International Journal of Mathematical Archive-9(6), 2018, 9-15
Available online through www.ijma.info ISSN 2229 – 5046

# SEQUENTIAL PYRAMIDAL GRAPHS

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*(Received On: 10-05-18; Revised & Accepted On: 29-05-18)*

***ABSTRACT***

*Let G = (V, E) be a graph with p vertices and q edges. A Graph G is said to admit Sequential Pyramidal labeling if its vertices can be labeled from the set of integers* $\{1,2\ldots pq\}$ *such that the induced edge labels obtained by the integral part of the division of the labels of the end vertices such that the numerators are greater than the denominators are the sequence of natural numbers from* $\{1,2\ldots q\}$ *in which the sum of the squares of the edge labels is the* $q^{th}$ *Pyramidal number. A Graph G which admits such a labeling is called a Sequential Pyramidal Graph. In this Paper we prove that all Cycles, Caterpillars, the graph* $C_n \Theta P_n$ *and all Jahangir graphs are Sequential Pyramidal graphs and also introduce some classes of Non-Sequential Pyramidal graphs. By a graph we mean a finite, undirected graph without multiple edges or loops. For graph theoretic terminology, we refer to Bondy and Murty [2] and Harary [4]. For number theoretic terminology, we refer to M. Apostal [1] and for graph labeling we refer to J.A. Gallian [3].*

***Key words****: Pyramidal number, Sequential Pyramidal, Caterpillars, Jahangir graphs.*

## I. INTRODUCTION

Graph labeling is an assignment of labels to the vertices or edges or to both the vertices and edges of a graph subject to certain conditions. Labeled graphs serve as models for a broad range of applications such as Coding Theory, Communication Networks, Radio Astronomy and many other Scientific fields. Most graph labeling trace their origin to one introduced by Rosa in 1967.Graham and Sloane defined harmonious labeling of a graph. Sequential labeling are the variations of harmonious labeling. In a Sequential Pyramidal labeling all the edge labels are the sequence of natural numbers from 1, 2,…,q and the sum of the squares of these natural numbers yield a Pyramidal number. Hence for every Sequential Pyramidal graph there is a constant pyramidal number associated with the edge labels. In this paper we prove the Sequential Pyramidal labeling of some graphs and also investigate certain classes of Non-Sequential Pyramidal graphs.

## II. SEQUENTIAL PYRAMIDAL LABELING

**Definition 2.1:** A Triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n. If $n^{th}$ Triangular number is denoted by $T_n$ then $T_n = \frac{n(n+1)}{2}$. Triangular numbers are found in the third diagonal of Pascal's Triangle starting at row 3. They are 1, 3, 6, 10, 15, 21...

**Definition 2.2:** The sum of Consecutive triangular numbers is known as tetrahedral numbers. They are found in the fourth diagonal of Pascal's Triangle. These numbers are 1, 1 + 3, 1 + 3 + 6, 1 + 3 + 6 + 10...
(i.e.) 1, 4, 10, 20, 35…

**Definition 2.3:** The Pyramidal numbers or Square Pyramidal numbers are the sums of consecutive pairs of tetrahedral numbers. The following are some Pyramidal numbers.1. 1 + 4, 4 + 10, 10 + 20, 20 + 35...
(i.e.) 1, 5, 14, 30, 55...

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**Remark 2.4:** The Pyramidal numbers are also calculated by the following formula: $p_n = \frac{n(n+1)(2n+1)}{6}$

**Notation:** The notation $p_i$ is used for each Pyramidal number where i = 1, 2, 3...

**Definition 2.5:** A Sequential Pyramidal labeling of a graph $G(p,q)$ is a one-one function $f:V(G)\rightarrow\{1,2,3,\ldots\ldots..pq\}$ that induces a bijection $f^+:E(G)\rightarrow\{1,2,3\ldots q\}$ of the edges of G defined by $f^+(uv) = \left[\frac{f(u)}{f(v)}\right]$ where $f(u) > f(v)$ such that $\sum_{i=1}^{q} {e_i}^2 = p_q$ where $p_q$ is the $q^{th}$ Pyramidal number. The graph which admits such a labeling is called a Sequential Pyramidal Graph.

**Theorem 2.6:** All cycles $C_n$ of length n are Sequential Pyramidal graphs.

**Proof:** Let G = $C_n$ be a Cycle of length n. We discuss the following cases.

**Case-(i):** Let the length of the Cycle $C_n$ be such that $3\le n\le 4$. Let $v_1, v_2, \ldots v_n$ be the vertices and $e_1, e_2 \ldots e_n$ be the edges of the Cycle $C_n$. Define $f:V(G)\rightarrow\{1,2,3,\ldots\ldots..pq\}$ as follows:

$$f(v_1) = 1,\ f(v_2) = n,\ f(v_i) = i-1 \text{ where } 3 \le i \le 4.$$

**Case-(ii):** Let $C_n$ be a Cycle of length n such that $5\le n\le 6$. Define $f: V(G)\rightarrow\{1,2,3,\ldots\ldots..pq\}$ as follows:
$f(v_1) = 1$, $f(v_2) = n$

$$f(v_i) = \begin{cases} (n-i+1)f(v_{i-1}) & for\ i = 3 \\ \left[\frac{f(v_{i-1})}{(n-i+1)}\right] & for\ i = 4 \\ i-1 & for\ i = 5,6 \end{cases}$$

**Case-(iii):** Let $C_n$ be a Cycle of length n such that $n\ge 7$, n is odd. Let $v_1, v_2, \ldots, v_n$ be the vertices of the Cycle and $e_i = v_i v_{i+1}$ be the edges of the Cycle. Define $f: V(G)\rightarrow\{1,2,3,\ldots\ldots..pq\}$ as follows:
$f(v_1) = 1$, $f(v_2) = n$

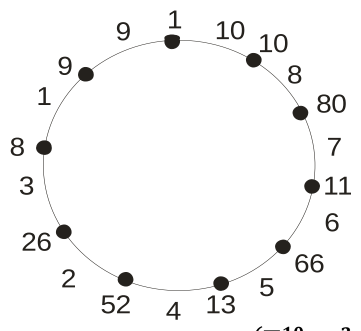
$$f(v_i) = \begin{cases} (n-i+1)f(v_{i-1}) & for\ 3 \le i \le n-4, i\ is\ odd \\ \left[\frac{f(v_{i-1})}{n-i+1}\right] & for\ 4 \le i \le n-3, i\ is\ even \\ 2f(v_{i-1}) & for\ i = n-2 \\ i-1 & for\ i = n-1, n \end{cases}$$

**Case-(iv):** Let $C_n$ be a Cycle of length n such that $n\ge 8$, n is even. Let $v_1, v_2, \ldots. v_n$ be the vertices of the Cycle.
Define $f: V(G)\rightarrow\{1,2,3,\ldots\ldots..pq\}$ as follows:
$f(v_1) = 1$, $f(v_2) = n$,

$$f(v_i) = = \begin{cases} (n-i+1)f(v_{i-1}) & for\ 3 \le i \le n-3, i\ is\ odd \\ \left[\frac{f(v_{i-1})}{n-i+1}\right] & for\ 4 \le i \le n-4, i\ is\ even \\ \frac{f(v_{i-1})}{2} & for\ i = n-2 \\ i-1 & for\ i = n-1, n \end{cases}$$

By defining $f^+(e_i) = \left[\frac{f(u)}{f(v)}\right]$, $f(u) > f(v)$ where f(u) and f(v) denote the labels of the end vertices for each edge $e_i$, i varies from 1 to n all the edges of the cycle $C_n$ are labeled with the first n natural numbers such that $\sum_{i=1}^{n} {e_i}^2 = p_n$ where $p_n$ is the $n^{th}$ Pyramidal number. Hence all cycles $C_n$ are Sequential Pyramidal graphs.

**Example:**




**Sequential Pyramidal labeling of $C_{10}$ $\left(\sum_{i=1}^{10} {e_i}^2 = p_{10} = 385\right)$**

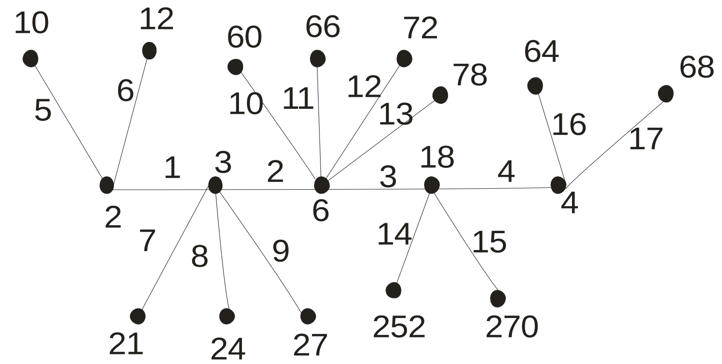
**Theorem 2.7:** All caterpillars are Sequential Pyramidal graphs .

**Proof:** A caterpillar is a tree, the removal of whose end points results in a Path. It is obtained by identifying the roots of n stars $K_{1,k_1}$, $K_{1,k_2}$ , …………$K_{1,k_n}$ with the n vertices of a Path $P_n$. Denote the identified vertices by $v_1$, $v_2$,….$v_n$ and the pendent vertices of the Star $K_{1,k_i}$ by $V_{i,j_i}$ where i = 1, 2, …….n and $j_i$ = 1,2,………$k_i$ .

$$\text{Define } f(v_i) = \begin{cases} i+1 & for\ i = 1,2 \\ 2i & for\ i = 3 \\ (i-1)f(v_{i-1}) & for\ 4 \le i \le n, i\ is\ even \\ \left\lceil \frac{f(v_{i-1})}{i-1} \right\rceil & for\ 5 \le i \le n,\ i\ is\ odd \end{cases}$$

Define f $(v_{i,j_i})$ = f($v_i$) (n+k-1) where k =1, 2, 3,… and for each fixed i, $j_i$ varies from 1.2.3……$k_i$. By defining $f^+(e_i) = \left\lceil \frac{f(u)}{f(v)} \right\rceil$, f(u) > f(v) where f(u) and f(v) denote the labels of the end vertices for each edge $e_i$, i varies from 1 to q, all the q edges of the Caterpillar are labeled with the first q natural numbers such that $\sum_{i=1}^{q} {e_i}^2$ = $p_q$ where $p_q$ is the $q^{th}$ Pyramidal number. Hence all Caterpillars are Sequential Pyramidal graphs.

**Example:**




**Sequential Pyramidal labeling of a Caterpillar $\left(\sum_{i=1}^{17} {e_i}^2 = \mathbf{p_{17}} = \mathbf{1785}\right)$**

**Theorem 2.8:** The graph $C_n \odot P_n$ is a Sequential Pyramidal graph for n ≥ 3.

**Proof:** Let G be a $C_n \odot P_n$ graph with $2n-1$ vertices and $2n-1$ edges.
Let V(G) = $\{v_n, v_{n-1}, \ldots.. v_1 = u_1, u_2, \ldots. u_n\}$ and E(G) = $\{e_i = v_iv_{i+1} \cup v_nv_1 \cup e_i' = u_iu_{i+1}, i = 1, 2 \ldots.. n-1\}$ be the Vertex set and Edge set of the given graph G. We discuss the following cases.

**Case-(i):** Let n= 3, 4. Define f: V$(G)$→$\{1,2,3, \ldots \ldots.. pq\}$ as follows:

f($u_1$) = f($v_1$) = 1, f($v_2$) = n

f($v_i$) = i-1 where 3 ≤ i ≤ 4 ,

$$f(u_i) = \begin{cases} (2n-1)+j & for\ even\ i\ and\ j = 0,1 \\ (2n-i+1)f(u_{i-1}) & for\ odd\ i \end{cases}$$

**Case-(ii):** Let n= 5, 6. Define f: V$(G)$→$\{1,2,3, \ldots \ldots.. pq\}$ as follows:

f($u_1$) =f($v_1$) = 1, f($v_2$) = n

$$f(v_i) = \begin{cases} (n-i+1)f(v_{i-1}) & for\ i = 3 \\ \left\lceil \frac{f(v_{i-1})}{(n-i+1)} \right\rceil & for\ i = 4 \\ i-1 & for\ i = 5,6 \end{cases}$$

$$f(u_i) = \begin{cases} (2n-1)+j & for\ even\ i\ and\ j = 0,1,2 \\ (2n-i+1)f(u_{i-1}) & for\ odd\ i \end{cases}$$

**Case-(iii):** Let n ≥ 7, n is odd. Define f: V$(G)$→$\{1,2,3, \ldots \ldots.. pq\}$ as follows:

f($v_1$) =f($u_1$) = 1, f($v_2$) = n

$$f(v_i) = \begin{cases} (n-i+1)f(v_{i-1}) & for\ 3 \le i \le n-4, i\ is\ odd \\ \left\lceil \frac{f(v_{i-1})}{n-i+1} \right\rceil & for\ 4 \le i \le n-3, i\ is\ even \\ 2f(v_{i-1}) & for\ i = n-2 \\ i-1 & for\ i = n-1, n \end{cases}$$

$$f(u_i) = \begin{cases} (2n-1)+j & for\ 2 \le i \le n,\ i\ is\ even\ \ and\ for\ each\ j = 0,1,2 \ldots \\ (2n-i+1)f(u_{i-1}) & for\ 3 \le i \le n,\ i\ is\ odd \end{cases}$$

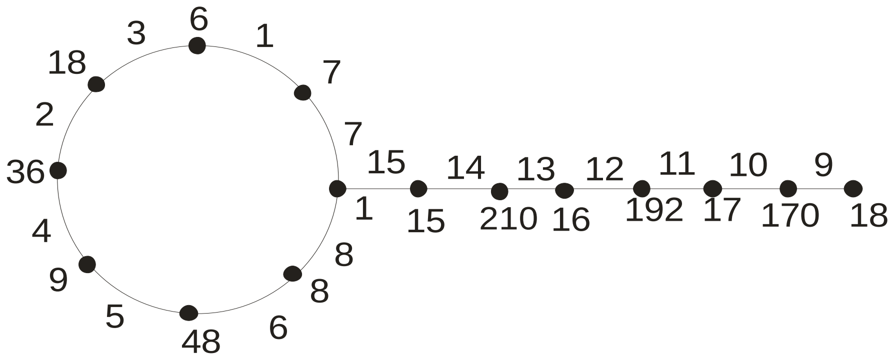
**Case-(iv):** Let $n \geq 8$, n is even. Define f: $V(G) \rightarrow \{1, 2, 3, \ldots\ldots..pq\}$ as follows:

$f(v_1) = f(u_1) = 1$, $f(v_2) = n$

$$f(v_i) = \begin{cases} (n-i+1)f(v_{i-1}) & for\ 3 \leq i \leq n-3, i\ is\ odd \\ \left[\frac{f(v_{i-1})}{n-i+1}\right] & for\ 4 \leq i \leq n-4, i\ is\ even \\ \frac{f(v_{i-1})}{2} & for\ i = n-2 \\ i-1 & for\ i = n-1, n \end{cases}$$

Define the labels $f(u_i)$ as in case(iii). By defining $f(e_i) = \left[\frac{f(u)}{f(v)}\right]$, $f(u) > f(v)$ where f(u) and f(v) denote the labels of the end vertices for each edge $e_i$, $i$ varies from 1 to (2n-1), all the (2n-1) edges of the graph $C_n \odot P_n$ are labeled with the first $2n-1$ natural numbers such that $\sum_{i=1}^{2n-1} {e_i}^2 = p_{2n-1}$ where $p_{2n-1}$ is the $(2n-1)^{th}$ Pyramidal number. Hence $C_n \odot P_n$ is a Sequential Pyramidal graph for all $n \geq 3$.

**Example:**




**Sequential Pyramidal labeling** of $C_8 \odot P_8$ $\left(\sum_{i=1}^{15} \boldsymbol{e_i}^2 = \mathbf{p_{15} = 1240}\right)$

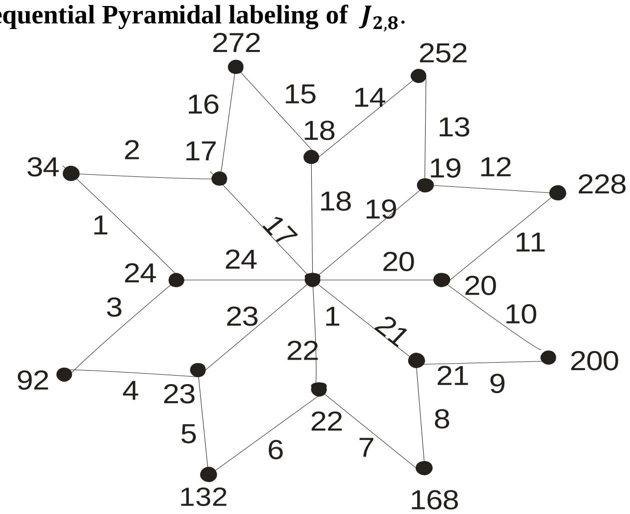
**Theorem 2.9:** All Jahangir graphs $J_{2,m}$, $m \geq 3$ are Sequential Pyramidal graphs .

**Proof:** Let $G = J_{2,m}$ be a Jahangir graph on 2m+1 vertices consisting of a Cycle $C_{2m}$ with one additional vertex which is adjacent to m vertices of $C_{2m}$ at a distance 2 to each other on $C_{2m}$. Let $v_1, v_2, \ldots, v_{2m}$ be the vertices that are incident clockwise on Cycle $C_{2m}$ . Let $v_{2m+1}$ denote the central vertex. Define $f: V(G) \rightarrow \{1,2,3, \ldots\ldots..pq\}$ as follows:

$$f(v_i) = \begin{cases} 2m + (i-j) & for\ 1 \leq i \leq 2m-1,\ 0 \leq j \leq m-1, i\ is\ odd \\ \{2m-(i-2)\}f(v_{i-1}) & for\ 2 \leq i \leq 2m-2, i\ is\ even \\ 2f(v_1) & for\ i = 2m \\ 1 & for\ i = 2m+1 \end{cases}$$

Hence all the 2m +m edges of the Jahangir graph $J_{2,m}$ are labeled with the first 2m+m natural numbers.

**Sequential Pyramidal labeling of $\boldsymbol{J_{2,8}}$.**

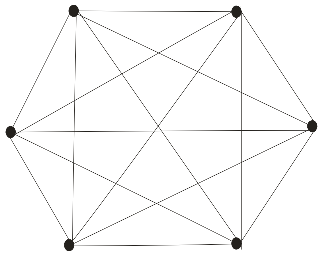



## III. NON-SEQUENTIAL PYRAMIDAL GRAPHS

**Definition 3.1:** The graphs which do not admit Sequential Pyramidal labeling are termed as Non-Sequential Pyramidal graphs. The following are some classes of Non-Sequential Pyramidal graphs.

**Theorem 3.2:** All Complete graphs $K_p$, p ≥ 4 are Non-Sequential Pyramidal graphs.

**Proof:** Let $K_p$ be a Complete graph with p vertices and q edges where p ≥ 4. In a Complete graph any two distinct points are adjacent and q = $\binom{p}{2}$. If $K_p$ admits a Sequential Pyramidal labeling, to receive the edge label q the following are the possibilities:



**Complete graph $K_6$**

**Case-1:** Any two of the vertices of $K_p$ are assigned the labels 1 and q. Let $v_o$ be a vertex labeled with 1. Since deg $v_o$ = $p$ – 1, there are $p - 1$ vertices adjacent to $v_o$. According to the condition of a Sequential Pyramidal labeling these $p - 1$ vertices must be labeled as q, q – 1, q – 2, …,q – $(p-2)$. On dividing by 1, the $p - 1$ edges incident with $v_o$ receive the labels q, q – 1, q – 2, …, q – $(p-2)$. Now we observe that $\left\lfloor\frac{q}{q-1}\right\rfloor$ = 1, $\left\lfloor\frac{q-1}{q-2}\right\rfloor$ = 1 for any q ≥ 6. Hence there are two edges receiving the same label 1 which is a contradiction.

**Case-2:** Suppose two of the vertices of $K_p$ are assigned the labels 2 and 2q. Let $v_1$ be a vertex labeled with 2. There are $p - 1$ vertices adjacent to $v_1$. To receive the other edge labels these $p - 1$ vertices adjacent to $v_1$ must be labeled as 2q, 2(q – 1), 2(q – 2), …,$2\big(q-(p-2)\big)$. On dividing by 2 the edge labels are again q, q – 1, q – 2, …,q – $(p-2)$. Now, $\left\lfloor\frac{2q}{2(q-1)}\right\rfloor$ = 1, $\left\lfloor\frac{2(q-1)}{2(q-2)}\right\rfloor$ = 1 for any q ≥ 6. Hence there are two edges receiving the same label 1 which is a contradiction. The above proof will hold good for the case with vertex labels 3q and 3, 4q and 4,…,$(p-1)q$ and $(p-1)$, pq and p. For, considering the vertex labels $pq$ and p proceeding as before we have the edge labels as q, q – 1, q – 2, …,q – $(p-2)$ in which again $\left\lfloor\frac{pq}{p(q-1)}\right\rfloor$ = 1, $\left\lfloor\frac{p(q-1)}{p(q-2)}\right\rfloor$ = 1 for q ≥ 6 which is again a contradiction.

**Case 3:** If x is any other vertex label such that $p+1 \le x \le pq-1$.

**Subcase-3A:** To get the edge label q suppose any vertex adjacent to x is labeled as $qx$ so that $\left\lfloor\frac{qx}{q}\right\rfloor$ = q. Now for $x = p+1$ we have $qx = q(p+1) = pq+q > pq$ which is not possible as the maximum range of vertex labels in a Sequential Pyramidal graph is pq. Also when $x = pq-1$ we have $qx = q(pq-1) = pq^2 - q > pq$ for any $q \ge 6$, $p \ge 4$ which obviously exceeds the maximum range of vertex labels. Hence such a case is not possible.

**Subcase-3B:** To receive the edge labels 1,2,…,q suppose the vertices adjacent to x are labeled by successive multiplication and division it will be shown that there are atleast two edges receiving the same edge label thus giving a contradiction. Let $v_1, v_2,\ldots, v_p$ be the vertices of the outer cycle in the Complete graph $K_p$ in the clockwise direction. Let us consider the case $x = p+3$. Let $f(v_i)$ denote the labels of the vertices $v_i$, $i = 1\ to\ p$ and $E(v_iv_{i+1})$ denote the label of the edges incident with the vertices $v_i$, $v_{i+1}$. On repeated multiplication and division we have the following procedure: Without loss of generality assume that

$$x = p+3 = f(v_1), f(v_2) = 2(p+3). \text{ Hence } E(v_1v_2) = \left\lfloor\frac{f(v_2)}{f(v_1)}\right\rfloor = \left\lfloor\frac{2(p+3)}{p+3}\right\rfloor = 2 \quad (1)$$

Now, $f(v_3) = \left\lfloor\frac{f(v_2)}{3}\right\rfloor = \left\lfloor\frac{2(p+3)}{3}\right\rfloor$, $E(v_2v_3) = \left\lfloor\frac{f(v_2)}{f(v_3)}\right\rfloor = \left\lfloor\frac{2(p+3)3}{2(p+3)}\right\rfloor$ =3 and

$f(v_4)$ = 4 $f(v_3)$ = $\left\lfloor\frac{8(p+3)}{3}\right\rfloor$, $E(v_3v_4) = \left\lfloor\frac{f(v_4)}{f(v_3)}\right\rfloor = \left\lfloor\frac{8(p+3)3}{3\times 2(p+3)}\right\rfloor$ = 4.

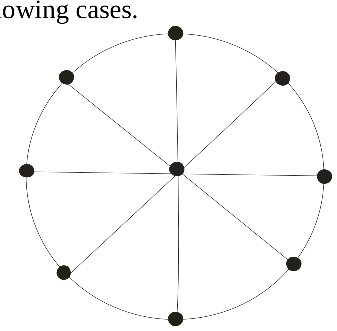
Now for the edge $E(v_1v_4)$. we have

$$E(v_1v_4) = \left\lfloor\frac{f(v_4)}{f(v_1)}\right\rfloor = \left\lfloor\frac{8(p+3)}{3(p+3)}\right\rfloor = 2 \quad (2)$$

Equations (1) and (2) imply that there are two edges receiving the same label 2 which is a contradiction. Hence in all the cases discussed above we get a contradiction. Therefore we conclude that all Complete graphs $K_p$, p ≥ 4 are Non-Sequential Pyramidal graphs.

**Theorem 3.3:** All Wheel graphs $W_n, n \geq 4$ are Non-Sequential Pyramidal graphs.

**Proof:** Let $G = W_n$ be a Wheel graph with n vertices, $n \geq 4$. Let $v_0$ be the central vertex of the Wheel. Let $p, q$ denote the number of vertices and edges in the Wheel where $p = n$ and $q = 2n - 2$. Suppose $W_n$ admits a Sequential Pyramidal labeling, then we discuss the following cases.



**Wheel graph $W_9$**

**Case-1:** Suppose $f(v_0)$ = 1. Then we have the following subcases:

**Subcase-1A:** Suppose the $p-1$ vertices adjacent to $v_0$ are labeled with 2, 3,…,$n$ then on dividing by 1, the inner edges of the Wheel incident with $v_0$ receive the labels 2,3,…,$n$. Now, for any two edges on the boundary of the Wheel we have $\left[\frac{3}{2}\right]$ = 1, $\left[\frac{4}{3}\right]$ = 1. This implies that two edges receive the same label 1 which is a contradiction.

**Subcase-1B:** Suppose the $p-1$ vertices adjacent to $v_0$ are labeled as $q, q-1, q-2, \ldots q-(n-2)$ then on dividing by 1, the inner edges of the Wheel incident with $v_0$ again receive the labels $q, q-1, \ldots\ q-(n-2)$. Now, $\left[\frac{q}{q-1}\right]$ = 1, $\left[\frac{q-1}{q-2}\right]$ = 1 for any integer q ≥ 6. Hence there are two edges receiving the same label 1 which is a contradiction.

**Case-2:** Suppose $f(v_0) = pq$. In this case the $p-1$ vertices adjacent to $v_0$ cannot be labeled with the integers 1,2,3,…,$n-1$ . For, in the above labeling the following are the edge labels: $\left[\frac{pq}{1}\right] = pq\ > q$, $\left[\frac{pq}{2}\right] = \left[\frac{n(2n-2)}{2}\right] = n(n-1) > 2(n-1) = q$. Proceeding like this, $\left[\frac{pq}{n-1}\right] = \left[\frac{n(2n-2)}{n-1}\right] = 2n\ > 2n-2 = q$. In this labeling the edge labels exceed the maximum range q which is not possible. Therefore the $p-1$ vertices adjacent to $v_0$ must be labeled only with the integers $n, n+1, n+2, \ldots, n+(n-2)$. Now, for any two edges on the boundary of the Wheel the labels are $\left[\frac{n+1}{n}\right] = 1$, $\left[\frac{n+2}{n+1}\right] = 1,\ n \geq 4$. Hence there are two edges receiving the same edge label 1 which is a contradiction.

**Case-3:** Suppose the central vertex $v_0$ is labeled with any of the integers $2,3, \ldots, pq-1$.

**Subcase-3A:** If $v_0$ is labeled with a label x such that $2 \leq x \leq p$ then the $p-1$ vertices adjacent to $x$ can be labeled either by multiplying $x$ by the integers 2,3,…,n to receive the edge labels 2, 3,..n which are of the lowest range or by multiplying by the integers $q, q-1, q-2, \ldots, q-(n-2)$ to receive the edge labels of the highest range. In either case for the vertices on the boundary the edge labels are $\left[\frac{3x}{2x}\right]$ = 1, $\left[\frac{4x}{3x}\right]$ = 1, $\left[\frac{qx}{(q-1)x}\right]$ = 1, $\left[\frac{(q-1)x}{(q-2)x}\right]$ = 1, for any $q \geq 6$. Therefore in both cases we have two edges receiving the same edge label 1 which is a contradiction.

**Subcase-3B:** Suppose x is a one digit number such that $p+1 \leq x \leq pq-1$. Then we have the following discussions:

(i) If the $p-1$ vertices adjacent to x are labeled by dividing x with the integers 2, 3,… atleast two of the edges receive the same edge label giving a contradiction.

(ii) If any vertex adjacent to x is given the label $qx$ to receive the edge label $\left[\frac{qx}{x}\right]$ = 1, for $x = p+1$ we have, $qx = q(p+1) = pq + q > pq$ that exceeds the maximum range of vertex labels pq which is not possible. This is true for any x in the range $p+1 \leq x \leq pq-1$.

(iii) If the $p-1$ vertices adjacent to x are labeled by multiplying x with the integers 2,3,…,n as in Subcase 1A we get $\left[\frac{3}{2}\right]$ = 1, $\left[\frac{4}{3}\right]$ = 1which implies that two edges receive the same label 1 giving a contradiction.

**Subcase-3C:** Suppose x is a two digit number such that $p+1 \leq x \leq pq-1$. As x is a two digit number the labels of the $p-1$ vertices adjacent to x cannot be assigned by multiplication process as it exceeds the maximum range $pq$. Hence in this case the labels are assigned by division process. Therefore the $p-1$ vertices adjacent to x can either be labeled by dividing out x by the integers 2,3,…,n or by dividing out x by the integers $q, q-1, q-2, \ldots, q-(n-2)$ to receive the corresponding edge labels. In either case there are atleast two edges receiving the same label giving a contradiction. Hence we conclude that all Wheel graphs $W_n,\ n \geq 4$ are Non-Sequential Pyramidal graphs.

**Remark 3.4:** The Friendship graph or the Dutch windmill graph got by the one point union of t copies of the Cycle $C_3$ where t $\geq 3$, are Non-Sequential pyramidal graphs.

## IV. CONCLUSION

If a graph has a vertex v adjacent to every vertex in the graph then the graph fails to be a Sequential Pyramidal graph. Also all graphs with atleast four Cycles behave as Non-Sequential Pyramidal graphs. Hence it is interesting to investigate such classes of graphs. This work has linked natural numbers with Pyramidal numbers as every Pyramidal number can be written as the sum of the squares of natural numbers. Also those graphs in which the set of vertex labels and the set of the squares of the edge labels have empty intersection can be popularly termed as Super Sequential Pyramidal graphs.

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**Source of support: Nil, Conflict of interest: None Declared.**