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EDGE COLOURING OF GENERALIZED PETERSEN GRAPH OF TYPE k

B. STEPHEN JOHN[1] AND J. C. JESSY*[2]

**Department of Mathematics,
Annai Velankanni College, Tholayavattam, Tamilnadu. India-629157.**

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ABSTRACT

Let $G = \{V, E\}$ be a connected simple graph. An edge colouring of a graph G is a function $f: E(G) \to C$, where C is a set of distinct colours. The edge colouring problem is one of the fundamental problem on graphs which often appears in various scheduling problems like the file transfer problem on computer networks. In this paper we determine the edge colouring of generalized petersen graph of type k.

Keywords: *Cycle, Chromatic number, Edge colouring, Generalized petersen graph.*

INTRODUCTION

The theory of graph colouring has existed for more than 150 years from its modest beginning of determining whether a geographic map can be coloured with four colours. The theory has become central in discrete mathematics with many contemporary generalization and application.

In this paper, we are concerned with finite, connected, simple graph. Let $G = \{V(G), E(G)\}$ be a graph, if there is an edge e joining any two vertices u and v of G, we say that u and v are adjacent. A k-edge colouring C of a graph G is an assignment of k-colours to the edges of G.

Definition 1.1: A **graph** G is an ordered pair $(V(G), E(G))$ consisting of a non-empty set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$ of edges together with an incidence function ψ_G that associates with each edge of G is an unordered pair of vertices of G.

Definition 1.2: An **edge colouring** C of a simple graph G is proper if no two adjacent edges are assigned the same colour.

A graph is **properly edge coloured** if it is coloured with the minimum possible number of colours.

Definition 1.3: The **edge chromatic number** of a graph G is the minimum number of colours required to colouring the edges of G in properly and is denoted by $\chi'(G)$.

Definition 1.4: The **generalized petersen graph** $GP(n,k)$ has vertices and edges of the form
$V\big(GP(n,k)\big) = \{a_i, b_i / 0 \le i \le n-1\}$, $E\big(GP(n,k)\big) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} / 0 \le i \le n-1\}$.

Definition 1.5: Walk is an alternating sequence of vertices and edges starting and ending with vertices.

A walk in which all the vertices are distinct is called a **path**. A path containing *n*-vertices is denoted by P_n.

A closed path is called **cycle**. A cycle containing *n*-vertices is denoted by C_n, the length of a cycle is the number of edges occurring on it.

Lemma 1: For any cycle C_n, $\chi'(C_n) = \begin{cases} 3, & if\ n\ is\ odd \\ 2, & if\ n\ is\ even \end{cases}$

***Corresponding Author: J. C. Jessy*[2], Department of Mathematics,
Annai Velankanni College, Tholayavattam, Tamilnadu. India-629157.***

Theorem 2.1: Let G be the generalized petersen graph of type k, that is $G = GP(n,k)$ for all $n \geq 6$ and k is odd, then
$\chi'(G) = \begin{cases} 4, & if\ n\ is\ odd \\ 3, & if\ n\ is\ even \end{cases}$

Proof: Let $G = GP(n,k)$ be the generalized petersen graph of type k. Let $V(G) = \{v_i, u_i / i = 1,2,\ldots,n\}$ be the vertex set of $G = GP(n,k)$ and $E(G)$ are the edges of G, it can be partitioned into three sets, E_1, E_2 and E_3 such that
$E_1(G) = \{v_i v_{i+1}, v_1 v_n / i = 1,2,\ldots,(n-1)\}$,
$E_2(G) = \{u_i u_{i+k}, u_1 u_{n-k+1} / i = 1,2,\ldots,(n-(k-1))\}$ and
$E_3(G) = \{v_i u_i / i = 1,2,\ldots,n\}$.

Clearly, the edges of $E_i, i = 1,2,3$ satisfies the condition

$$E_1(G) \cap E_2(G) \cap E_3(G) = \emptyset$$

The elements of E_1 and E_2 formed separated cycles of length n. Let it be C_1 and C_2.

The edges of $C_{i;(i=1,2)}$ are independent to each other, that is $\bigcap_{i=1}^{2} E(C_i) = \emptyset$ and the edges of E_3 are independent in G.

Type-I: If n is odd and $(n,k) = 1$.

For example, $GP(7,3)$ is represented in figure: 1.1

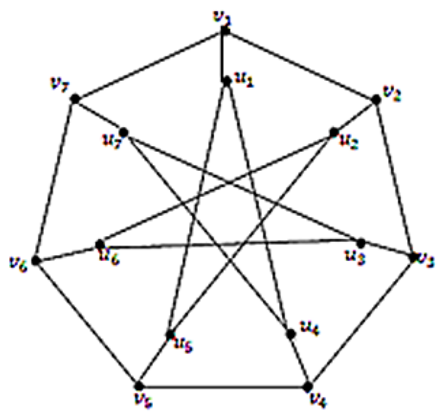

Figure:1.1

The cycle C_1 is of odd in length.

By lemma:1,

We need 3-colours to colour the edges of E_1, let it be c_1, c_2 and c_3.

The edge $v_1 u_1$ is adjacent to the edges $v_1 v_2$ and $v_n v_1$ which are coloured by the colours c_1 and c_3, so we can assign the colour c_2 to $v_1 u_1$.

Also, the edge $v_n u_n$ is adjacent to the edges $v_n v_1$ and $v_{n-1} v_n$, which are coloured by the colours c_3 and c_2 respectively, so we can colour $v_n u_n$ by the colour c_1.

The remaining edges $v_i u_i$ $(i = 2,\ldots,n-1)$, are adjacent with the edges of C_1 which are coloured by the colours c_1 and c_2, hence we can use the colour c_3 to the edges $v_i u_i$ $(i = 2,\ldots,n-1)$.

The edge $u_2 u_{2+k}$ is adjacent to the edge $v_2 u_2$ which is coloured by the colour c_3, so we can give either c_1 or c_2 to the edge $u_2 u_{2+k}$.

Without loss of generality, assume that the edge $u_2 u_{2+k}$ is coloured by the colour c_1.

Also, the edge $u_{n-k+2} u_2$ is adjacent to the edges $u_2 u_{2+k}$ and $v_2 u_2$, which are coloured by the colour c_1 and c_3, so we can assign the colour c_2 to the edge $u_{n-k+2} u_2$.

Repeat the above, to all the edges of E_2, which are adjacent with $v_i u_i$ $(i = 3,\ldots,n-1)$. except the edges adjacent with $v_n u_n$ and $v_1 u_1$.

Let $u_n u_{n-k}$ and $u_{n-k-1} u_n$ are the edges adjacent with $v_n u_n$.

The edge $v_n u_n$ is coloured by the colour c_1 and we cannot give the colour c_3 to the edges $u_n u_{n-k}$ and $u_{n-k-1} u_n$, since they are adjacent with the edges $u_{n-k} v_{n-k}$ and $u_{n-k-1} v_{n-k-1}$ respectively, which are already coloured by the colour c_3. So we can give the colour c_2 either to $u_n u_{n-k}$ or $u_{n-k-1} u_n$.

Without loss of generality, assume that the edge $u_{n-k-1} u_n$ is coloured by the colour c_2. Now, the edge $u_n u_{n-k}$ is adjacent with all the three existing colours say, c_1, c_2 and c_3.

Hence, we need a new colour c_4 to colour the edge $u_n u_{n-k}$.

Similarly, $u_1 u_{1+k}$ and $u_{n-k+1} u_1$ are the edges adjacent with $v_1 u_1$.

The edge $v_1 u_1$ is coloured by the colour c_2 and we cannot give the colour c_3 to the edges $u_1 u_{1+k}$ and $u_{n-k+1} u_1$, since they are adjacent with the edges $v_{1+k} u_{1+k}$ and $v_{2+k} u_{2+k}$ respectively, which are coloured by the colour c_3, so we can give the colour c_1 either to $u_1 u_{1+k}$ or $u_{n-k+1} u_1$.

Without loss of generality, assume that the edge $u_1 u_{1+k}$ is coloured by the colour c_1.

Now, the edge $u_{n-k+1} u_1$ is adjacent with all the three used colours c_1, c_2 and c_3.

Hence, we need a new colour c_4 to colour the edge $u_{n-k+1} u_1$.

Therefore, $\chi'(G) = 4$.

Type-II: If n is even and $(n, k) = 1$.

For example, $GP(8,3)$ is represented in figure: 1.2

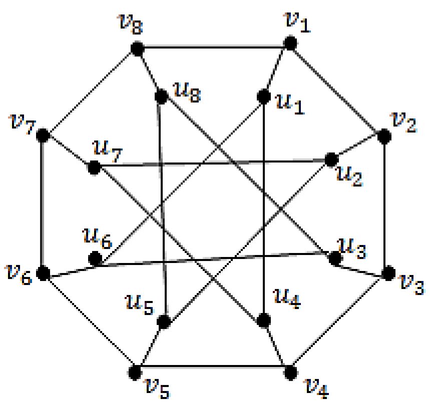

Figure:1.2

The cycles C_1 and C_2 are of even length.

By lemma:1,

The edges of C_1 and C_2 are coloured by two colours, let it be c_1 and c_2.

The edges $v_i u_i (i = 1,2, \dots, n-1)$, joining the cycles C_1 and C_2 are independent and adjacent with the existing coloured edges of C_1 and C_2, so we need a new colour c_3 to colour the edges of E_3.

Hence, we need 3-colours to colour all the edges.

Therefore, in this type $\chi'(G) = 3$.

Type-III: If $gcd(n, k) = k$

The edge set $E(G)$ can be partitioned into three sets E_1, E_2 and E_3 such that
$E_1(G) = \{v_i v_{i+1}, v_1 v_n / i = 1,2, \dots, n-1\}$,
$E_2(G) = \{u_i u_{i+k}, u_{n-k+i} u_i / i = 1,2, \dots, n-(k-1)\}$ and
$E_3(G) = \{v_i u_i / i = 1,2, \dots, n\}$.

In this type, the edge set of E_2 contains k-disjoint cycles of length-n/k. Let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_k}\}$. The cycles $C_{2_{i;\ (i=1,2,\dots,k)}}$ are either odd or even.

Case-(i): If n and ${}^{n}/_{k}$ are even, the cycles $C_{2_{i;\ (i=1,2,\dots,k)}}$ are of even in length.

For example, $GP(12,3)$ is represented in figure: 1.3

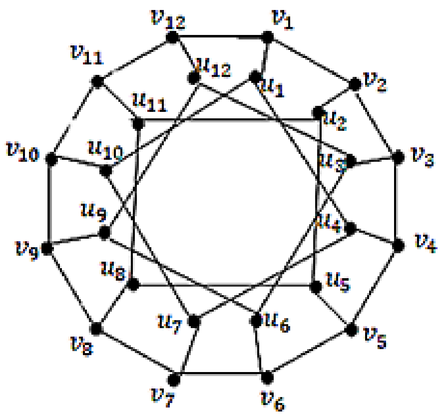

Figure: 1.3

By Type: II,

$$\chi'(G) = 3$$

Case-(ii): If n and ${}^{n}/_{k}$ are odd, the cycles $C_{2_{i;\ (i=1,2,\dots,k)}}$ are of odd length.

For example, GP (9, 3) is represented in figure: 1.4

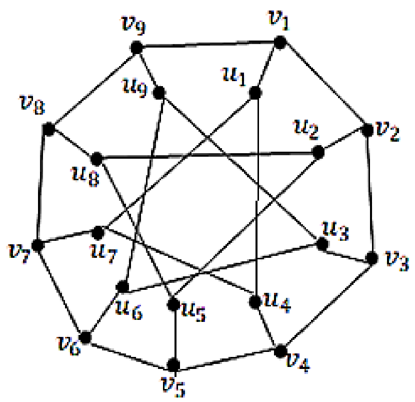

Figure: 1.4

By Type: I,

$$\chi'(G) = 4$$

Hence the theorem is proved.

Theorem 2.2: Let G be the generalized petersen graph of type k, that is $G = GP(n,k)$ for all n is odd and k is even, then $\chi'(G) = 4$.

Proof: Let $G = GP(n,k)$ be the generalized petersen graph of type k. Let $V(G) = \{v_i, u_i / i = 1,2,\dots,n\}$ be the vertex set of $G = GP(n,k)$.

Type-I: If n is odd and $(n,k) = 1$.

For example, $GP(9,2)$ is represented in figure: 1.5

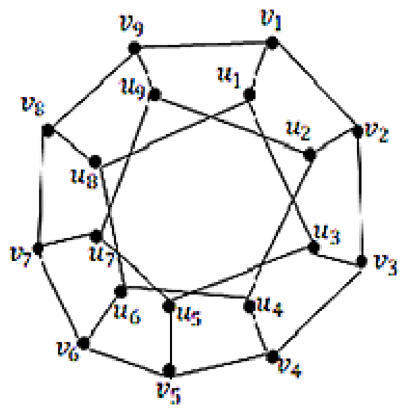

Figure:1.5

The edge set of $E(G)$ can be partitioned into three sets, E_1, E_2 and E_3 such that
$E_1(G) = \{v_i v_{i+1}, v_1 v_n / i = 1,2, \dots, n-1\}$,
$E_2(G) = \{u_i u_{i+k}, u_1 u_{n-k+1} / i = 1,2, \dots, n-(k-1)\}$ and
$E_3(G) = \{v_i u_i / i = 1,2, \dots, n\}$.

Clearly, the edges of $E_i, i = 1,2,3$ satisfies the condition
$E_1(G) \cap E_2(G) \cap E_3(G) = \emptyset$ and $E(G) = E_1(G) \cup E_2(G) \cup E_3(G)$.

The elements of E_1 and E_2 form separate cycles of length *n*. Let it be C_1 and C_2.

The edges of $C_{i,;\,(i=1,2)}$ are independent to each other, that is $\cap_{i=1}^{2} E(C_i) = \emptyset$ and the edges of E_3 are also independent in *G*.

Also, the cycles C_1 and C_2 are of odd in length.

By lemma:1,

We need *3*-colours to colour the edges of the cycles C_1 and C_2, let it be c_1, c_2 and c_3.

The edges $(v_i u_i (i = 1,2, \dots, n))$ of E_3 which are formed by joining the cycles C_1 and C_2 are coloured according to there adjacency with the edges of E_1 and E_2. But the edge say, $v_s u_s (1 \le s \le n)$ is adjacent to the edges of E_1 and E_2 which are coloured by the colours c_1, c_2 and c_3. So, we cannot give these three colours to colour the edge $v_s u_s (1 \le s \le n)$.
Hence, we need another new colour c_4 to colour the edge $v_s u_s \in E_3(G)$.
Thus, we need *4*-colours to colour all the edges of $E(G)$.
Therefore, $\chi'(G) = 4$.

Type-II: If $gcd\,(n,k) = t$, n and $^{n}/_{t}$ are odd.
For example, $GP(15,6)$ is represented in figure: 1.6

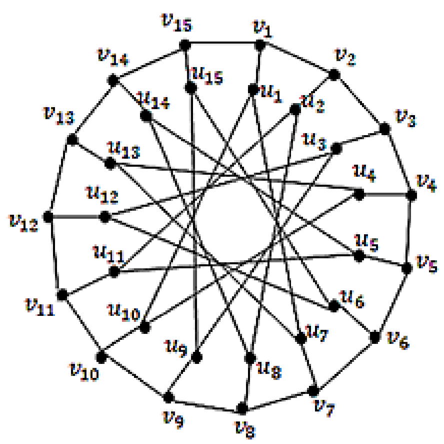

Figure:1.6

The edge set of $E(G)$ can be partitioned into three sets, E_1, E_2 and E_3 such that
$E_1(G) = \{v_i v_{i+1}, v_1 v_n / i = 1,2,\dots,n-1\}$,
$E_2(G) = \{u_i u_{i+k}, u_1 u_{n-k+1} / i = 1,2,\dots,n-(k-1)\}$ and
$E_3(G) = \{v_i u_i / i = 1,2,\dots,n\}$.

Clearly, the edges of $E_{i;\,(i=1,2,3)}$ satisfies the condition
$E_1(G) \cap E_2(G) \cap E_3(G) = \emptyset$ and $E(G) = E_1(G) \cup E_2(G) \cup E_3(G)$.
The edges of E_2 forms t-disjoint cycles in C_2, let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$. Each cycle is of odd length. Also, the cycle C_1 formed by the elements of E_1 is of odd length.
By type: I

$$\chi'(G) = 4.$$

Therefore, for all n is odd and k is even, $\chi'(G) = 4$.
Hence the theorem is proved.

Theorem 2.3: Let G be the generalized petersen graph of type k, that is $G = GP(n,k)$ for all n and k are even, then

$$\chi'(G) = \begin{cases} 4, & if\ {}^{n}/_{\gcd(n,k)}\ is\ odd \\ 3, & if\ {}^{n}/_{\gcd(n,k)}\ is\ even \end{cases}$$

Proof: Let $G = GP(n,k)$ be the generalized petersen graph of type k.
Let $V(G) = \{v_i, u_i; i = 1,2,\dots,n\}$ be the vertex set of $G = GP(n,k)$ and $E(G)$ can be partitioned into three sets, E_1, E_2 and E_3 such that
$E_1(G) = \{v_i v_{i+1}, v_1 v_n / i = 1,2,\dots,n-1\}$,
$E_2(G) = \{u_i u_{i+k}, u_i u_{n-k+i} / i = 1,2,\dots,n-(k-1)\}$ and
$E_3(G) = \{v_i u_i / i = 1,2,\dots,n\}$
Clearly, the edges of $E_i, i = 1,2,3$ satisfies the condition
$E_1(G) \cap E_2(G) \cap E_3(G) = \emptyset$ and $E(G) = E_1(G) \cup E_2(G) \cup E_3(G)$.

The elements of E_1 form an even cycle of length n. Let it be C_1.

By lemma:1,

We need 2-colours say, c_1 and c_2 to colour the edges of the cycle C_1.

Clearly, $gcd\ (n,k) = t$. Hence the edge set of E_2 forms t-disjoint cycles of length, let it be $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$ and these cycles are independent.

Also, the edges of E_3 are independent on G and the edges of E_1 and E_2 are independent in G.

Type-I: If ${}^{n}/_{t}$ is odd, then the cycles $C_{2_{i\,;\,(i=1,2,\dots,t)}}$ are of odd length.
For example, $GP(10,4)$ is represented in figure: 1.7

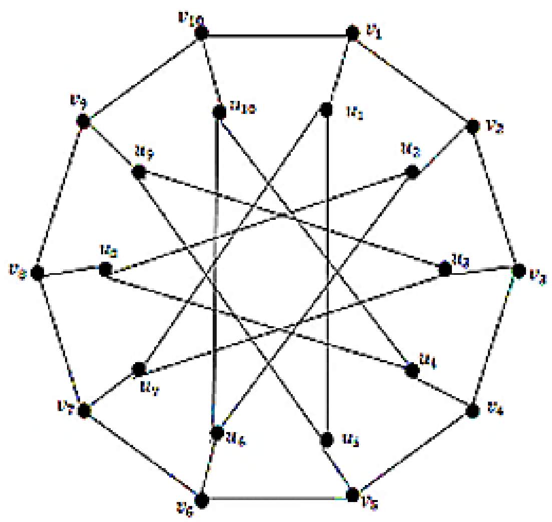

Figure: 1.7

Since, the edges of E_1 and E_2 are independent, by lemma: 1, the cycles $C_{2_{i\,;\,(i=1,2,\ldots,t)}}$ are coloured by *3*-colours say, c_1, c_2 and c_3.
The edges $v_iu_i(i = 1,2,\ldots,n)$ of E_3 which are connected by the cycles C_1 and $C_{2_{i\,;\,(i=1,2,\ldots,t)}}$ they are coloured according to there adjacency with the edges of E_1 and E_2. The edges $v_su_s(1 \le s \le n)$ is adjacent to the edges of E_1 and E_2 which are coloured by the colours c_1, c_2 and c_3. So, we cannot give these three colours to colour the edge $v_su_s(1 \le s \le n)$.
Hence, we need another new colour c_4 to colour the edge $v_su_s \in E_3(G)$.
Thus, in this type we need *4*-colours to colour all the edges of $E(G)$.
Therefore, $\chi'(G) = 4$.

Type-II: If $^n/_t$ is even, then the cycles $C_{2_{i\,;\,(i=1,2,\ldots,t)}}$ are of even length.
For example, $GP(16,6)$ is represented in figure: 1.8

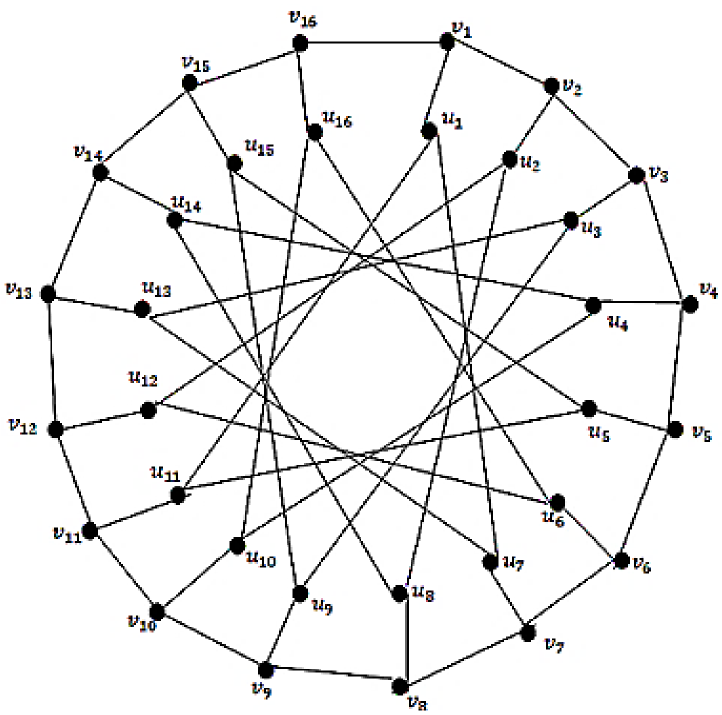

Figure: 1.8

Since, the edges of E_1 and E_2 are independent, by lemma:1, the cycles $C_{2_{i\,;\,(i=1,2,\ldots,t)}}$ are coloured by *2*-colours say, $c_1$and c_2.
The edges $(v_iu_i(i = 1,2,\ldots,n))$ of E_3 which are formed by joining the cycles C_1 and $C_{2_{i\,;\,(i=1,2,\ldots,t)}}$ are adjacent with the existing colours. So, we need a new colour c_3 to colour the edges of E_3.
Hence, we need *3*-colours to colour the edges of type: II.
Therefore, $\chi'(G) = 3$.
Hence the theorem is proved.

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