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ON WRAPPED NEGATIVE BINOMIAL MODEL

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ABSTRACT

In this paper, a new discrete circular model called Wrapped Negative Binomial distribution is constructed by applying the method of wrapping for Negative Binomial distribution. The characteristic function of the Wrapped Negative Binomial Distribution is derived and the population characteristics such as Mean, Variance, Standard Deviation, Skewness and Kurtosis are studied. The graph of probability mass function for various values of parameters are plotted using MATLAB.

Keywords: *Circular model, Characteristic function, Trigonometric moments, Mean direction.*

1. INTRODUCTION

Circular data arise in many diverse scientific contexts. Many examples of this kind of data are found in earth science, Ecology (as wind direction analysis), Biology (study of animal movement direction), Physics and more. in general any context where the study of data recorded in radians or degrees on the unit circle.

Many good number of continuous circular models are available in the literature. Wrapped circular models (Girija (2010)), stereographic circular and semicircular models (Phani (2013)) and Offset circular and semicircular models (Radhika (2014)) and l – axial models (Sastry (2016)) provide a rich and very useful class of models for circular as well as l-axial data. Scant attention was paid in construction of discrete circular models. There is a need to develop discrete circular models which are invariant of zero direction and sense of rotation. (Girija *et al.* (2014)) derived Wrapped Discrete Binomial model and studied its characteristics.

2. CIRCULAR DISTRIBUTIONS

A circular distribution is a probability distribution whose total probability is defined on the unit circle. Probabilities to different directions are assigned by representing each direction as point on the unit circle. The range of circular random variable θ measured in radians, can be taken as $[0, 2\pi)$ or $[-\pi, \pi)$.

Circular distributions are of two types, they may be discrete or continuous versions.

2.1. Wrapped Discrete Circular Random Variables

If X is a discrete random variable on the set of integers, then reduction modulo $2\pi m\left(m \in \mathbb{Z}^{+}\right)$ wraps the integers on to the group of m^{th} roots of unity which is a sub group of unit circle.

i.e $\theta = 2\pi x\left(\bmod 2\pi m\right)$

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More precisely θ is a mapping from a set of integers G which is a group with respect to '+' to the set of m^{th} roots of unity G' which is a group with respect to '.' is defined as

$$\theta(x) = e^{\frac{2\pi ix}{m}} \quad \text{where} \quad x \in G,\ e^{\frac{2\pi ix}{m}} \in G'$$

Then θ is called wrapped discrete circular random variable.

Clearly θ is a homomorphism

1) $\theta(x+y) = e^{\frac{2\pi i(x+y)}{m}} = e^{\frac{2\pi ix}{m}} e^{\frac{2\pi iy}{m}} = \theta(x)\theta(y)$

2) $\theta(0) = e^{\frac{2\pi i(0)}{m}} = e^{0} = 1$ where $0 \in G$,$1 \in G'$

Since θ contains a finite number of elements they are denoted by $\theta = \left\{ \frac{2\pi r}{m} \middle/ r = 0,1,2.....m-1 \right\}$ which is lattice on the unit Circle.

3. PROBABILITY MASS FUNCTION

Suppose if θ is a wrapped discrete circular random variable then probability mass function of θ is denoted by $Pr\left(\theta = \frac{2\pi r}{m}\right)$ which is defined as

$$Pr\left(\theta = \frac{2\pi r}{m}\right) = \sum_{k=-\infty}^{\infty} P(r+km) \quad \text{where } r = 0,1,...m-1 \text{ Such that } m \in \mathbb{Z}^{+} \tag{3.1}$$

Further if it is to be a circular probability mass function it should satisfy the following properties.

1. $Pr\left(\theta = \frac{2\pi r}{m}\right) \geq 0$
2. $\sum_{r=0}^{m-1} Pr\left(\theta = \frac{2\pi r}{m}\right) = 1$
3. $Pr(\theta) = Pr(\theta + 2\pi l)$ For any integer l (i.e Pr is periodic)

4. DISTRIBUTION FUNCTION

Suppose if θ is a wrapped discrete circular random variable then distribution function of θ is

Denoted by $F_w(\theta)$ which is defined as

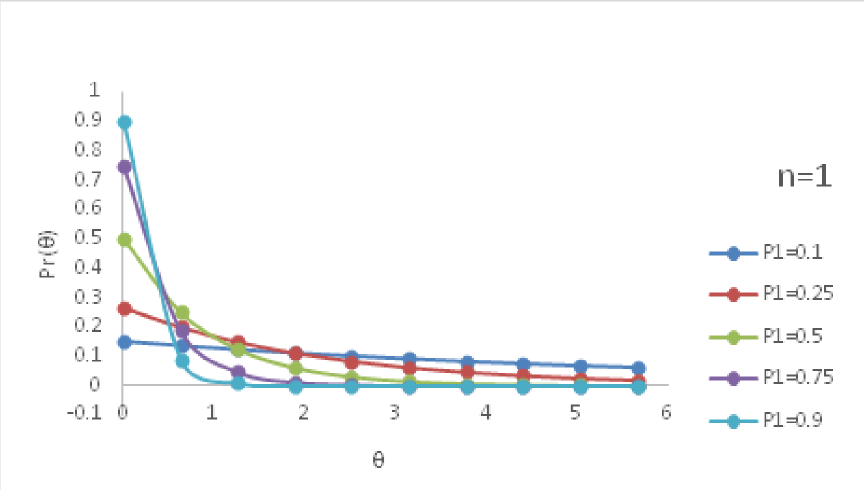
$$F_w(\theta) = \sum_{r=0}^{y} Pr\left(\theta = \frac{2\pi r}{m}\right) \quad \text{where } y = 0,1,...m-1$$

5. WRAPPED NEGATIVE BINOMIAL DISTRIBUTION

Suppose if X follows Negative Binomial Distribution with the parameters n, p_1 where $n \in \mathbb{Z}^{+}$ and $0 < p_1 < 1$ then the probability mass function of the wrapped Negative Binomial distribution defined from (3.1) as

$$Pr(\theta = \frac{2\pi r}{m}) = \sum_{k=0}^{\infty} (r+km+n-1)_{c_{n-1}}\, p_1^{\,n} q_1^{\,r+km}$$

Where p_1 =probability of success and q_1 =probability of failure $\ni$ $p_1 + q_1 = 1$



Graph for Probability Mass Function of Wrapped Negative Binomial Distribution

6. DISTRIBUTION FUNCTION OF WRAPPED NEGATIVE BINOMIAL DISTRIBUTION

The distribution function of the Wrapped Negative Binomial Distribution is defined as

$$F_w(\theta) = \sum_{r=0}^{y} Pr\left(\theta = \frac{2\pi r}{m}\right)$$

$$= \sum_{r=0}^{y} (r+km+n-1)_{c_{n-1}} p_1^{\,n} q_1^{\,r+km} \quad \text{where} \quad y = 0,1,\ldots m-1$$

7. CHARACTERISTIC FUNCTION OF THE WRAPPED NEGATIVE BINOMIAL DISTRIBUTION

The Characteristic function of the Wrapped Negative Binomial Distribution is defined as

$$\varphi_\theta(p) = \sum_{r=0}^{m-1} e^{ip\theta}\, Pr(\theta = \frac{2\pi r}{m}) \text{ where } \quad p \in \mathbb{Z}$$

$$= \sum_{r=0}^{m-1} e^{ip\theta}\left[\sum_{k=0}^{\infty} (r+km+n-1)_{c_{n-1}} p_1^{\,n} q_1^{\,r+km}\right]$$

$$= \left[\frac{p_1}{1-e^{\frac{2\pi ip}{m}} q_1}\right]^n = \rho_p e^{i\mu_p}$$

$$= \left[\frac{p_1}{\left(\left(1-q_1\cos\frac{2\pi p}{m}\right)-iq_1\sin\frac{2\pi p}{m}\right)} \frac{\left(\left(1-q_1\cos\frac{2\pi p}{m}\right)+iq_1\sin\frac{2\pi p}{m}\right)}{\left(\left(1-q_1\cos\frac{2\pi p}{m}\right)+iq_1\sin\frac{2\pi p}{m}\right)}\right]^n$$

$$= \left[\frac{p_1\left(1-q_1\cos\frac{2\pi p}{m}\right)+ip_1q_1\sin\frac{2\pi p}{m}}{1+q_1^2-2q_1\cos\frac{2\pi p}{m}}\right]^n$$

$$= (x+iy)^n \quad \text{Where} \quad x = \frac{p_1\left(1-q_1\cos\frac{2\pi p}{m}\right)}{1+q_1^2-2q_1\cos\frac{2\pi p}{m}} \quad \textit{and} \quad y = \frac{p_1q_1\sin\frac{2\pi p}{m}}{1+q_1^2-2q_1\cos\frac{2\pi p}{m}}$$

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By taking $x = R\cos\theta$ and $y = R\sin\theta$ we get $x^2 + y^2 = R^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\begin{aligned}(x+iy)^n &= (R\cos\theta + iR\sin\theta)^n \\ &= (R^n\cos n\theta + iR^n\sin n\theta) \\ &= \alpha_p + i\beta_p\end{aligned}$$

where $\alpha_p = R^n\cos n\theta$ and $\beta_p = R^n\sin n\theta$

Clearly

$$R = \sqrt{\left[\frac{p_1^2\left(1-q_1\cos\frac{2\pi p}{m}\right)^2 + p_1^2q_1^2\sin^2\frac{2\pi p}{m}}{\left(1+q_1^2-2q_1\cos\frac{2\pi p}{m}\right)^2}\right]}$$

$$R = \sqrt{\left[\frac{p_1^2\left(1+q_1^2-2q_1\cos\frac{2\pi p}{m}\right)}{\left(1+q_1^2-2q_1\cos\frac{2\pi p}{m}\right)^2}\right]}$$

$$R = \frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\frac{2\pi p}{m}}}$$

$$\text{Now}\quad \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left[\frac{p_1q_1\sin\frac{2\pi p}{m}}{\left(p_1\left(1-q_1\cos\frac{2\pi p}{m}\right)\right)}\right]$$

Then

$$\alpha_p = \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\frac{2\pi p}{m}}}\right]^n \cos n\left[\tan^{-1}\left[\frac{p_1q_1\sin\frac{2\pi p}{m}}{\left(p_1\left(1-q_1\cos\frac{2\pi p}{m}\right)\right)}\right]\right]$$

$$\beta_p = \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\frac{2\pi p}{m}}}\right]^n \sin n\left[\tan^{-1}\left[\frac{p_1q_1\sin\frac{2\pi p}{m}}{\left(p_1\left(1-q_1\cos\frac{2\pi p}{m}\right)\right)}\right]\right]$$

$$\text{Now}\ \rho_p = \sqrt{\alpha_p^2+\beta_p^2} = \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\frac{2\pi p}{m}}}\right]^n$$

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and $$\mu_p = \tan^{-1}\left[\frac{\beta_p}{\alpha_p}\right] = \left[\tan^{-1}\left[\tan n\left[\tan^{-1}\left[\left(\frac{p_1 q_1 \sin\frac{2\pi p}{m}}{p_1\left(1-q_1\cos\frac{2\pi p}{m}\right)}\right)\right]\right]\right]\right]$$

$$= \left[n\tan^{-1}\left[\frac{p_1 q_1 \sin\frac{2\pi p}{m}}{\left(p_1\left(1-q_1\cos\frac{2\pi p}{m}\right)\right)}\right]\right]$$

Now the circular mean direction is denoted by μ_1 and it is defined as $\mu_1 = n\tan^{-1}\left[\frac{p_1 q_1 \sin\frac{2\pi}{m}}{p_1\left(1-q_1\cos\frac{2\pi}{m}\right)}\right]$

Now ρ_1 represents the concentration towards mean direction which is defined as $\rho_1 = \left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\frac{2\pi}{m}}}\right]^n$

In general $\mu_1 = \mu$ and $\rho_1 = \rho$

Now the circular variance is denoted by V_o which is defined as

$$V_o = 1-\rho$$

$$= 1-\left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\frac{2\pi}{m}}}\right]^n$$

The circular standard deviation is denoted by σ_o and it is defined as

$$\sigma_o = \sqrt{-2\log(1-V_o)}$$

$$= \sqrt{-2\log\left[\frac{p_1}{\sqrt{1+q_1^2-2q_1\cos\frac{2\pi}{m}}}\right]^n}$$

$$= \sqrt{-2\left[\log p_1^n - \log(1+q_1^2-2q_1\cos\frac{2\pi}{m})^{\frac{n}{2}}\right]}$$

8. CENTRAL TRIGONOMETRIC MOMENTS

The p[th] Central trigonometric moment of θ is defined as

$$\varphi_\theta^*(p) = E\left[e^{ip(\theta-\mu)}\right] = E\left[e^{ip\theta}e^{-ip\mu}\right] = e^{-ip\mu}E\left[e^{ip\theta}\right]$$

$$= e^{-ip\mu}[\alpha_p + i\beta_p]$$

$$= (\cos p\mu - i\sin p\mu)[\alpha_p + i\beta_p]$$

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$$= \alpha_p \cos p\mu + i\beta_p \cos p\mu - i\alpha_p \sin p\mu + \beta_p \sin p\mu$$
$$= \left(\alpha_p \cos p\mu + \beta_p \sin p\mu\right) + i\left(\beta_p \cos p\mu - \alpha_p \sin p\mu\right)$$
$$= \alpha_p^* + i\beta_p^*$$

where $\alpha_p^* = \alpha_p \cos p\mu + \beta_p \sin p\mu$ and $\beta_p^* = \beta_p \cos p\mu - \alpha_p \sin p\mu$

For Wrapped Negative Binomial distribution Skewness is defined as

$$\gamma_1 = \frac{\beta_2^*}{\left(V_o\right)^{\frac{3}{2}}}$$
$$= \frac{\beta_2 \cos 2\mu - \alpha_2 \sin 2\mu}{\left(1 - \left[\frac{p_1}{\sqrt{1 + q_1^2 - 2q_1 \cos \frac{2\pi}{m}}}\right]^n\right)^{\frac{3}{2}}}$$

And the kurtosis is defined as

$$\gamma_2 = \frac{\alpha_2^* - (1 - V_o)^4}{\left(V_o\right)^2}$$
$$= \frac{\left(\alpha_2 \cos 2\mu + \beta_2 \sin 2\mu\right) - \left(\left[\frac{p_1}{\sqrt{1 + q_1^2 - 2q_1 \cos \frac{2\pi}{m}}}\right]^n\right)^4}{\left(1 - \left[\frac{p_1}{\sqrt{1 + q_1^2 - 2q_1 \cos \frac{2\pi}{m}}}\right]^n\right)^2}$$

REFERENCES

1. Abramowitz, M. and Stegun, I.A. (1965), Handbook of Mathematical Functions, Dover, New York.
2. Dattatreya Rao, A.V., RamabhadraSarma, I. and Girija, S.V.S. (2007), On Wrapped Version of Some Life Testing Models, Comm Statist,-Theor.Meth., 36, issue # 11, pp.2027-2035.
3. Girija, S.V.S. (2010). Construction of New Circular Models, VDM VERLAG, Germany. ISBN 978-3-639-27939-9.
4. Girija.S.V.S, A.V.DattatreyaRao and G.V.L.N.Srihari (2014) On Wrapped Binomial Model - Characteristics, Horizon Research PublicationsVolume2(7),pp.231-234(2014).
5. Mardia, K.V. and Jupp, P.E. (2000), Directional Statistics, John Wiley, Chichester.
6. Phani Y, (2013) "On Stereographic Circular and Semicircular Models", unpublished thesis submitted to AcharyaNagarjuna University for the award of Ph.D.
7. Phani, Y.,Girija, S.V.S., DattatreyaRao and G.V.L.N.Srihari (2015). Symmetric Circular Model Induced by Inverse Stereographic Projection On Double weibull Distribution, International Journal of soft computing, Mathematics and Control (IJSCMC), Vol 4 No.1,.pp.67-74
8. Radhika A.J.V, (2014) "Mathematical Tools in the Construction of New Circular Models", unpublished thesis submitted to AcharyaNagarjuna University for the award of Ph.D.
9. Rao Jammalamadaka S. and Sen Gupta, A. (2001), Topics in Circular Statistics, World Scientific Press, Singapore.
10. Sastry Ch.V. (2016). On *l* – Axial Models, unpublished thesis submitted to AcharyaNagarjuna University for the award of Ph.D.

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