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A NOTE ON THE USE OF PRODUCT MEAN (E, q)B IN THE TRIGONOMETRIC FOURIER APPROXIMATION<br>PANKAJINI TRIPATHY ${ }^{1}$ AND BIRUPAKHYA PRASAD PADHY ${ }^{2} \dagger$<br>1,2Department of Mathematics, School of Applied Sciences, Kalinga Institute of Industrial Technology, Deemed to be University, Bhubaneswar, Odisha, India.

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#### Abstract

A. Zygmund introduced trigonometric Fourier approximation where as L. Mcfadden [2] introduced Lipchitz class. Dealing with degree of approximation of conjugate series of a Fourier series Padhy et al. have established some theorems. In this paper, we have extended their result and established a theorem on the use of product mean $(E, q) B$ in the degree of Approximation of the conjugate series of Fourier series of a function of weighted Lipchitz $\operatorname{class} W\left(L^{p}, \xi(u)\right)$.


Keywords: Degree of Approximation, $W\left(L^{p}, \xi(u)\right)$ class of function, $(E, q)$ - mean, $B$ - mean, $(E, q) B$-product mean, Conjugate Fourier series, Lebesgue integral.

AMS subject classification: 42B05, 42B08.

## 1. INTRODUCTION

The sequence $\left\{u_{n}\right\}$ of the $B$-mean of the sequence $\left\{s_{n}\right\}$ is given by

$$
\begin{equation*}
u_{n}=\sum_{\lambda=0}^{n} b_{m \lambda} s \lambda, n=1,2, \cdots \tag{1.1}
\end{equation*}
$$

is the sequence -to-sequence transformation, where $B=\left(b_{m n}\right)_{\infty \times \infty}$ be a $\infty \times \infty$ matrix and $\sum b_{n}$ be a given infinite series with the sequence of partial sums $\left\{s_{n}\right\}$.

The series $\sum b_{n}$ is said to be $B$ - summable to $S$ if

$$
\begin{equation*}
u_{n} \rightarrow s \text { as } n \rightarrow \infty \tag{1.2}
\end{equation*}
$$

The regularity conditions for $B$-summability are:
(i) $\sup \sum^{\infty}\left|b_{m n}\right|<L$ where $L$ is an absolute constant.
$m \quad n=0$
(ii) $\lim b_{m n}=0$
$m \rightarrow \infty$
(iii) $\lim _{m \rightarrow \infty} \sum_{n=0}^{\infty} b_{m n}=1$

The sequence $\left\{v_{n}\right\}[1]$ of the $(E, q)$ mean of the sequence $\left\{s_{n}\right\}$ is given by

$$
\begin{equation*}
v_{n}=\frac{1}{(1+q)^{n}} \sum_{\lambda=0}^{n}\binom{n}{\lambda} q^{n-\lambda} s \lambda \tag{1.3}
\end{equation*}
$$

The series $\sum b_{n}$ is said to be $(E, q)$ summable to $s$ if

$$
\begin{equation*}
v_{n} \rightarrow s \text { as } n \rightarrow \infty \tag{1.4}
\end{equation*}
$$

Clearly $(E, q)$ method is regular [7].
Further, let $w_{n}$ be the $(E, q)$ transform of the $B$-transform of $\left\{s_{n}\right\}$ defined by

$$
\begin{align*}
w_{n} & =\frac{1}{(1+q)^{n}} \sum_{k=0}^{n}\binom{n}{k} q^{n-k} u_{k} \\
& =\frac{1}{(1+q)^{n}} \sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\lambda=0}^{k} b_{k \lambda} s \lambda\right\} \tag{1.5}
\end{align*}
$$

The series $\sum b_{n}$ is said to be $(E, q) B$-summable to $s$ if

$$
\begin{equation*}
w_{n} \rightarrow s \quad \text { as } \quad n \rightarrow \infty \tag{1.6}
\end{equation*}
$$

Let $g(u)$ be a Lebesgue integrable function with period $2 \pi$ in $(-\pi, \pi)$, The Fourier series of $g$ at any point ' $x$ ' is given by

$$
\begin{equation*}
g(x) \sim \frac{c_{0}}{2}+\sum_{n=1}^{\infty}\left(c_{n} \cos n x+d_{n} \sin n x\right) \equiv \sum_{n=0}^{\infty} G_{n}(x) \tag{1.7}
\end{equation*}
$$

where $c_{0}, c_{n}$ and $d_{n}$ are the Fourier coefficients and the conjugate series of the Fourier series (1.7) is

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(c_{n} \cos n x-d_{n} \sin n x\right) \equiv \sum_{n=1}^{\infty} H_{n}(x) \tag{1.8}
\end{equation*}
$$

Let $\overline{S_{n}}(g ; x)$ be the n-th partial sum of (1.8).
For a function $g: R \rightarrow R$, the $L_{\infty}$-norm of is defined by

$$
\begin{equation*}
\|g\|_{\infty}=\sup \{|g(x)|: x \in R\} \tag{1.9}
\end{equation*}
$$

and the $L_{\mathcal{V}}$-norm is defined by

$$
\begin{equation*}
\|g\|_{v}=\left(\int_{0}^{2 \pi} \mid g(x)^{v}\right)^{\frac{1}{v}}, v \geq 1 \tag{1.10}
\end{equation*}
$$

The degree of approximation of a function $g: R \rightarrow R$ by a trigonometric polynomial $Q_{n}(x)$ of degree n under the norm $\|\cdot\|_{\infty}$ is defined by [6]

$$
\begin{equation*}
\left\|Q_{n}-g\right\|_{\infty}=\sup \left\{\left|Q_{n}(x)-g(x)\right|: x \in R\right\} \tag{1.11}
\end{equation*}
$$

and the degree of approximation $E_{n}(g)$ of a function $g \in L_{V}$ is given by [6]

$$
\begin{equation*}
E_{n}(g)=\min _{Q_{n}}\left\|Q_{n}-g\right\|_{V} \tag{1.12}
\end{equation*}
$$

This method of approximation is called Trigonometric Fourier approximation.
A function $g \in \operatorname{Lip} \alpha$ if [2]

$$
\begin{equation*}
|g(x+u)-g(x)|=O\left(|u|^{\alpha}\right), 0<\alpha \leq 1 \tag{1.13}
\end{equation*}
$$

and $g(x) \in \operatorname{Lip}(\alpha, r)$, for $0 \leq x \leq 2 \pi$,if[2]

$$
\begin{equation*}
\left(\int_{0}^{2 \pi}|g(x+u)-g(x)|^{r} d x\right)^{\frac{1}{r}}=O\left(|u|^{\alpha}\right), 0<\alpha \leq 1, r \geq 1, u>0 \tag{1.14}
\end{equation*}
$$

For a given positive increasing function $\xi(u)$, the function $g(x) \in \operatorname{Lip}(\xi(u), r)$, if[2]

$$
\begin{equation*}
\left(\int_{0}^{2 \pi}|g(x+u)-g(x)|^{r} d x\right)^{\frac{1}{r}}=O(\zeta(u)), r \geq 1, u>0 \tag{1.15}
\end{equation*}
$$

For a given positive increasing function $\xi(u)$ and an integer $p>1$ the function $g(x)$ belongs to $W\left(L^{p}, \zeta(u)\right)$, if [2]

$$
\begin{equation*}
\left(\int_{0}^{2 \pi}|g(x+u)-g(x)|^{p}(\sin x)^{p \beta} d x\right)^{\frac{1}{p}}=O(\zeta(u)), \beta \geq 0 . \tag{1.16}
\end{equation*}
$$

We use the following notation throughout this paper:

$$
\psi(u)=\frac{1}{2}\{g(x+u)-g(x-u)\}
$$

and

$$
\begin{equation*}
\overline{K_{n}}(u)=\frac{1}{\pi(1+q)^{n}} \sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{\lambda=0}^{k} b_{k} \lambda \frac{\cos \frac{u}{2}-\cos \left(\lambda+\frac{1}{2}\right) u}{\sin \frac{u}{2}}\right\} \tag{1.17}
\end{equation*}
$$

Further, the method $(E, q) B$ is assumed to be regular and this case is supposed through out the paper.

## 2. KNOWN THEOREM

Dealing with the degree of approximation by the product $(E, q)(C, 1)$-mean of Fourier series, Nigam [3] proved the following theorem:

Theorem 2.1: If $g$ is a $2 \pi$-Periodic function belonging to class Lip $\alpha$, then its degree of approximation by $(E, q)(c, 1)$ summability mean on its Fourier series $\sum_{n=0}^{\infty} G_{n}(x)$ is given by

$$
\left\|E_{n}^{q} c_{n}^{1}-g\right\|_{\infty}=O\left(\frac{1}{(n+1)^{\alpha}}\right), 0<\alpha<1
$$

where $E_{n}^{q} c_{n}^{1}$ represents the $(E, q)$ transform of $(C, 1)$ transform of $s_{n}(g ; x)$.
Padhy et al. [5] proved the following theorem using $(E, q) B$ mean of the conjugate series of the Fourier series.
Theorem 2.2: If $g$ is a $2 \pi$ - Periodic function of class Lip $\alpha$, then degree of approximation by the product $(E, q) B$ summability means on its conjugate series of Fourier series (1.8) is given by $\left\|w_{n}-g\right\|_{\infty}=O\left(\frac{1}{(n+1)^{\alpha}}\right), 0<\alpha<1$, where $w_{n}$ as defined in (1.5).
Recently, Padhy et al [4] proved the following theorem using $(E, s)\left(N, p_{n}, q_{n}\right)$ mean of the conjugate series of the Fourier series of a function of class $\operatorname{Lip}(\alpha, r)$ in the following form:

Theorem 2.3: If $g$ is a $2 \pi$-Periodic function of class $\operatorname{Lip}(\alpha, r)$, then degree of approximation by the product $(E, s)\left(N, p_{n}, q_{n}\right)$ summability means on its conjugate series of Fourier series (1.8) is given by $\left\|w_{n}-g\right\|_{\infty}=O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right), 0<\alpha<1, r \geq 1$,where $w_{n}$ as defined in (1.5).

## 3. MAIN THEOREM

In this paper, we have proved a theorem on degree of approximation by the product mean $(E, q) B$ of Fourier series (1.8).We prove

Theorem 3.1: Let $\xi(u)$ be a positive increasing function and $g$ be a $2 \pi$-Periodic function of the class $W\left(L^{p}, \zeta(u)\right), p>1, u>0$. Then degree of approximation by the product $(E, q) B$ means of the conjugate series of the Fourier series (1.8) is given by

$$
\begin{equation*}
\left\|w_{n}-f\right\|_{r}=O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right), r \geq 1 \tag{3.1}
\end{equation*}
$$

provided

$$
\begin{equation*}
\left(\int_{0}^{\frac{1}{n+1}}\left(\frac{u \psi(u) \sin \beta u}{\xi(u)}\right)^{r} d u\right)^{\frac{1}{r}}=O\left(\frac{1}{n+1}\right) \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{u^{-\delta}|\psi(u)|}{\xi(u)}\right)^{r} d u\right)^{\frac{1}{r}}=O\left((n+1)^{\delta}\right) \tag{3.3}
\end{equation*}
$$

hold uniformly in $x$ with $\frac{1}{r}+\frac{1}{s}=1$, where $\delta$ is an arbitrary number such that $s(1-\delta)-1>0$ and $w_{n}$ is as defined in (1.5).
4. Lemma: We require the following Lemma to prove the theorem.

Lemma -4.1[5]:

$$
\left|\overline{K_{n}}(u)\right|=\left\{\begin{array}{l}
O(n), \quad 0 \leq u \leq \frac{1}{n+1} \\
O\left(\frac{1}{u}\right), \\
\frac{1}{n+1} \leq u \leq \pi
\end{array}\right.
$$

Where $\overline{K_{n}}(u)$ is as defined in (1.17).

## 5. PROOF OF THEOREM- 3.1:

Using Riemann -Lebesgue theorem, we have the n-th partial sum $\overline{s_{n}}(g ; x)$ of the Fourier series (1.8) of $g(x)$,

$$
\overline{s_{n}}(g ; x)-g(x)=\frac{2}{\pi} \int_{0}^{\pi} \psi(u) \frac{\cos \frac{u}{2}-\sin \left(n+\frac{1}{2}\right) u}{2 \sin \left(\frac{u}{2}\right)} d \mu
$$

The $B-$ transform of $\overline{S_{n}}(g ; x)$ [2] using (1.1) is given by

$$
u_{n}-g(x)=\frac{2}{\pi} \int_{0}^{\pi} \psi(u) \sum_{k=0}^{n} b_{n k} \frac{\cos \frac{u}{2}-\sin \left(n+\frac{1}{2}\right) u}{2 \sin \left(\frac{u}{2}\right)} d u
$$

If $w_{n}$ be the $(E, q) B$ transform of $\overline{S_{n}}(g ; x)$, then we have

$$
\begin{aligned}
\left\|w_{n}-g\right\| & =\frac{2}{\pi(1+q)^{n}} \int_{0}^{\pi} \psi(u) \sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{v=0}^{k} b_{n k} \frac{\cos \frac{u}{2}-\sin \left(n+\frac{1}{2}\right) u}{2 \sin \left(\frac{u}{2}\right)}\right\} d u \\
& =\int_{0}^{\pi} \psi(u) \overline{K_{n}}(u) d u
\end{aligned}
$$

$$
=\left\{\begin{array}{c}
\frac{1}{n+1}+\int_{0}^{\pi} \\
\frac{1}{n+1}
\end{array}\right\} \overline{K_{n}}(u) d u
$$

$$
\begin{equation*}
=I_{1}+I_{2}, \text { say } \tag{5.1}
\end{equation*}
$$

Now

Next

$$
\left|I_{2}\right|=\frac{2}{\pi(1+q)^{n}}\left|\int_{\frac{1}{n+1}}^{\pi} \psi(u) \sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{v=0}^{k} b_{n k} \frac{\cos \frac{u}{2}-\sin \left(n+\frac{1}{2}\right) u}{2 \sin \left(\frac{u}{2}\right)}\right\} d u\right|
$$

$$
\begin{aligned}
& \left|I_{1}\right|=\frac{2}{\pi(1+q)^{n}}\left|\int_{0}^{\frac{1}{n+1}} \psi(u) \sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\sum_{v=0}^{k} b_{n k} \frac{\cos \frac{u}{2}-\sin \left(n+\frac{1}{2}\right) u}{2 \sin \left(\frac{u}{2}\right)}\right\} d u\right| \\
& =\left|\begin{array}{l}
\frac{1}{n+1} \\
\int 0 \\
0
\end{array} \psi(u) \overline{K_{n}}(u) d u\right|
\end{aligned}
$$

$$
\begin{align*}
& =\left(\int_{0}^{\frac{1}{n+1}}\left(\frac{\xi(u)}{u^{1+\beta}}\right)^{s} d u\right)^{\frac{1}{s}} \\
& =O\left(\xi\left(\frac{1}{n+1}\right)\left(\int_{0}^{\frac{1}{n+1}} \frac{d u}{u^{(1+\beta) s}}\right)^{\frac{1}{s}}\right. \\
& =O\left(\xi\left(\frac{1}{n+1}\right)\right) O\left((n+1)^{-\frac{1}{s}+1+\beta}\right) \\
& =O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{r}}\right) \tag{5.2}
\end{align*}
$$

$$
\begin{align*}
& =\left|\begin{array}{l}
\left.\int_{\frac{1}{n+1}}^{\pi} \psi(u) \overline{K_{n}}(u) d u \right\rvert\,
\end{array}\right| \\
& \left.\leq\left(\int_{\frac{1}{n+1}}^{\pi}\left|\frac{u^{-\delta} \psi_{\psi(u) \sin } \beta_{u}}{\xi(u)}\right|^{r} d u\right)^{\frac{1}{r}} \int_{\frac{1}{n+1}}^{\pi}\left|\frac{\xi(u) \overline{K_{n}}(u)}{u^{-\delta} \sin \beta_{u}}\right|^{S} d u\right)^{\frac{1}{s}}, \\
& \text { (Where } \frac{1}{r}+\frac{1}{s}=1 \text {, using Hölder's inequality) } \\
& =O\left((n+1)^{\delta}\right)\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(u)}{u^{1+\beta-\delta}}\right)^{s} d u\right)^{\frac{1}{s}} \text {, using Lemma-4.1 and (3.3) } \\
& =O((n+1) \delta)\left(\int_{\frac{1}{\pi}}^{n+1}\left(\frac{\xi\left(\frac{1}{z}\right)}{z^{\delta-(1+\beta)}}\right)^{s} \frac{d z}{z^{2}}\right)^{\frac{1}{s}}, \\
& \text { ( } \xi\left(\frac{1}{z}\right) /\left(\frac{1}{z}\right) \text { is a positive increasing function as } \xi(u) \text { is a positive increasing function ) } \\
& =O\left((n+1)^{1+\delta} \xi\left(\frac{1}{n+1}\right)\left(\int_{\varepsilon}^{n+1} \frac{d y}{z^{s(\delta-\beta-1)+2}}\right)^{\frac{1}{s}} \text {, For some } \frac{1}{\pi} \leq \varepsilon \leq n+1\right. \text {. } \\
& \text { (Using second mean value theorem) } \\
& =O\left((n+1)^{1+\delta} \xi\left(\frac{1}{n+1}\right)\right) O\left((n+1)^{\beta+1-\delta-\frac{1}{s}}\right) \\
& =O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right) \tag{5.3}
\end{align*}
$$

Then from (5.2) and (5.3), we have

$$
\begin{aligned}
& \left|w_{n}-g\right|=O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right), r \geq 1 \\
& \left\|w_{n}-g\right\|_{r}=\int_{0}^{2 \pi} O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)^{r} d x\right)^{\frac{1}{r}}, r \geq 1
\end{aligned}
$$

$$
\begin{aligned}
& =O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right)\left(\int_{0}^{2 \pi} d x\right)^{\frac{1}{r}} \\
& =O\left((n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right)
\end{aligned}
$$

This completes the proof of the theorem.
Remark: If we put $\beta=0$ and $\xi(u)=u^{\alpha}$ in the main theorem then the degree of approximation of a function $g$ belonging to the class $\operatorname{Lip}(\alpha, r), 0<\alpha \leq 1, r \geq 1$ is given by

$$
\left\|w_{n}-g\right\|_{r}=O\left((n+1)^{-\alpha+\frac{1}{r}}\right)
$$

and if we take $r \rightarrow \infty$ then the degree of approximation of a function $g$ belonging to the class $\operatorname{Lip}(\alpha), 0<\alpha \leq 1$ is given by

$$
\left\|w_{n}-g\right\|_{r}=O\left((n+1)^{-\alpha}\right)
$$

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