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EVERY FINITELY PRESENTED TORSION GROUP IS FINITE

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ABSTRACT

Four lemmas are requisite to the presented proof that every finitely presented torsion group is finite (equivalently, some non-finitely presented torsion group H is infinite), namely, H is an abelian subgroup of the real numbers modulo 1 (proved using the Subgroup Criterion), every element of H is of finite order (equivalently, some element not in H is of infinite order), H is not finitely presented, and H is infinite; all of which are shown in this paper. The proof that some element not in H is of infinite order relies on the proof that the sum of a rational number and an irrational number is equal to an irrational number (Khan, n.d.). In the proof that H is not finitely presented, we first show that H is not finitely generated utilizing the important proposition that states that "the subgroup generated by the set A is equal to closure of A" (Dummit & Foote, 2004, p. 63).

Keywords: *finite groups, finitely presented group, torsion group, periodic group, real numbers modulo 1, Burnside problem.*

The conjecture that every finitely presented torsion (periodic) group is finite is commonly referred to as the Burnside problem. It is regarded as "one of the oldest and most influential in group theory" (Burnside problem, n.d.). Unlike prior research on this problem conducted by Burnside, Golod-Shafarevich, Kurosh, Schur, etc., the proof of the Burnside problem in this paper is not contingent upon these previous related findings. E.g., it does not involve Burnside groups, matrix groups, and/or p-groups.

Proposition 1: (*The Subgroup Criterion*) *A subset H of a group G is a subgroup if and only if*
(1) *$H \neq \emptyset$, and*
(2) *for all $x, y \in H$, $xy^{-1} \in H$* (Dummit & Foote, 2004, p. 47)

Proof: Omitted.

Lemma 1: *Let $(G,\star)$ be the real numbers modulo 1 and let $H = G \cap \mathbb{Q}$. $(H,\star)$ is an abelian subgroup of $(G,\star)$.*

Proof: Let $x \in H$, and $y \in H - \{0\}$. Since $0 \in H$, H is a nonempty subset of G. Note that $1 - y$ is the inverse of y with respect to $\star$ (cf. Appendix). Since H is also a subset of $\mathbb{Q}$ which is closed under addition, it follows that

$$x \star (1 - y) = x + (1 - y) - [x + (1 - y)]$$

belongs to H and since 0 is the identity element of G with respect to $\star$ (0 is its own inverse) (cf. Appendix), $x \star 0 = x \in H$. Hence, the Subgroup Criterion gives that $(H,\star)$ is a subgroup of $(G,\star)$ and since $(G,\star)$ is abelian, $(H,\star)$ is abelian by definition. We may refer to $(H,\star)$ as the *rational numbers modulo 1*.

Theorem 1: *If $A(x)$ is an open sentence with variable x, then $\sim(\forall x)A(x)$ is equivalent to $(\exists x){\sim}A(x)$*

Proof: Let U be any universe.
The sentence $\sim(\forall x)A(x)$ is true in U
iff $(\forall x)A(x)$ is false in U
iff the truth set of $A(x)$ is not the universe
iff the truth set of $\sim A(x)$ is nonempty
iff $(\exists x){\sim}A(x)$ is true in U.
(Smith, Eggen, Andre, & Richard, 1.3 Quantifiers, 2011, p. 23)

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Lemma 2: *For all $x \in H$, x is of finite order.*

Proof: By Theorem 1, the statement 'for some $x \notin H$, x is of infinite order' is logically equivalent to Lemma 2 (cf. proof of Theorem 4). Thus, we prove the equivalent statement: for some $x \notin H$, x is of infinite order. Consider, $\sqrt{2}/2 \in G - H$ which belongs to $G - H$ because the product of an irrational number $\sqrt{2}$ and a rational number $1/2$ is an irrational number (Khan, n.d.). It is easy to see that operating $\sqrt{2}/2$ repeatedly with respect to $\star$ produces elements of the form

$$j\frac{\sqrt{2}}{2} - k \cdot 1 \text{ with } j \in \mathbb{Z}^{+} \text{ and } k \in \mathbb{Z}^{+} \cup \{0\}$$

and the product/sum of an irrational number and a rational number (integer) is an irrational number (Khan, n.d.). The identity element $\mathbb{0}$ is a rational number, therefore $\sqrt{2}/2$ is of infinite order.

Theorem 2: Euclid. *There is an infinite number of primes.* (Burton, 2011, p. 46)

Proof: Omitted.

Definition 1: *If A is any subset of the group G define*

$$\langle A \rangle = \bigcap_{\substack{A \subseteq H \\ H \leq G}} H$$

This is called the subgroup of G generated by A. (Dummit & Foote, 2004, p. 62)

Definition 2: $\bar{A}$**,** called the closure of A, *is the set of all finite products (called words) of elements of A and inverses of elements of A.*
(Dummit & Foote, 2004, p. 63)

Proposition 2: $\bar{A} = \langle A \rangle$. (Dummit & Foote, 2004, p. 63)

Proof: Omitted.

Definition 3: *Let S be a subset of a group G such that $G = \langle S \rangle$. We say G is finitely generated if there is a presentation (S, R) such that S is a finite set and we say is finitely presented if there is a presentation with both and finite sets. R finite sets.* (Dummit & Foote, 2004, p. 218)

From Definition 3 we deduce that any group G is not finitely generated if there is an infinite set S such that $G = \langle S \rangle$.

Lemma 3: $(H,\star)$ *is not finitely presented.*

Proof: By Theorem 2, there is an infinitude of primes. Thus, $A = \left\{\frac{1}{p} \mid p \text{ is prime}\right\}$ is an infinite set. By Proposition 2, assume arbitrary $a_1^{\epsilon_1} \star a_2^{\epsilon_2} \star \ldots \star a_n^{\epsilon_n} \in \langle A \rangle$ where $a_i \in A$ and $\epsilon_i = \pm 1$. The ϵ_i are notation for the "non-inverse" or inverse of the a_i with respect to $\star$; i.e., it is not multiplicative notation. Since $0 < a_i < 1$ such that the a_i are rational, $a_i \in H$. Moreover, Lemma 1 proves that $a_i^{-1} \in H$. Thus $a_i^{\epsilon_i} \in H$. By closure, $a_1^{\epsilon_1} \star a_2^{\epsilon_2} \star \ldots \star a_n^{\epsilon_n} \in H$. But $a_1^{\epsilon_1} \star a_2^{\epsilon_2} \star \ldots \star a_n^{\epsilon_n}$ was arbitrary. Therefore $\langle A \rangle \subseteq H$.

Now suppose arbitrary $a_1^{\epsilon_1} \star a_2^{\epsilon_2} \star \ldots \star a_n^{\epsilon_n} \in H$. Since $a_i \in A$, by Proposition 2, $a_1^{\epsilon_1} \star a_2^{\epsilon_2} \star \ldots \star a_n^{\epsilon_n} \in \langle A \rangle$. But $a_1^{\epsilon_1} \star a_2^{\epsilon_2} \star \ldots \star a_n^{\epsilon_n}$ was arbitrary. Hence, $H \subseteq \langle A \rangle$.

Therefore $H = \langle A \rangle$ or $(H,\star)$ is not finitely generated because A is not a finite set by Definition 3. Consequently, $(H,\star)$ is not finitely presented by Definition 3.

Theorem 3: *(Density of $\mathbb{Q}$ in $\mathbb{R}$) If x and y are real numbers with $x < y$, then there exists a rational number r such that $x < r < y$.* (Lay, 2005, p. 125)

Proof: Omitted
From Theorem 3, we deduce the following corollary:

Corollary 1: *If x and y are real numbers with $x < y$, then there are infinitely many rational numbers in the interval* $[x, y]$. (Lay, 2005, p. 127)

Proof: Omitted.

Definition 4: *Let G be an abelian group.* $\{g \in G \mid |g| < \infty\}$ *is the torsion subgroup of G.* (Dummit & Foote, 2004, p. 48)

Definition 5: *Two quantified statements are **equivalent in a given universe** iff they have the same truth value in that universe.* (Smith, Eggen, Andre, & Richard, 1.3 Quantifiers, 2011, p. 22)

We now prove the following:

Theorem 4: *Every finitely presented torsion group is finite.*

Proof: By Theorem 1, the statement 'some non-finitely presented torsion group is infinite' is logically equivalent to Theorem 4. To see this more clearly, let $\mathcal{F}$ be the set of all finitely generated torsion groups, and let $\mathcal{I}$ be the set of all non-finitely (infinitely) generated torsion groups. Hence, the *universe* would be $\mathcal{T} = \mathcal{F} \cup \mathcal{I}$, the set of all torsion groups. Thus we rewrite the statements in symbolic form

$$\begin{gathered} \forall T \in \mathcal{F}, T \text{ is finite} \\ \Leftrightarrow \\ \exists T \in \mathcal{I}, T \text{ is infinite} \end{gathered} \tag{1}$$

which, to reiterate, is valid by Theorem 1. To clarify, set the set "U" in Theorem 1 equal to $\mathcal{I}$ and recall that the universe is $\mathcal{T} = \mathcal{F} \cup \mathcal{I}$ which is not U. Assume the latter statement in (1) is true in U, then statements 2 and 5 in the proof of Theorem 1 show that (1) is true because the truth set of 'T is finite' is $\mathcal{F}$ and not the universe; most notably the statement $(\forall x)A(x)$ is false in U, or in our case, $\forall T \in \mathcal{F}$, T is finite (it is false in $U = \mathcal{I}$ and true in $\mathcal{T} - U = \mathcal{F}$). I.e., $\forall T \in \mathcal{F}$, T is finite is true because there is no other choice for the truth set except $\mathcal{F}$ (it cannot be the universe $\mathcal{T}$ according to the statement in the proof of Theorem 1) and the statement 'T is finite' is clearly false in the empty set. By Definition 5, the statements in (1) are equivalent in the universe $\mathcal{T}$, we now prove they are true.

Thus, we prove the equivalent statement: some non-finitely presented torsion group is infinite. By Lemmas 1 and 2, $(H,\star)$ is a torsion group. By Lemma 3, $(H,\star)$ is non-finitely presented. Finally, fixing $x = 0$ and $y = 1$, we see that H is infinite by Corollary 1.

APPENDIX

Section 1.1, Exercise 7: Let $G = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ and for $x, y \in G$ let $x \star y$ be the fractional part of $x + y$ (i.e., $x \star y = x + y - [x + y]$ where $[a]$ is the greatest integer less than or equal to a). Prove that $\star$ is a well defined binary operation on G and that G is an abelian group under $\star$ (called the *real numbers mod 1*). (Dummit & Foote, 2004, p. 21)

Proof: Let $a, b, c \in G$. When $0 \leq a + b < 1$, then $[a + b] = 0$ (see "Identity" section for proof that $-0 = 0$). When $1 \leq a + b < 2$, then $[a + b] = 1$. Thus G is closed under $\star$. The mapping $a \star b : G \times G \to G$ is defined for all $a, b \in G$ because G is closed under $\star$.

Define a binary relation $\sim$ on G as follows:

$a \sim b$ if and only if $a \star b = a + b - [a + b]$ (i.e., $(a, b) \in G \times G$)

Reflexive:

$(a, a) \in G \times G$ because $a \star a$ is defined for all $a \in G$.

Symmetric:

$$\begin{aligned} (a, b) \in G \times G \Rightarrow a \star b &= a + b - [a + b] \\ &= b + a - [b + a] \text{ (ring axiom (i))} \\ &= b \star a \\ \Rightarrow (b, a) &\in G \times G \end{aligned}$$

Transitive:

$(a, b), (b, c) \in G \times G$

$$\begin{aligned} \Rightarrow a \star b &= a + b - [a + b] \\ &= b + a - [b + a] \text{ (ring axiom (i))} \\ &= b + (a - [b + a]) \text{ (Proposition 1.1(5))} \end{aligned}$$

and

$b \star c = b + c - [b + c]$

$= b + (c - [b + c])$ (Proposition 1.1(5))

$\Rightarrow a \star b + \big(-(a - [b + a])\big) = b + (a - [b + a]) + \big(-(a - [b + a])\big)$

$= b + 0$ (group axiom (ii))

and

$b \star c + \big(-(c - [b + c])\big) = b + (c - [b + c]) + \big(-(c - [b + c])\big)$

$= b + 0$ (group axiom (ii))

$\Rightarrow b \star c + \big(-(c - [b + c])\big) = a \star b + \big(-(a - [b + a])\big)$

$\Rightarrow c \star b + \big(-(c - [b + c])\big) = a \star b + \big(-(a - [b + a])\big)$ ($\star$ is symmetric)

$\Rightarrow c = a$

$\Rightarrow c + c - [a + c] = a + c - [a + c]$

$= a \star c$

$\Rightarrow (a, c) \in G \times G$

Therefore since $\star$ is reflexive, symmetric, and transitive, $\star$ is a binary equivalence relation. By Proposition 0.2(1), the set of equivalence classes of $\star$ form a partition of $G \times G$. This shows that $\star$ is well defined. Since $a \star b$ is defined for all $a, b \in G$, $\star$ is well defined, and G is closed under $\star$, thus $\star$ is a well defined binary operation.

Associativity:

$(a \star b) \star c = (a + b - [a + b]) + c - [(a + b - [a + b]) + c]$

$= a + (b + (-[a + b]) + c) - [a + (b + (-[a + b]) + c)]$ (group axiom (i))

$= a + \big(b + c + (-[a + b])\big) - \big[a + \big(b + c + (-[a + b])\big)\big]$ (ring axiom (i))

$= a + (b + c - [a + b]) - [a + (b + c - [a + b])]$

$= a + (b + c - [c + b]) - [a + (b + c - [c + b])]$

($a = c$ by transitivity of $\star$ since $a \sim b$ and $b \sim c$)

$= a + (b + c - [b + c]) - [a + (b + c - [b + c])]$ (ring axiom (i))

$= a \star (b \star c)$

Identity:

$0 \star a = a \star 0$ (see "Symmetric")

$= a + 0 - [a + 0]$

$= a - [a]$ (group axiom (ii))

$= a - 0$

$= a + (-0)$

$= a + (-1)0$ (Proposition 7.1(4))

$= a + 0$ (Proposition 7.1(1))

$= a$ (group axiom (ii))

Inverses:

Let $d \in G - \{0\}$ and note that $1 - d \in G - \{0\}$ for all $d \in G - \{0\}$. The "Identity" section shows that the inverse of 0 is 0.

$(1 - d) \star d = d \star (1 - d)$ (see "Symmetric")

$= d + (1 - d) - [d + (1 - d)]$

$= d + \big(1 + (-d)\big) - \big[d + \big(1 + (-d)\big)\big]$

$= d + \big((-d) + 1\big) - \big[d + \big((-d) + 1\big)\big]$ (ring axiom (i))

$= d + (-d) + 1 - [d + (-d) + 1]$ (Proposition 1.1(5))

$= 0 + 1 - [0 + 1]$ (group axiom (iii))

$= 1 - [1]$ (group axiom (ii))

$= 1 - 1$

$= 1 + (-1)$

$= 0$ (group axiom (iii))

Abelian:

See "Symmetric.

Hence $(G, \star)$ is an abelian group."

(Nava, 2017)

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