\circ x)\in I \Leftrightarrow (w\circ t)\circ x \in I \Leftrightarrow (w\circ t)\circ y \in I$ $\Leftrightarrow (t\circ w)\circ y \in I \Leftrightarrow y\circ(t\circ w)\in I \Leftrightarrow (y\circ t)\circ w \in I \Leftrightarrow (y\circ t)\circ z \in I \Leftrightarrow (t\circ y)\circ z \in I \Leftrightarrow$ $t\circ(y\circ z)\in I.$ Therefore $t\circ(x\circ w)\in I \Leftrightarrow t\circ(y\circ z)\in I$. Hence $(x\circ w, y\circ z)\in R^I$. Hence R^{I} is congruence relation on L. Suppose $(x,0)\in R^I$. Then $a\circ x \in I$, for every $a \in L$. Inparticular $x\circ x \in I$ and hence $x \in I$. Now, suppose $t \in L$ such that $t\circ 0^* \in I$. It follows that $t\circ 0^*\circ x^* \in I$. Since 0^* unimaximal, $t\circ x^* \in I$. Conversely, suppose $t\circ x^* \in I$. Since $x \in I$ and I is kernel ideal, $(x^*\circ(t\circ x^*)^*)^* \in I$. This implies $((t\circ x^*)^*\circ x^*)^* \in I$. It follows that $(t^*\circ x^*)^* \in I$. This implies $t^{**}\in I$, since $x^*\circ t^* \le t^*$ we get $t^{**}\le (t^*\circ x^*)^*$. Therefore $t^{**}\circ t \in I$. Hence $t \in I$. Therefore $t\circ 0^* \in I$. Thus $(x^*,0^*)\in R^I$. Therefore R^{I} is $*$-congruence on L. Now, we shall prove that $\mathrm{I} = (\mathrm{R}^{\mathrm{I}})_0$. Suppose $\mathrm{x} \in (\mathrm{R}^{\mathrm{I}})_0$. It follows that, clearly $(\mathrm{R}_{\mathrm{I}})_0 \subseteq \mathrm{I}$. Conversely, suppose $x \in I$ and $a \in L$. Then we have $x\circ a \in I$ and hence $a\circ x \in I.$ It follows that $(x,0)\in R^I$. Therefore $I \subseteq (R^I)_0$. Hence $(R^I)_0 = I$. Thus R^I is a $*$-congruence with kernel I.

Theorem 4.3: Let L be a $*$-commutative $PCASL$ and let I be a kernel ideal of L. Then $R_I \vee \psi$ is the largest $*$-congruence with kernel I.

Proof: Suppose I is a kernel ideal of L. Since R_I and ψ are $*$-congruences on L, it follows that $R_I \vee \psi$ is $*$-congruence. Since R_I is the smallest $*$-congruence with kernel I. It follows that $(R_I \vee \psi)_0 = I$. Finally, we shall prove that $R_I \vee \psi$ is the largest $*$-congruence on L with kernel I. Suppose θ is a $*$-congruence on L with kernel I. Let $(x, y)\in\theta$. Then we have $(x\circ x^*, y\circ x^*)\in\theta, (x\circ y^*, y\circ y^*)\in\theta.$ This implies $(0, y\circ x^*)\in\theta, (x\circ y^*,0)\in\theta.$ Therefore $x\circ y^*, y\circ x^* \in I.$ Since I is kernel ideal,
$((x\circ y^*)^*\circ(y\circ x^*)^*)^* \in I.$ Now, put $i = ((x\circ y^*)^*\circ(y\circ x^*)^*)^*.$

Then $i \in I$. Now, consider $i^* \circ x^{**} = ((x \circ y^*)^* \circ (y \circ x^*)^*)^{**} \circ x^{**} =$
$((x \circ y^*)^* \circ (y \circ x^*)^* \circ x)^{**} = ((x \circ y^*)^* \circ x)^{**} = ((y^* \circ x)^* \circ x)^{**} = (y^{**} \circ x)^{**} = y^{****} \circ x^{**} = x^{**} \circ y^{**}.$
Therefore $i^* \circ x^{**} = x^{**} \circ y^{**}$. Similarly, we can prove that $i^* \circ y^{**} = x^{**} \circ y^{**}$. Hence $(x^{**}, y^{**}) \in R_I$. Now, since $x^{**} = x^{****}, (x, x^{**}) \in \psi$. Similarly, we get $(y, y^{**}) \in \psi$. Therefore there exist a finite sequence x, x^{**}, y^{**}, y such that $(x, x^{**}) \in \psi, (x^{**}, y^{**}) \in R_I$ and $(y^{**}, y) \in \psi$. Therefore $(x, y) \in R_I \vee \psi$. Hence $\theta \subseteq R_I \vee \psi$. Thus $R_I \vee \psi$ is the largest $*$-congruence on L with kernel I.

In the following we prove that R^I is the largest $*$-congruence on L with kernel I.

Theorem 4.4: Let L be a $*$-commutative $PCASL$ and let I be a kernel ideal of L. Then $R^I = R_I \vee \psi.$

Proof: Suppose I is a kernel ideal. Since R^I is $*$-congruence with kernel I on L and $R_I \vee \psi$ is the largest $*$-congruence with kernel I on L, $R^I \subseteq R_I \vee \psi$. Now, we have R_I is the smallest $*$-congruence on L with kernel I and R^I is a $*$-congruence with kernel I, we get $R_I \subseteq R^I$. Suppose $(x, y) \in \psi$. Then $x^{**} = y^{**}$. Suppose $t \in L$ such that $t \circ x \in I$. Now, we have I is a kernel ideal.

Therefore $(t \circ x)^{**} \in I \Rightarrow t^{**} \circ x^{**} \in I$
$$\Rightarrow t^{**} \circ y^{**} \in I$$
$$\Rightarrow (t^{**} \circ y^{**}) \circ t \circ y \in I$$
$$\Rightarrow ((t^{**} \circ y^{**}) \circ t) \circ y) \in I$$
$$\Rightarrow (t^{**} \circ (y^{**} \circ t)) \circ y \in I$$
$$\Rightarrow (t^{**} \circ (t \circ y^{**})) \circ y \in I$$
$$\Rightarrow ((t^{**} \circ t) \circ y^{**}) \circ y \in I$$
$$\Rightarrow (t^{**} \circ t) \circ (y^{**} \circ y) \in I$$
$$\Rightarrow t \circ y \in I.$$

Similarly, we can prove that if $t \circ y \in I$ then $t \circ x \in I$. Therefore $(x, y) \in R^I$. Hence $\psi \subseteq R^I$. It follows that $R_I \vee \psi \subseteq R^I$. Thus $R^I = R_I \vee \psi$.

Next, we prove if I is a non-empty subset of PCASL then $I^* = \{x \in L : x \circ i = 0,\ for\ all\ i \in I\}$ is a kernel ideal. For this, first we prove the following lemma.

Lemma 4.5: Let L be a $*$-commutative $PCASL$. Then $(x, y) \in R^I$ if and only if $((x \circ y^*)^* \circ (x^* \circ y)^*)^* \in I$.

Proof: Suppose I is a kernel ideal of L and suppose $(x, y) \in R^I$. Then $(x^*, y^*) \in R^I$. It follows that $(x \circ x^*, x \circ y^*) \in R^I$. This implies $(0, x \circ y^*) \in R^I$. Similarly, we get $(x^* \circ y, 0) \in R^I$. Therefore $x \circ y^*, x^* \circ y \in I$. Hence by theorem 3.12, we get $((x \circ y^*)^* \circ (x^* \circ y)^*)^* \in I$. Conversely, suppose $((x \circ y^*)^* \circ (x^* \circ y)^*)^* \in I$. Now, put $i = ((x \circ y^*)^* \circ (x^* \circ y)^*)^* \in I$. Then clearly, $i \in I$. Now, consider $i^* \circ x = ((x \circ y^*)^* \circ (x^* \circ y)^*)^{**} \circ x = ((x \circ y^*)^{***} \circ (x^* \circ y)^{***}) \circ x =$
$((x \circ y^*)^* \circ ((x^* \circ y)^* \circ x)) = (x \circ y^*)^* \circ x\ (since\ (x^* \circ y) \circ x = 0) = (y^* \circ x)^* \circ x = y^{**} \circ x.$
Therefore $i^* \circ x = y^{**} \circ x$. Similarly, we get $i^* \circ y = x^{**} \circ y$. Now, $i^* \circ x = y^{**} \circ x$. This implies $(i^* \circ x)^{**} = (y^{**} \circ x)^{**}$. It follows that $i^{***} \circ x^{**} = y^{****} \circ x^{**}$. Therefore $i^* \circ x^{**} = y^{**} \circ x^{**}$. Similarly, we get $i^* \circ y^{**} = x^{**} \circ y^{**} = y^{**} \circ x^{**}$. Hence $i^* \circ x^{**} = i^* \circ y^{**}$. Thus $(x^{**}, y^{**}) \in R_I$. Now, we have $(x, x^{**}) \in \psi$, $(x^{**}, y^{**}) \in R_I$ and $(y^{**}, y) \in \psi.$ Therefore $(x, y) \in R_I \vee \psi = R^I.$ Hence $(x, y) \in R^I.$

Theorem 4.6: If I is non-empty subset of a $PCASL\ \ L,$ then $I^* = \{x \in L : x \circ i = 0,\ \ for\ \ all\ \ i \in I\}$ the set of elements disjoint from I is a kernel ideal of L.

Proof: Suppose I is an ideal of L. Since $0 = 0 \circ i$ for all $i \in I,\ 0 \in I^*$. Therefore I^* is a non-empty subset of L. Suppose $x \in I^*$ and $a \in L$. Then $x \circ i = 0$, for all $i \in I$. Now, consider $(x \circ a) \circ i = (a \circ x) \circ i = a \circ (x \circ i) = a \circ 0 = 0$, for all $i \in I$. Therefore $x \circ a \in I^*$. Hence I^* is an ideal of L. Let $x, y \in I^*$. Then $x \circ i = 0, y \circ i = 0$, for all $i \in I$. It follows that $x^* \circ i = i, y^* \circ i = i$, for all $i \in I$. Let $i \in I$. Now, consider $(x^* \circ y^*)^* \circ i = (x^* \circ y^*)^* \circ (x^* \circ i) = (y^* \circ x^*)^* \circ (x^* \circ i) =$ $((y^* \circ x^*)^* \circ x^*) \circ i = ((y^{**} \circ x^*) \circ i =$ (since $((x \circ y)^* \circ y = x^* \circ y) = ((y^{**} \circ x^*) \circ (y^* \circ i) =$ $((y^{**} \circ (x^* \circ y^*)) \circ i = ((y^{**} \circ (y^* \circ x^*)) \circ i = ((y^{**} \circ y^*) \circ x^*)) \circ i = (0 \circ x^*) \circ i = 0 \circ i = 0.$ Therefore $(x^* \circ y^*)^* \circ i = 0$. Hence $(x^* \circ y^*)^* \in I^*$. Thus I^* is a kernel ideal of L.

Corollary 4.7: If I is an ideal of a $*$-commutative $PCASL$ L then I^* is a kernel ideal of L.

Recall that if I is a kernel ideal of $*$-commutative $PCASL\ \ L,$ then R^I is the largest $*$- congruence with kernel I. Also, we have if I is a non-empty subset of L then I^* is a kernel ideal of L and hence R^{I^*} is the largest $*$-congruence with kernel I^*. In the following, we give another form of R^{I^*} and prove that $\psi = R^I \cap R^{I^*}$.

First observe that for any a in PCASL, $(a^*] = (a]^* = [a]^*$ and hence $((a \circ b)^{**}] = (a^{**}] \cap (b^{**}].$ Now, we have the following theorem.

Theorem 4.8: Let L be a $*$-commutative $PCASL$ and I be an ideal of L. Define a relation R^{I^*} on L by $(x, y) \in R^{I^*}$ if and only if $(x^{**}] \cap I = (y^{**}] \cap I$. Then R^{I^*} is the largest $*$-congruence with kernel I^*.

Proof: Clearly, R^{I^*} is an equivalence relation on L. Suppose (x, y) and $(w,t) \in R^{I^*}$. Then $(x^{**}] \cap I = (y^{**}] \cap I$ and $(w^{**}] \cap I = (t^{**}] \cap I$. Now, consider $((x \circ w)^{**}] \cap I =$ $((x^{**}] \cap (w^{**}]) \cap I = \ \ ((x^{**}] \cap I) \cap ((w^{**}] \cap I)) = ((y^{**}] \cap I) \cap ((t^{**}] \cap I)) = ((y^{**}] \cap (t^{**}]) \cap I =$ $((y \circ t)^{**}] \cap I =$ Therefore $(x \circ w, y \circ t) \in R^{I^*}$. Hence R^{I^*} is a congruence relation on L. Suppose $(x,0) \in R^{I^*}$. Then $(x^{**}] \cap I = (0^{**}] \cap I = (0] \cap I = (0]$. Therefore $(x^{**}] \cap I = (0]$. Now, let $i \in I$. This implies $i \circ x \in I$. Now, consider $i \circ x = i \circ (x^{**} \circ x) = (i \circ x^{**}) \circ x = (x^{**} \circ i) \circ x = x^{**} \circ (i \circ x) \in (x^{**}]$. This implies $i \circ x \in (x^{**}] \cap I = (0]$. Thorefore $i \circ x = 0$ and hence $x \circ i = 0$. It follows that $x^* \circ i = i$. Now, we have $x^* \circ i \in (x^*]$ and $i \circ x^* \in I$ and hence $x^* \circ i \in I$. Therefore $x^* \circ i \in (x^*] \cap I$. Hence $i \in (x^*] \cap I$. Thus $I \subseteq (x^*] \cap I$. It follows that $(x^*] \cap I = I$. This implies $(x^{***}] \cap I = L \cap I$. It follows that $(x^{***}] \cap I = (0^*] \cap I$. Therefore $(x^{***}] \cap I = (0^{***}] \cap I$. Hence $(x^*,0^*) \in R^{I^*}$. Thus R^{I^*} is a $*$-congruence on L.

Now, we shall prove that $(R^{I^*})_0 = I^*$. Suppose $t \in (R^{I^*})_0$. Then $(t,0) \in R^{I^*}$. This implies $(t^{**}] \cap I = (0^{**}] \cap I$. Hence $(t^{**}] \cap I = (0]$. Now, we have $t^{**} \circ t = t$ and hence $t \in (t^{**}]$. Let $x \in I$. Then we have $x \circ t \in I.$ Therefore $t \circ x \in I.$ Since $t \in (t^{**}],\ t \circ x \in (t^{**}]$. Therefore $t \circ x \in (t^{**}] \cap I = (0]$. Hence $t \circ x = 0$. Thus $t \in I^*$. Therefore $(R^{I^*})_0 \subseteq I^*$. Conversely, suppose $t \in I^*$. Then $t \circ x = 0$, for all $x \in I$. It is enough to prove that $(t^{**}] \cap I = (0]$. Suppose $a \in (t^{**}] \cap I.$ This implies $a \in (t^{**}]$ and $a \in I$. It follows that $a = t^{**} \circ a$ and $a \in I$. Since $a \in I, t \circ a = 0$. It follows that $t^* \circ a = a$.

Now, consider $a = t^{**} \circ a = t^{**} \circ t^{*} \circ a = 0 \circ a = 0$. Therefore $(t^{**}] \cap I = (0]$. Therefore $I^{*} \subseteq (R^{I^{*}})_0$. Hence $(R^{I^{*}})_0 = I^{*}$. Now, we shall prove that $R^{I^{*}}$ is the largest $*$-congruence with kernel I^{*}. That is enough to prove that $R^{I^{*}} = R_{I^{*}} \vee \psi$. Clearly, we get $R^{I^{*}} \subseteq R_{I^{*}} \vee \psi$. Since $R_{I^{*}}$ is a smallest $*$-congruence with kernel I^{*}, we have $R_{I^{*}} \subseteq R^{I^{*}}$. Suppose $(x, y) \in \psi$. Then $x^{**} = y^{**}$. Since $x^{**} = y^{**}$, $(x^{**}] \cap I = (y^{**}] \cap I$. It follows that $(x, y) \in R^{I^{*}}$. Therefore $R_{I^{*}} \vee \psi \subseteq R^{I^{*}}$. Hence $R_{I^{*}} \vee \psi = R^{I^{*}}$. Thus $R^{I^{*}}$ is the largest $*$-congruence with kernel I^{*}.

Finally, we prove the following theorem which characterize the congruence ψ.

Theorem 4.9: Let L be a $*$-commutative $PCASL$ and I be a kernel ideal of L. Then $\psi = R^{I} \cap R^{I^{*}}$.

Proof: Suppose I is a kernel ideal of L. Then clearly, $\psi \subseteq R^{I} \cap R^{I^{*}}$. Conversely, suppose $(x, y) \in R^{I} \cap R^{I^{*}}$. Then $(x, y) \in R^{I}$ and $(x, y) \in R^{I^{*}}$. Since by lemma(4.5), we get $((x \circ y^{*})^{*} \circ (x^{*} \circ y)^{*})^{*} \in I$ and also, $((x \circ y^{*})^{*} \circ (x^{*} \circ y)^{*})^{*} \in I^{*}$. Therefore $((x \circ y^{*})^{*} \circ (x^{*} \circ y)^{*})^{*} \in I \cap I^{*} = \{0\}$.
Hence $((x \circ y^{*})^{*} \circ (x^{*} \circ y)^{*})^{*} = 0$.
Now, we have $(x \circ y^{*})^{**} \leq ((x \circ y^{*})^{*} \circ (x^{*} \circ y)^{*})^{*}$ and $(x^{*} \circ y)^{**} \leq ((x \circ y^{*})^{*} \circ (x^{*} \circ y)^{*})^{*}$. It follows that $(x \circ y^{*})^{**} = 0$ and $(x^{*} \circ y)^{**} = 0$. This implies $x^{**} \circ y^{***} = 0$ and $x^{***} \circ y^{**} = 0$. Therefore $x^{**} \circ y^{*} = 0$ and $x^{*} \circ y^{**} = 0$. Hence $y^{*} \circ x^{**} = 0$ and $x^{*} \circ y^{**} = 0$. It follows that $y^{**} \circ x^{**} = x^{**}$ and $x^{**} \circ y^{**} = y^{**}$. Therefore $x^{**} = y^{**}$. Hence $(x, y) \in \psi$. Therefore $R^{I} \cap R^{I^{*}} \subseteq \psi$. Thus $\psi = R^{I} \cap R^{I^{*}}$.

REFERENCES

1. Blyth,T.S.: Ideals and filters of Pseudo-Complemented Semilattices, Proceedings of the Edinburgh Mathematical Society,(1980) , 23, 301-316.
2. Nanaji Rao,G., Sujatha kumari, S.: Pseudo-complemented Almost Semilattices, International Journal of Mathematical Archive - 8(10), 2017, 94-102.
3. Nanaji Rao, G.,Terefe, G.B.: Almost Semilattice, International Journal of Mathematical Archive – 7(3), 2016, 52-67.
4. Szasz, G.: Lattice Theory: Introduction to lattice theory, Accadamic Press, New York, (1963).

Source of support: Nil, Conflict of interest: None Declared.