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A NOTE ON THE USE OF PRODUCT MEAN (E, q)B IN THE TRIGONOMETRIC FOURIER APPROXIMATION

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ABSTRACT

A. Zygmund introduced trigonometric Fourier approximation where as L. Mcfadden [2] introduced Lipchitz class. Dealing with degree of approximation of conjugate series of a Fourier series Padhy et al. have established some theorems. In this paper, we have extended their result and established a theorem on the use of product mean $(E,q)B$ in the degree of Approximation of the conjugate series of Fourier series of a function of weighted Lipchitz class $W(L^p,\xi(u))$.

Keywords: *Degree of Approximation, $W(L^p,\xi(u))$ class of function, (E,q)- mean, B - mean, $(E,q)B$-product mean, Conjugate Fourier series, Lebesgue integral.*

AMS subject classification: *42B05, 42B08.*

1. INTRODUCTION

The sequence $\{u_n\}$ of the B -mean of the sequence $\{s_n\}$ is given by

$$u_n = \sum_{\lambda=0}^{n} b_{m\lambda}\, s_\lambda \ , n = 1,2,\cdots \tag{1.1}$$

is the sequence –to-sequence transformation, where $B = (b_{mn})_{\infty\times\infty}$ be a $\infty\times\infty$ matrix and $\sum b_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$.

The series $\sum b_n$ is said to be B - summable to s if

$$u_n \to s \text{ as } n \to \infty, \tag{1.2}$$

The regularity conditions for B -summability are:

(i) $\sup_m \sum_{n=0}^{\infty} |b_{mn}| < L$ where L is an absolute constant.

(ii) $\lim_{m\to\infty} b_{mn} = 0$

(iii) $\lim_{m\to\infty} \sum_{n=0}^{\infty} b_{mn} = 1$

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The sequence $\{v_n\}$ [**1**] of the (E,q) mean of the sequence $\{s_n\}$ is given by

$$v_n = \frac{1}{(1+q)^n} \sum_{\lambda=0}^{n} \binom{n}{\lambda} q^{n-\lambda} s_\lambda \quad . \tag{1.3}$$

The series $\sum b_n$ is said to be (E,q) summable to s if

$$v_n \to s \text{ as } n \to \infty \tag{1.4}$$

Clearly (E,q) method is regular [**7**].

Further, let w_n be the (E,q) transform of the B -transform of $\{s_n\}$ defined by

$$\begin{aligned} w_n &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} u_k \\ &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\lambda=0}^{k} b_{k\lambda} s_\lambda \right\} \end{aligned} \tag{1.5}$$

The series $\sum b_n$ is said to be $(E,q)B$-summable to s if

$$w_n \to s \text{ as } n \to \infty \tag{1.6}$$

Let $g(u)$ be a Lebesgue integrable function with period 2π in (-π,π), The Fourier series of g at any point 'x' is given by

$$g(x) \sim \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos nx + d_n \sin nx\right) \equiv \sum_{n=0}^{\infty} G_n(x) \tag{1.7}$$

where c_0, c_n and d_n are the Fourier coefficients and the conjugate series of the Fourier series (1.7) is

$$\sum_{n=1}^{\infty} \left(c_n \cos nx - d_n \sin nx\right) \equiv \sum_{n=1}^{\infty} H_n(x) \tag{1.8}$$

Let $\overline{s_n}(g;x)$ be the n-th partial sum of (1.8).

For a function $g: R \to R$, the L_∞ -norm of is defined by

$$\|g\|_\infty = \sup\{|g(x)| : x \in R\} \tag{1.9}$$

and the L_υ -norm is defined by

$$\|g\|_\nu = \left(\int_0^{2\pi} |g(x)|^\nu \right)^{\frac{1}{\nu}}, \ \nu \ge 1. \tag{1.10}$$

The degree of approximation of a function $g: R \to R$ by a trigonometric polynomial $Q_n(x)$ of degree n under the norm $\| \cdot \|_\infty$ is defined by [**6**]

$$\|Q_n - g\|_\infty = \sup\{|Q_n(x) - g(x)| : x \in R\} \tag{1.11}$$

and the degree of approximation $E_n(g)$ of a function $g \in L_\nu$ is given by [**6**]

$$E_n(g) = \min_{Q_n} \left\| Q_n - g \right\|_\nu \tag{1.12}$$

This method of approximation is called Trigonometric Fourier approximation.
A function $g \in Lip\ \alpha$ if [**2**]

$$\left| g(x+u) - g(x) \right| = O\left(|u|^\alpha \right) ,0 < \alpha \le 1 \tag{1.13}$$

and $g(x) \in Lip\ (\alpha, r)$, for $0 \le x \le 2\pi$,if[**2**]

$$\left(\int_0^{2\pi} \left| g(x+u) - g(x) \right|^r dx \right)^{\frac{1}{r}} = O\left(|u|^\alpha \right) ,0 < \alpha \le 1, r \ge 1, u > 0. \tag{1.14}$$

For a given positive increasing function $\xi(u)$, the function $g(x) \in Lip(\xi(u), r)$, if[**2**]

$$\left(\int_0^{2\pi} \left| g(x+u) - g(x) \right|^r dx \right)^{\frac{1}{r}} = O(\zeta(u)), r \ge 1, u > 0 \tag{1.15}$$

For a given positive increasing function $\xi(u)$ and an integer $p > 1$ the function $g(x)$ belongs to $W\left(L^p, \zeta(u) \right)$, if [**2**]

$$\left(\int_0^{2\pi} \left| g(x+u) - g(x) \right|^p (\sin x)^{p\beta} dx \right)^{\frac{1}{p}} = O(\zeta(u)), \beta \ge 0. \tag{1.16}$$

We use the following notation throughout this paper:

$$\psi(u) = \frac{1}{2} \left\{ g(x+u) - g(x-u) \right\}$$

and

$$\overline{K_n}(u) = \frac{1}{\pi (1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\lambda=0}^{k} b_{k\lambda} \frac{\cos\frac{u}{2} - \cos\left(\lambda + \frac{1}{2} \right) u}{\sin\frac{u}{2}} \right\} \tag{1.17}$$

Further, the method $(E, q)B$ is assumed to be regular and this case is supposed through out the paper.

2. KNOWN THEOREM

Dealing with the degree of approximation by the product $(E, q)(C,1)$-mean of Fourier series, Nigam [**3**] proved the following theorem:

Theorem 2.1: *If* g *is a* $2\pi-$*Periodic function belonging to class* $Lip\,\alpha$, *then its degree of approximation by* $(E,q)\,(c,1)$ *summability mean on its Fourier series* $\sum_{n=0}^{\infty} G_n(x)$ *is given by*

$$\left\| E_n^q c_n^1 - g \right\|_\infty = O\left(\frac{1}{(n+1)^\alpha} \right), 0<\alpha<1,$$

where $E_n^q c_n^1$ *represents the* (E,q) *transform of* $(C,1)$ *transform of* $s_n\,(g;x)$.

Padhy *et al.* [**5**] proved the following theorem using $(E,q)B$ mean of the conjugate series of the Fourier series.

Theorem 2.2: *If* g *is a* $2\pi-$ *Periodic function of class* $Lip\,\alpha$, *then degree of approximation by the product* $(E,q)B$ *summability means on its conjugate series of Fourier series (1.8) is given by* $\|w_n - g\|_\infty = O\left(\frac{1}{(n+1)^\alpha} \right), 0<\alpha<1$, *where* w_n *as defined in (1.5)*.

Recently, Padhy *et al* [**4**] proved the following theorem using $(E,s)(N,p_n,q_n)$ mean of the conjugate series of the Fourier series of a function of class $Lip(\alpha,r)$ in the following form:

Theorem 2.3: *If* g *is a* $2\pi-$*Periodic function of class* $Lip(\alpha,r)$, *then degree of approximation by the product* $(E,s)(N,p_n,q_n)$ *summability means on its conjugate series of Fourier series (1.8) is given by*

$$\|w_n - g\|_\infty = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}} \right), \ 0<\alpha<1, r\ge 1$$ *,where* w_n *as defined in (1.5)*.

3. MAIN THEOREM

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,q)B$ of Fourier series (1.8).We prove

Theorem 3.1: *Let* $\xi(u)$ *be a positive increasing function and* g *be a* $2\pi-$ *Periodic function of the class* $W\left(L^p,\zeta(u)\right), p>1, u>0$. *Then degree of approximation by the product* $(E,q)B$ *means of the conjugate series of the Fourier series (1.8) is given by*

$$\|w_n - f\|_r = O\left((n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right) \right), \ r\ge 1 \tag{3.1}$$

provided

$$\left(\int_0^{\frac{1}{n+1}} \left(\frac{u\,\psi(u)\sin^\beta u}{\xi(u)} \right)^r du \right)^{\frac{1}{r}} = O\left(\frac{1}{n+1} \right) \tag{3.2}$$

and

$$\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{u^{-\delta} |\psi(u)|}{\xi(u)} \right)^{r} du \right)^{\frac{1}{r}} = O\left((n+1)^{\delta} \right) \tag{3.3}$$

hold uniformly in x *with* $\frac{1}{r}+\frac{1}{s}=1$, *where* δ *is an arbitrary number such that* $s(1-\delta)-1>0$ *and* w_n *is as defined in (1.5).*

4. Lemma: We require the following Lemma to prove the theorem.

Lemma -4.1[5]:

$$\left|\overline{K_n}(u)\right| = \begin{cases} O(n), & 0 \le u \le \frac{1}{n+1} \\ O\left(\frac{1}{u}\right), & \frac{1}{n+1} \le u \le \pi \end{cases},$$

Where $\overline{K_n}(u)$ *is as defined in (1.17).*

5. PROOF OF THEOREM- 3.1:

Using Riemann –Lebesgue theorem, we have the n-th partial sum $\overline{s_n}(g;x)$ of the Fourier series **(**1.8) of $g(x)$,

$$\overline{s_n}(g;x) - g(x) = \frac{2}{\pi}\int_0^{\pi} \psi(u) \frac{\cos\frac{u}{2} - \sin\left(n+\frac{1}{2}\right)u}{2\sin\left(\frac{u}{2}\right)} du$$

The $B-$ transform of $\overline{s_n}(g;x)$ [**2**] using (1.1) is given by

$$u_n - g(x) = \frac{2}{\pi}\int_0^{\pi} \psi(u) \sum_{k=0}^{n} b_{nk} \frac{\cos\frac{u}{2} - \sin\left(n+\frac{1}{2}\right)u}{2\sin\left(\frac{u}{2}\right)} du,$$

If w_n be the $(E,q)B$ transform of $\overline{s_n}(g;x)$, then we have

$$\|w_n - g\| = \frac{2}{\pi(1+q)^n}\int_0^{\pi} \psi(u) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} b_{nk} \frac{\cos\frac{u}{2} - \sin\left(n+\frac{1}{2}\right)u}{2\sin\left(\frac{u}{2}\right)} \right\} du$$

$$= \int_0^{\pi} \psi(u)\, \overline{K_n}(u)\, du$$

$$= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right\} \overline{K_n}(u)\, du$$

$$= I_1 + I_2 \text{, say} \tag{5.1}$$

Now

$$|I_1| = \frac{2}{\pi(1+q)^n} \left| \int_0^{\frac{1}{n+1}} \psi(u) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} b_{nk} \frac{\cos\frac{u}{2} - \sin\left(n+\frac{1}{2}\right)u}{2\sin\left(\frac{u}{2}\right)} \right\} du \right|$$

$$= \left| \int_0^{\frac{1}{n+1}} \psi(u)\overline{K_n}(u)\, du \right|$$

$$\leq \left(\int_0^{\frac{1}{n+1}} \left| \frac{u\psi(u)\sin^{\beta} u}{\xi(u)} \right|^r du \right)^{\frac{1}{r}} \left(\int_0^{\frac{1}{n+1}} \left| \frac{\xi(u)\,\overline{K_n}(u)}{u\sin^{\beta} u} \right|^s du \right)^{\frac{1}{s}} \text{, where } \frac{1}{r} + \frac{1}{s} = 1,$$

(using Hölder's inequality)

$$= \left(\int_0^{\frac{1}{n+1}} \left(\frac{\xi(u)}{u^{1+\beta}} \right)^s du \right)^{\frac{1}{s}} \qquad \text{(Using lemma-4.1 and (3.2))}$$

$$= O\left(\xi\left(\frac{1}{n+1} \right) \right) \left(\int_0^{\frac{1}{n+1}} \frac{du}{u^{(1+\beta)s}} \right)^{\frac{1}{s}}$$

$$= O\left(\xi\left(\frac{1}{n+1} \right) \right) O\left((n+1)^{-\frac{1}{s}+1+\beta} \right)$$

$$= O\left(\xi\left(\frac{1}{n+1} \right) (n+1)^{\beta+\frac{1}{r}} \right) \tag{5.2}$$

Next

$$|I_2| = \frac{2}{\pi(1+q)^n} \left| \int_{\frac{1}{n+1}}^{\pi} \psi(u) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \sum_{\nu=0}^{k} b_{nk} \frac{\cos\frac{u}{2} - \sin\left(n+\frac{1}{2}\right)u}{2\sin\left(\frac{u}{2}\right)} \right\} du \right|$$

$$= \left| \int_{\frac{1}{n+1}}^{\pi} \psi(u)\overline{K_n}(u)\,du \right|$$

$$\le \left(\int_{\frac{1}{n+1}}^{\pi} \left| \frac{u^{-\delta}\psi(u)\sin^{\beta} u}{\xi(u)} \right|^r du \right)^{\frac{1}{r}} \left(\int_{\frac{1}{n+1}}^{\pi} \left| \frac{\xi(u)\,\overline{K_n}(u)}{u^{-\delta}\sin^{\beta} u} \right|^s du \right)^{\frac{1}{s}},$$

(Where $\frac{1}{r}+\frac{1}{s}=1$, using Hölder's inequality)

$$= O\left((n+1)^{\delta}\right) \left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(u)}{u^{1+\beta-\delta}} \right)^s du \right)^{\frac{1}{s}}, \text{ using Lemma-4.1 and (3.3)}$$

$$= O\left((n+1)^{\delta}\right) \left(\int_{\frac{1}{\pi}}^{n+1} \left(\frac{\xi\left(\frac{1}{z}\right)}{z^{\delta-(1+\beta)}} \right)^s \frac{dz}{z^2} \right)^{\frac{1}{s}},$$

($\xi\left(\frac{1}{z}\right)/\left(\frac{1}{z}\right)$ is a positive increasing function as $\xi(u)$ is a positive increasing function)

$$= O\left((n+1)^{1+\delta}\,\xi\left(\frac{1}{n+1}\right)\right) \left(\int_{\varepsilon}^{n+1} \frac{dy}{z^{s(\delta-\beta-1)+2}} \right)^{\frac{1}{s}}, \text{ For some } \frac{1}{\pi} \le \varepsilon \le n+1.$$

(Using second mean value theorem)

$$= O\left((n+1)^{1+\delta}\,\xi\left(\frac{1}{n+1}\right)\right) O\left((n+1)^{\beta+1-\delta-\frac{1}{s}}\right)$$

$$= O\left((n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right) \tag{5.3}$$

Then from (5.2) and (5.3), we have

$$|w_n - g| = O\left((n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right), \quad r \ge 1$$

$$\|w_n - g\|_r = \left(\int_0^{2\pi} O\left((n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right)^r dx \right)^{\frac{1}{r}}, \quad r \ge 1$$

$$= O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right)\left(\int_0^{2\pi} dx\right)^{\frac{1}{r}}$$

$$= O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of the theorem.

Remark: If we put $\beta = 0$ and $\xi(u) = u^{\alpha}$ in the main theorem then the degree of approximation of a function g belonging to the class $Lip(\alpha, r), 0 < \alpha \le 1, r \ge 1$ is given by

$$\|w_n - g\|_r = O\left((n+1)^{-\alpha+\frac{1}{r}} \right).$$

and if we take $r \to \infty$ then the degree of approximation of a function g belonging to the class $Lip(\alpha), 0 < \alpha \le 1$ is given by

$$\|w_n - g\|_r = O\left((n+1)^{-\alpha} \right).$$

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