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THE UPPER CONNECTED EDGE DETOUR NUMBER OF AN EDGE DETOUR GRAPH

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ABSTRACT

For any two vertices u and v in a connected graph G, the detour distance $D(u,v)$ is the length of a longest $u-v$ path in G. A $u-v$ path of length $D(u,v)$ is called a $u-v$ detour. A set $S \subseteq V$ is called an edge detour set of G if every edge in G lies on a detour joining a pair of vertices of S. A connected edge detour set of a graph G is a edge detour set S such that the sub graph $< S >$ induced by S is connected. The minimum cardinality of a connected edge detour set of G is a connected edge detour number, denoted by $cdn_1(G)$ of G and any connected edge detour set of order $cdn_1(G)$ is called a connected edge detour basis of G. A connected edge detour set S in a connected graph G is called a minimal connected edge detour set of G if no proper subset of S is a connected edge detour set of G. The upper connected edge detour number $cdn_1^+(G)$ of G is the maximum cardinality of a minimal connected edge detour set of G. In this paper the upper connected edge detour number of certain classes of graphs is determined. It is proved that for each pair a,b of integers with $5 \le a \le b$ \$5, there is a connected graph G with $cdn_1(G) = a$ and $cdn_1^+(G) = b$. It is also proved that for every pair a,b of integers with $2 < a < b$, there exists a connected graph G with $cdn_1^+(G) = a$ and $cdn_1^+(G) = b$ S.

Keywords: *detour, edge detour number, connected edge detour number, upper connected edge detour number.*

AMS Subject Classification: *05C12.*

INTRODUCTION

By a *graph* $G = (V, E)$, we mean a finite undirected connected simple graph. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies, we refer to [1, 4].

For vertices u and v in a connected graph G, the *distance* $d(u,v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is called a $u-v$ *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v.

The minimum eccentricity among the vertices of G is the *radius*, $rad(G)$ of G and the maximum eccentricity is its *diameter*, $diam(G)$ of G.

A vertex x is said to lie on a $u-v$ *detour* P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum cardinality of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A detour set S in a connected graph G is called a *minimal detour set* if no proper subset of S is a detour set. The *upper*

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detour number $dn^{+}(G)$ is the maximum cardinality of a minimal detour set of G. The upper detour number of a graph was introduced and studied by Chartrand *et.al* [3].

A set $S \subseteq V$ is called a *connected detour set* of G if S is a detour set of G and the subgraph $G < S >$ induced by S is connected. The connected *detour number* $cdn(G)$ of G is the minimum cardinality of its connected detour set and any connected detour set of order $cdn(G)$ is called a *connected detour basis* of G. A connected detour set S in a connected graph G is called a *minimal connected detour set* of G if no proper subset of S is a connected detour set of G. The *upper connected detour number* $cdn^{+}(G)$ of G is the maximum cardinality of a minimal connected detour set of G. In 2009, the upper connected detour number of a graph was introduced and studied by Santhakumaran and Athisayanathan [5].

A set $S \subseteq V$ is called an *edge detour set* of G if every edge in G lies on detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum cardinality of its edge detour sets and any edge set of order $dn_1(G)$ is an *edge detour basis* of G. An edge detour set S in a connected graph G is called a *minimal edge detour set* of G if no proper subset of S is an edge detour set of G. The *upper edge detour number* $dn^{+}(G)$ of G as the maximum cardinality of a minimal edge detour set of G. In 2011, the upper edge detour number of a graph was introduced and studied by Santhakumaran and Athisayanathan [6, 7].

A set $S \subseteq V$ is called a *connected edge detour set* of G if S is an edge detour set G and the subgraph $< S >$ induced by S is connected. The *connected edge detour number* $cdn_1(G)$, of G is the minimum order of its connected edge detour sets and any connected edge detour set of order $cdn_1(G)$ is called a *connected edge detour basis* of G. These concepts were studied by Prabakar and Athisayanathan [8].

The following theorems are used in sequel.

Theorem 1.1[7]: All the end-vertices and the cut-vertices of an edge detour graph G belong to every connected edge detour set of G.

Theorem 1.2[7]: Let G be the complete graph K_n $(n \geq 3)$ or cycle $C_n(n \geq 3)$. Then a set $S \subseteq V$ is a connected edge detour basis of G if and only if S consists of any three adjacent vertices of G.

Theorem 1.3[7]: Let G be the complete bipartite graph $K_{m.n}(2 \leq m \leq n)$. Then a set $S \subseteq V$ is a connected edge detour basis of G if and only if S consists of any three vertices of G such that two vertices from one partition and one from other partition of G.

Theorem 1.4[7]: If T is a tree of order $n \geq 2$, then $cdn_1(T) = n$.

Theorem 1.5[7]:

- **(a)** If G is the complete graph K_n, then $cdn_1(G) = 3$.
- (b) If G is the complete bipartite graph $K_{m,n}(2 \leq m \leq n)$, then G $cdn_1(G) = 3$.
- (c) If G is the cycle C_n, then $cdn_1(G) = 3$.

Throughout this paper G denotes an edge detour graph with at least two vertices.

2. UPPER CONNECTED EDGE DETOUR NUMBER

Definition 2.1: A connected edge detour set S in an edge detour graph G is called a *minimal connected edge detour set* of G if no proper subset of S is a connected edge detour set of G. The upper connected edge detour number $cdn_1(G)$ of G is the maximum cardinality of a minimal connected edge detour set of G.

Example 2.2: For the edge detour graph G given in Figure 2.1, $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ $S_2 = \{v_1, v_2, v_3, v_9, v_{10}, v_5, v_6\}$, $S_3 = \{v_1, v_2, v_3, v_7, v_8, v_4, v_5, v_6\}$ are the minimal connected edge detour sets of G so that $cdn_1(G) = 7$. Moreover, the sets S_1, S_2 and S_3 contain the cut-vertices v_3 and v_5 of G and end-vertices v_1 and v_2 of G. Thus, every minimal connected edge detour sets of an edge detour graph must contain cut-vertex and end-vertex of an edge detour graph G.

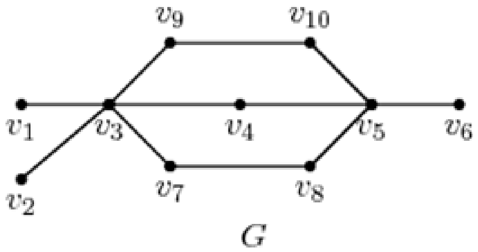

Figure-2.1

Example 2.3: For the edge detour graph G given in Figure 2.1, it is clear that the set $S_1 = \{v_1, v_2, v_7, v_4, v_9\}$ is a minimal connected detour set of G so that $dn_1(G) = 5$. Also the set $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the minimal connected edge detour set of G so that $cdn_1(G) = 7$. Hence the minimal edge detour set and minimal connected edge detour set of an edge detour graph G are different.

Example 2.3: For the edge detour graph G given in Figure 2.2, $S_1 = \{u, s, x, y, t, v\}$ and $S_1 = \{u, s, w, t, v\}$ are the minimal connected edge detour sets of G so that $cdn_1(G) = 5$ and $cdn_1^+(G) = 6$.

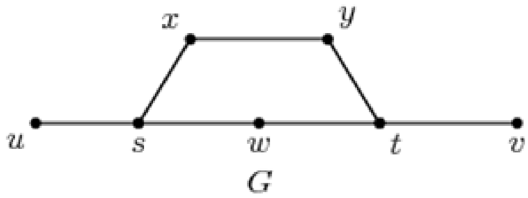

Figure-2.2

Remark 2.4: Every minimum connected edge detour set is a minimal connected edge detour set, but the converse is not true. For the edge detour graph G given in Figure 2.2, $S_1 = \{u, v, s, t, x, y\}$ is a minimal connected edge detour set of G but not a minimum connected edge detour set of G.

Theorem 2.5: For any edge detour graph G of order $n \geq 2$, $2 \leq cdn_1(G) \leq cdn_1^+(G) \leq n$.

Proof: A connected edge detour set needs at least two vertices so that $cdn_1(G) \geq 2$. Let S be any connected edge detour basis of G. Then S is also a minimal connected edge detour set of G and hence the result follows.

Corollary 2.6: Let G be an edge detour graph G of order n. If $cdn_1(G) = n$, then $cdn_1^+(G) = n$.

Remark 2.7: The bounds in Theorem 2.5 are sharp. For the complete graph $K_n(n \geq 2)$ and the cycle $C_n(n \geq 3)$, $cdn_1(G) = cdn_1^+(G) = 3$. Also for the edge detour graph G given in the Figure 2.2, $cdn_1(G) < cdn_1^+(G)$.

Now, we proceed to determine $cdn_1^+(G)$ for some classes of an edge detour graphs.

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Theorem 2.8: Let G be the complete graph $K_n(n \geq 2)$ or cycle $C_n(n \geq 3)$. Then a set $S \subseteq V$ is a minimal connected edge detour set of G if and only if S consists of any three adjacent vertices of G.

Proof: If S consists of any three adjacent vertices of G, then by Theorem 1.2, S is a connected edge detour basis of G so that S is minimal. Conversely, assume that $S \subseteq V$ be a minimal connected edge detour set of G. If $|S| = 2$, then S is not a connected edge detour set of G. Let $|S| \geq 4$. Let S_1 be any subset of S with $|S_1| = 3$. Then by Theorem 1.2, $|S_1|$ is a connected edge detour set of G so that S is not a minimal connected edge detour set of G, which is contradiction. Thus S consists of any three adjacent vertices of G.

Theorem 2.9: Let G be the complete bipartite graph $K_{n,m}(2 \leq n \leq m)$. Then a set $S \subseteq V$ is a minimal connected weak edge detour set of G if and only if S consists of any three vertices of G such that two vertices from one partition and one from other partition of G.

Proof: If S consists of any three vertices of G such that two vertices from one partition and one from other partition of G, then by Theorem 1.3, S is a connected edge detour basis of G so that S is minimal. Conversely, assume that $S \subseteq V$ be a minimal connected edge detour set of G such that $|S| = 4$. Then there exists a subset $S_1 = \{u, v, w\}$ of S such that two vertices from one partition and one from other partition of G. Then by Theorem 1.3, S_1 is a connected edge detour set of G, which is a contradiction.

Theorem 2.10: Let G be an edge detour graph of order n.

(a) If G is the complete graph $K_n(n \geq 2)$ then $cdn_1(G) = cdn_1^+(G) = 3$.

(b) If G is the complete bipartite graph $K_{n,m}(2 \leq n \leq m)$, then $cdn_1(G) = cdn_1^+(G) = 3$.

(c) If G is the cycle $C_n(n \geq 3)$ then $cdn_1(G) = cdn_1^+(G) = 3$.

(d) If G is any tree of order $n \geq 2$, then $cdn_1(G) = cdn_1^+(G) = n$.

Proof:

(a) This follows from Theorem 1.5(a) and Theorem 2.8.
(b) This follows from Theorem 1.5(b) and Theorem 2.9.
(c) This follows from Theorem 1.5(c) and Theorem 2.8.
(d) This follows from Theorem 1.4 and Corollary 2.6.

The following theorem gives a realization result.

Theorem 2.1: For every pair a, b of integers with $5 < a < b$, there exists an edge detour graph G with $cdn_1(G) = a$ and $cdn_1^+(G) = b$.

Proof: Let $5 < a < b$. Let G be an edge detour graph obtained from the cycle $C : v_1, v_2 .... v_{b-a+4}, v_1$ of order $b - a + 4$ by adding $a - 3$ new vertices $u_1, u_2, ... u_{a-3}$ and joining u_1 to v_1 and each $u_i(2 < i < (a-3))$ to v_{b-a+3} of C. The edge detour graph G is connected of order $b + 1$ and is shown in Figure 2.4. Let $X = \{v_2, v_3, ... v_{b-a+2}\}$, $Y = \{u_2, u_3, ... u_{a-3}, v_1, v_2 .... v_{b-a+3}\}$ and $Z = \{v_{b-a+4}\}$.

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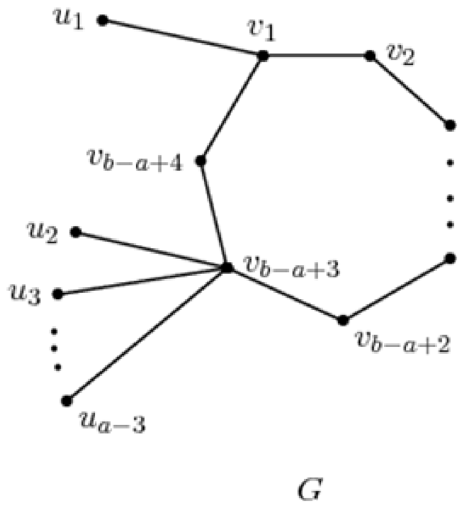

Figure-2.4

First, we show that $cdn_1(G)=a$. By Corollary 1.1, every connected edge detour set of G contains Y. Clearly Y is not a connected edge detour set of G and so $cdn_1(G) \geq |Y|+1=a$. On the other hand, let $S=Y\cup Z$. Then every edge of G lies on detour joining a pair of vertices S. Also, $G<S>$ is connected. Hence S is a connected edge detour set of G and so $cdn_1(G)\leq |S|=a$. Therefore $cdn_1(G)=a$.

Now, we show that $cdn_1^+(G)=b$. Let $S'=X\cup Y$. Then it is clear that S' is a connected edge detour set of G. We show that S' is a minimal connected edge detour set of G. Assume, to the contrary, that S' is not a minimal connected edge detour set of G. Then there is a proper subset T of such that T is a connected edge detour set of G. Since T is a proper subset of S', there exists a vertex $v\in S'$ and $v\notin T$. By Theorem 1.1, every connected edge detour set contains Y and so we must have $v=v_i\in X$ for some $i(2<i<(b-a+2))$. Then it is clear that $G<T>$ is not connected and so T is not a connected edge detour set of G, which is a contradiction. Thus S' is a minimal connected edge detour set of G and so $cdn^+(G)>|S'|=b$. Now, if $cdn^+(G)>b$, then let M be a minimal connected edge detour set of G with $|M|\geq b+1$. Since G has $b+1$ elements and S' is a minimal connected edge detour set of G, it follows that M is not a minimal connected edge detour set of G, which is a contradiction. Therefore $cdn^+(G)=b$.

Theorem 2.12: For every pair a,b of integers with $3<a<b$, there exists an edge detour graph G with $dn_1(G)=a$ and $cdn^+(G)=b$.

Proof: For any tree of order b with a end-vertices is the desired graph.

Theorem 2.13: For every pair a,b \$a\$, \$b\$ of integers with $2<a<b$, there exists an edge detour graph G \$G\$ with $dn^+(G)=a$ and $cdn^+(G)=b$.

Proof: Let G be an edge detour graph obtained from the path $P: v_1, v_2, ...v_a$ of order a by adding $w_1, w_2, ...w_{b-a}$ of order $b-a$ new vertices joining v_1 and v_2. Let $u_1, u_2, ...u_{b-a}$ of order $b-a$ \$ new vertices joining v_1 and w_{b-a}. The edge detour graph G is connected of order $2b-a$ and shown in the Figure 2.5. Let $X=\{v_1, v_2, ...v_a\}$, $Y=\{w_1, w_2, ...w_{b-a}\}$, $Z=\{u_1, u_2, ...u_{b-a}\}$. For we show that $cdn^+(G)=a$. By Corollary 1.8, $S=X$ is connected detour set of G. Clearly S is also a minimal connected detour number of G.

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Now, we show that $cdn^+(G) = b$. Let $S' = \{X \cup Y\}$. Then it is clear that S' is a connected edge detour set of G. We show that S' is a minimal connected edge detour set of G. Assume, to the contrary, that S' is not a minimal connected edge detour set of G. Then there is a proper subset T of S' such that T is a connected edge detour set of G. Since T is a proper subset of S', there exists a vertex $v \in S'$ and $v \notin T$. By Theorem 1.1, every connected edge detour set contains Y and so we must have $v = v_i \in X$ for some $i(1 < i < a)$. Then it is clear that $G < T >$ is not connected and so T is not a connected edge detour set of G, which is a contradiction. Thus S' is a minimal connected edge detour set of G and so $cdn^+(G) > |S'| = b$. Now, if $cdn^+(G) > b$, then let M be a minimal connected edge detour set of G with $|M| \geq b+1$. Since G has $b+1$ elements and S' is a minimal connected edge detour set of G, it follows that M is not a minimal connected edge detour set of G, which is a contradiction. Therefore, $cdn^+(G) = b$.

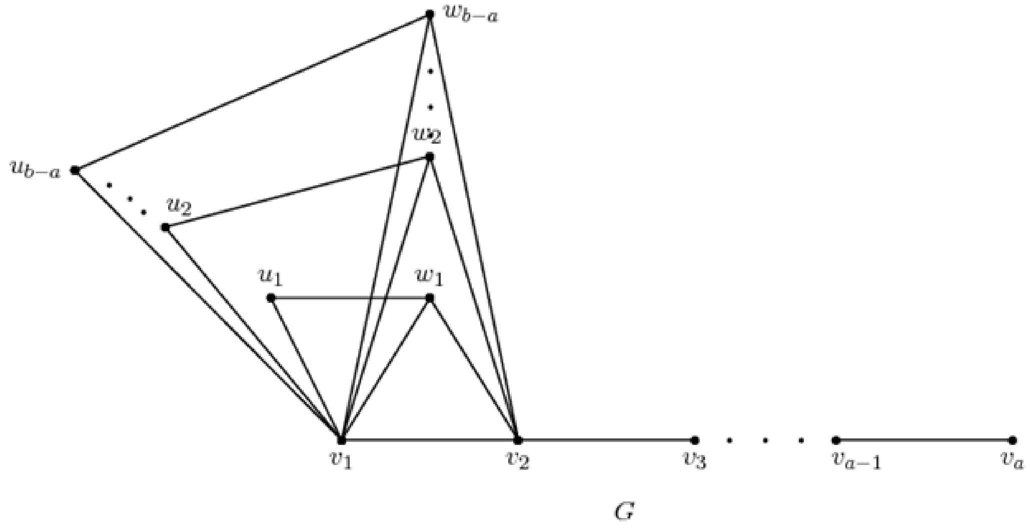

Figure-2.5

Remark 2.14: The edge detour graph G in Figure 2.6 contains exactly 2 minimal connected edge detour sets namely $X \cup Y$ and $Y \cup Z$. Hence this example shows that there is no ``Intermediate Value Theorem'' for minimal connected edge detour sets, that is, if k is an integer such that $cdn_1(G) < k < cdn_1^+(G)$, then there need not exist a minimal connected edge detour set of cardinality k in G.

Using the structure of the graph G constructed in the proof of Theorem 2.12, we can obtain a graph H_n of order n with $cdn_1(G) = 5$ and $cdn^+(G) = n-1$ for all $n > 6$. Thus we have the following. There is an infinite sequence H_n of edge detour graphs H_n G of order $n > 6$ such that $cdn_1(H_n) = 5$, $cdn^+(H_n) = n-1$, $\lim_{n\to\infty} \frac{cdn_1(H_n)}{n} = 0$ and $\lim_{n\to\infty} \frac{cdn_1^+(H_n)}{n} = 1$.

Let H_n be the graph obtained from the cycle $C : v_1, v_2 .....v_{n-1}, v_1$ of order $n-2$ by adding two new vertices u_1, u_2 and joining u_1 to v_1 and each u_2 to v_{n-3} of C. The graph H_n is connected and is shown in Figure 2.6. Let $X = \{v_2, v_3 .....v_{n-4}\}, Y = \{u_2, v_1 .....v_{n-3}\}$ and $Z = \{v_{n-2}\}$. It is clear from the proof of Theorem 2.12 that the graph H_n contains exactly 2 minimal connected edge detour sets namely $X \cup Y$ and $Y \cup Z$ so that $cdn^+(H_n) = n-1$ and $cdn_1(G) = 5$. Hence the theorem follows.

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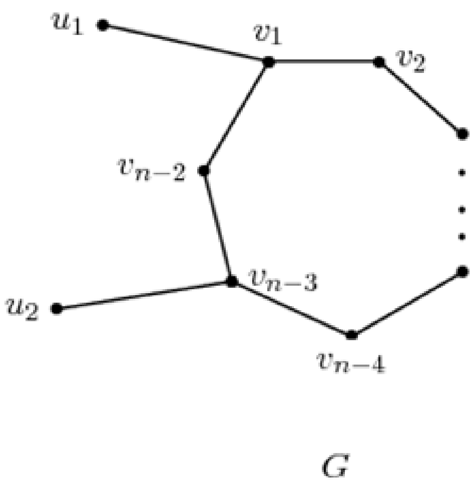

Figure-2.6

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