International Journal of Mathematical Archive-9(5), 2018, 98-101
Available online through www.ijma.info ISSN 2229 – 5046

GENERALIZED COMMON FIXED POINT THEOREM OF COMPATIBLE MAPPING OF TYPE (R) IN COMPLETE METRIC SPACE

AKASH SINGHAL[1*], Dr. RAJESH KUMAR SHARMA AND Dr. ANIL AGRAWAL

[1*]Department of Mathematics,
Institute of Technology & Management, ITM Campus, Gwalior (M.P.), India.
&
Research Scholar MGCGV Chitrakoot Satna (M.P.), India.

[2]Department of Mathematics, Govt. Holkar Science College, Indore (M.P.), India.

[3]Faculty of Science & Environment MGCGV, Chitrakoot Satna (M.P.), India.

(Received On: 23-03-18; Revised & Accepted On: 02-05-18)

ABSTRACT

The purpose of this paper is to present a common fixed theorem for compatible mapping of type (R) in complete metric space satisfying a generalized inequality. we also present a example that shows the applicability and validity of our result.

***2000 AMS classification**: 47H10, 54H25.*

***Keywords:** Common Fixed Point, Cauchy Sequence, Compatible Mapping, Complete Metric Space.*

1. INTRODUCTION AND PRELIMINARIES

Initially Jungck in 1976 [2] proved a common fixed point theorem for commuting maps, this result was extended and generalized in various ways by many authors. Recently Jungck in 1986 [3] introduced the generalized concept of weak commutativity which is called compatibility. In 1994 Pathak, Chang and Cho [5] gave the idea of compatible mapping of type (P). Rohen, Singh and Shambu [9] in 2004 gave the idea of compatible mapping of type (R) by combining the definition of compatible mapping and compatible mapping type (P).

The aim of this paper is to present a common fixed point theorem of compatible mapping type (R) in complete metric space by in view of four maps. This result revise the result of Bijendra and Chauhan [1] and others.

Before starting our main result following definitions and propositions are required in the sequel.

Definition 1.1: Let P and Q be self maps of a complete metric space (X, d) are said to be compatible on X if $\lim_{n\to\infty} d(PQx_n, QPx_n) = 0$ when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = t = \lim_{n\to\infty} Qx_n$ for some $t \in X$.

Definition 1.2: Let P and Q be self maps of a complete metric space (X, d) are said to be compatible of type (P) on X if $\lim_{n\to\infty} d(PPx_n, QQx_n) = 0$ when when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = t = \lim_{n\to\infty} Qx_n$ for some $t \in X$.

Definition 1.3: Let P and Q be self maps of a complete metric space (X, d) are said to be compatible of type (R) on X if $\lim_{n\to\infty} d(PQx_n, QPx_n) = 0$ when when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = t = \lim_{n\to\infty} Qx_n$ for some $t \in X$.

2. MAIN RESULT

We need the following proposition for our main result.

Corresponding Author: Akash singhal[1], [1*]Department of Mathematics,
Institute of Technology & Management, ITM Campus, Gwalior (M.P.), India.*

Proposition 2.1: Let P and Q be self maps of a complete metric space (X, d). if a pair (P,Q) is compatible type R on X and Pt = Qt for t ϵ X. then PQt =QPt = PPt = QQt .

Proposition 2.2: P and Q be self maps of a complete metric space (X, d). if a pair (P,Q) is compatible type R on X and and limn→∞ Px_n = t = limn→∞ Qx_n for some t ϵ X. then

(i) limn→∞ d (PQx_n, Qt)=0 if Q is continuous
(ii) limn→∞ d (QPx_n, Pt)=0 if P is continuous
(iii) PQt = QPt and Pt = Qt If P and Q are continuous at t.

Theorem 2.3: Let P, Q, R, S be self maps of a complete metric space (X, d) satisfying the following conditions .

(1)P(X) $\subseteq$ S(X) and Q(X) $\subseteq$ R(X)
(2) $[d(Px, Qy)]^2 \leq k_1[d(Rx, Qy)d(Rx, Sy) + d(Rx, Sy)d(Px, Rx)] + k_2[d(Px, Qy)d(Px, Rx) + d(Px, Qy)d(Px, Sy]$
Where $0 \leq k_1 + 3k_2 < 1$ and $k_{1,}k_{2} \geq 0$
(3) One of P, Q, R, S is continuous
(4) [P, R] and [Q, S] are compatible type (R) on X.
Then P, Q, R, S have a unique common fixed point in X.

Proof: let $x_0 \epsilon$ X then by (1) ∃ $x_1 \epsilon$ X such that $Sx_1 = Px_0$ and for $x_1 \epsilon$ X ∃ $x_2 \epsilon$ X Such that $Rx_2 = Qx_1$ and so on. Continuing this way we can define a sequence $\{y_n\}$ in X such that

$Y_{2n+1} = Sx_{2n+1} = Px_{2n}$ and $y_{2n} = Rx_{2n} = Qx_{2n-1}$

$d[(y_{2n+1},y_{2n})]^2 = [d(Px_{2n}, Qx_{2n-1})]^2 \leq K_1[d(Rx_{2n}, Qx_{2n-1})d(Rx_{2n}, Sx_{2n-1}) + d(Rx_{2n}, Sx_{2n-1})d(Px_{2n}, Rx_{2n})]$
$+ K_2[d(Px_{2n}, Qx_{2n-1})d(Px_{2n}, Rx_{2n}) + d(Px_{2n}, Qx_{2n-1})d(Px_{2n}, Sx_{2n-1})]$
$= K_1[d(y_{2n},y_{2n})d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n-1})d(y_{2n+1},y_{2n})]$
$+ K_2[d(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n-1})]$
$= K_1[d(y_{2n}, y_{2n-1})d(y_{2n+1}, y_{2n})] +K_2[d(y_{2n+1}, y_{2n})^2 + d(y_{2n}, y_{2n+1})d(y_{2n+1}, y_{2n-1})]$

$(1-k_2)d[(y_{2n+1},y_{2n})]^2 \leq d[(y_{2n+1},y_{2n})] [K_1d(y_{2n},y_{2n-1})+K_2d(y_{2n+1},y_{2n-1})]$
$(1-k_2)d[(y_{2n+1},y_{2n})] \leq K_1d(y_{2n},y_{2n-1})+K_2d(y_{2n+1},y_{2n-1})]$
$(1-k_2)d[(y_{2n},y_{2n+1})] \leq K_1d(y_{2n},y_{2n-1})+K_2d(y_{2n+1},y_{2n-1})]$
$(1-k_2)d[(y_{2n},y_{2n+1})] \leq K_1d(y_{2n},y_{2n-1})+K_2[d(y_{2n+1},y_{2n}) +d(y_{2n},y_{2n-1})]$
$(1-2k_2)d[(y_{2n},y_{2n+1})] \leq (K_1+K_2) d(y_{2n},y_{2n-1})]$
$d[(y_{2n},y_{2n+1})] \leq \rho d(y_{2n},y_{2n-1})]$

where $\rho = \dfrac{k_1 + k_2}{1 - 2k_2} < 1$

$\{y_n\}$ is a Cauchy sequence.

Since $\{y_n\}$ is a Cauchy sequence and since X is a complete metric then ∃ a point z ϵ X such that lim y_n = z as n→∞ consequently subsequences Px_{2n}, Rx_{2n}, Qx_{2n-1} and Sx_{2n+1} converges to z .

Let R be a continuous, since P and R are compatible type (R) on X, then by proposition (2.2) we have $R^2 x_{2n} \to Rz$ and $PRx_{2n} \to Rz$ as n→∞.

Now by condition (2) of theorem, we have

$[d(PRx_{2n},Qx_{2n-1})]^2 \leq K_1[d(R^2x_{2n}, Qx_{2n-1})d(R^2x_{2n}, Sx_{2n-1}) + d(R^2x_{2n}, Sx_{2n-1})d(PRx_{2n}, R^2x_{2n})]$
$+K_2[d(PRx_{2n}, Qx_{2n-1})d(PRx_{2n}, R^2x_{2n}) + d(PRx_{2n}, Bx_{2n-1})d(PRx_{2n}, Sx_{2n-1})]$

as n→∞ we have

$[d(Rz, z)]^2 \leq K_1[d(Rz, z)d(Rz, z) + d(Rz, z)d(Rz, Rz)] + K_2[d(Rz, z)d(Rz, Rz) + d(Rz, z)d(Rz, z)]$
$[d(Rz, z)]^2 \leq K_1[d(Rz, z)^2] + K_2[d(Rz, z)^2)]$
$[d(Rz, z)]^2 \leq (K_1+K_2)[d(Rz, z)^2)]$
This is a contradiction.
Than d(Rz, z) = 0
hence Rz = z

Now
$[d(Pz,Qx_{2n-1})]^2 \leq K_1[d(Rz, Qx_{2n-1})d(Rz, Sx_{2n-1}) + d(Rz, Sx_{2n-1})d(Pz, Rz)]$
$+ K_2[d(Pz,Qx_{2n-1})d(Pz, Rz) + d(Pz, Qx_{2n-1})d(Pz, Sx_{2n-1})]$
Taking limas n→∞ we have
$[d(Pz, z)]^2 \leq K_1[d(z, z)d(z, z) + d(z,, z)d(Pz, z)] +K_2[d(Pz, z)d(Pz, z) + d(Pz, z)d(Pz, z)]$
$= 2k_2[d(Pz, z)]^2$
$[d(Pz, z)]^2 \leq 2k_2[d(Pz, z)]^2$

Which is a contradiction.
Then d(Pz, z) = 0
Hence Pz = z

Since by condition (1) zϵ S(X) also S is a self map of X so ∃ a point uϵ X such that z = P(z) = S(u) more over by condition (2), we have
$[d(z, Qu)]^2 = [d(Pz, Qu)]^2 \leq K_1[d(Rz, Qu)d(Rz, Su) + d(Rz,, Su)d(Pz, Rz)] + K_2[d(Pz, Qu)d(Pz, Rz) + d(Pz, Qu)d(Pz, Su)]$

Taking lim as n→∞.
$[d(z, Qu)]^2 \leq K_1[d(z, Qu)d(z, z)+d(z, z)d(z, z)] +K_2[d(z, Qu)d(z, z)+ d(z, Qu)d(z, z)]$
$[d(z, Qu)]^2 \leq 0$
[d(z, Qu)] = 0

i.e Qu = z

by condition (4) we have

[d(SQu, QSu)] = 0

Hence [d(Sz, Qz)]= 0
Sz = Qz

Now,
$[d(z, Sz)]^2 = [d(Pz, Sz)]^2 \leq K_1[d(Rz,Qz)d(Rz,Sz) + d(Rz, Sz)d(Pz, Rz)] + K_2[d(Pz, Qz)d(Pz, Rz) + d(Pz, Qz)d(Pz, z)]$

Taking lima sn→∞.

$[d(z, Sz)]^2 = \leq K_1[d(z, Qz)d(z, z) + d(z, z)d(z, z)] +K_2[d(z, Qz)d(z, z) + d(z, Qz)d(z, z)]$

$[d(z, Sz)]^2 \leq 0$

d (z, Sz) = 0
z = Sz

Hence Qz = Sz = z

Hence z is a common fixed of P, Q, R, S.

Uniqueness of z: let w is another common fixed point of P, Q, R, S, then we have

$[d(z, w)]^2 = [d(Pz, Qw)]^2 \leq K_1[d(Rz, Qw)d(Rz, Sw) + d(Rz,, Sw)d(Pz,Rz)] + K_2[d(Pz, Qw)d(Pz, Rz) + d(Pz, Qw)d(Pz, Sw)]$
$\leq K_1[d(z, w)d(z, w) + d(z,, w)d(z, z)] + K_2[d(z, w)d(z, z) + d(z, w)d(z, w)]$
$\leq (k_1+k_2) [d(z, w)]^2$
$[d(z, w)]^2 \leq (k_1+ k_2) [d(z, w)]^2$
Hence z = S(w) = w
z = w

Example 2.4: Let X=[0,∞) be endowed with a complete metric space (X, d) with metric $d(x,y) = |x-y|^2 = (x-y)^2$, define P, Q, R, S on X by $P(x) = \log\left(1+\frac{x}{4}\right)$, $Q(x) = \log\left(1+\frac{x}{6}\right)$ $R(x) = e^{3x} - 1,\ S(x) = e^{2x} - 1$.

Obviously P(x) = Q(x) = R(X) = S(X) = [0, ∞) .

We show that the pair (P, R) is compatible

Let $\{x_n\}$ be a sequence in X such that for some tϵX lim n→∞ d (Px_n, t) =0 and tϵX
sslim n→∞ d (Rx_n, t) = 0

i.e lim n→∞ $|Px_n - t|$=0 , lim n→∞ $|Rx_n - t|$=0. Since P and R are continuous, we have

lim n→∞ d(PRx$_n$, RPx$_n$) = lim n→∞ $|PRx_n - RPx_n|^2 = |Pt - Rt|^2 \left|\log(1+\frac{t}{4}) - (e^{3t}-1)\right|^2 = 0 \Leftrightarrow t = 0$

then (P, R) are compatible .

Similarly {Q, S} are compatible.
for each x, y ϵ X

$$[d\,(Px, Qy)]^2 = [(Px - Qy)^2]^2 = \left[\left\{\log\left(1+\frac{x}{4}\right) - \log\left(1+\frac{y}{6}\right)\right\}^2\right]^2 \le \left[\left(\frac{x}{4} - \frac{y}{6}\right)^2\right]^2 \le \frac{1}{(12)^4}[(3x-2y)^2]^2$$

$$\le \frac{1}{(12)^4}[e^{3x} - e^{2y})^2]^2$$

$$= \frac{1}{(12)^4}[d(Rx, Sy)]^2$$

$$\le \frac{10}{(12)^5}[d(Rx,Sy)d(Rx,Sy) + d(Rx,Sy)d(Rx,Rx)]$$

$$+ \frac{2}{(12)^5}[d(Rx,Sy)d(Rx,Rx) + d(Rx,Sy)d(Rx,Sy]$$

Where $k_1 = \frac{10}{(12)^5} \ge 0$ and $k_2 = \frac{2}{(12)^5} \ge 0$ and $k_1 + 3k_2 = \frac{10}{(12)^5} + 3\frac{2}{(12)^5} < 1$

Thus P, Q, R, S satisfy all condition of theorem (2.3) .moreover 0 is the unique common fixed point of P, Q, R, S.

This complete the proof.

REFERENCES

1. Bijendra Singh and M.S. Chauhan, On common fixed of four mappings, Bull. Cal math Soc.88, 1996, 451- 456.
2. G.Jungck, Commuting maps and fixed points, Amer.Math. Monthly 83, 1976, 261.
3. G.Jungck, Compatible maps and common fixed points, Inter.J. Math and Math.Sci.9, 7, 1986, 771-779
4. H.K.Pathak and M.S. khan, A comparison of various type of Compatible maps and common fixed points, Indian J.Pure Appl.Math.28(4), 1997, 471-485.
5. H.K.Pathak, S.S Chang and Y.J. Cho., Fixed point theorem for compatible mappings of type (P), Indian J. Math .36(2),1994, 151-166.
6. Koireng M., Leenthoi N., Rohen Y, Common fixed point of compatible Mappings of type (R), *General Mathematics* Notes10(1), 2012, 58-62.
7. Lohani.P.C. and Badshah, V.H., Compatibles mappings and common fixed point for four Mappings, Bull. Cal. Math. Soc.90, 1998, 301-308.
8. Shaidur Rahman, Yumnam Rohen, M.Poeshwar Singh, Generalized Common fixed point theorem of A-compatible and S-Compatible mappings, American J.Appl..Math. and Statistics 1, 2013. 27-29.
9. Y.Rohen, M.R.Singh and L.Shambhu, common fixed point of compatible mapping of type (C) in Banach Spaces, Proc.of Math. Soc. 20, 2004. 77-87.

Source of support: Nil, Conflict of interest: None Declared.