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THE TOTAL *x*-EDGE STEINER NUMBER OF A GRAPH

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ABSTRACT

F*or a vertex x of a connected graph G = (V, E) and W ⊂ V (G) is called total x-edge Steiner set if the subgraph < W > induced by W has no isolated vertex. The minimum cardinality of a total x-edge Steiner set of G is the total x-edge Steiner number of G and denoted by st1x(G). Some general properties satisfied by this concept are studied. The total x-edge Steiner number of certain classes of graphs are determined. Necessary conditions for connected graph of order p with total x-edge Steiner number to be p–1 is given. It is shown that for positive integers r, d and n > 2 with r ≤ d ≤ 2r, there exists a connected graph G with radG = r, diamG = d and st1x(G) = n for any vertex x in G. It is shown that for p,a and b are positive integers such that 4 ≤ a ≤ b ≤ p−1, then there exists a connected graph G of order p such that* $s_{1x}(G) = a$ *and* $s_{t1x}(G) = b$ *for some vertex x in G.*

Keywords: *Steiner distance, Steiner number, edge Steiner number, x-edge Steiner number, total x-edge Steiner number.*

AMS Subject Classification: *05C12.*

1. INTRODUCTION

By a graph G = (V, E), we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. The distance d(u, v) between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. An u–v path of length d(u, v) is called an u–v geodesic. It is known that the distance is a metric on the vertex set of G. For a vertex v of G, the eccentricity e(v) is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, radG and the maximum eccentricity is its diameter, diamG of G. For basic graph theoretic terminology, we refer to Harary [1]. For a nonempty set W of vertices in a connected graph G, the Steiner distance d(W) of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W - tree. It is to be noted that d(W) = d(u, v), when W = {u, v}. If v is an end vertex of a Steiner W-tree, then v ∈ W. Also if < W > is connected, then any Steiner W-tree contains the elements of W only. The set of all vertices of G that lie on some Steiner W-tree is denoted by S(W). If S(W) = V, then W is called a Steiner set for G. A Steiner set of minimum cardinality is a minimum Steiner set or simply a s-set of G and this cardinality is the Steiner number s(G) of G. If W is a Steiner set of G and v a cut vertex of G, then v lies in every Steiner W-tree of G and so W ∪ {v} is also a Steiner set of G. The Steiner number of a graph was introduced in [2] and further studied in [3, 4, 6, 7]. Let x be a vertex of a connected graph G and W ⊂ V (G) such that x ∉ W. Then W is called an x-edge Steiner set of G if every vertex of G lies on some Steiner W ∪ {x} - tree of G. The minimum cardinality of an x-edge Steiner set of G is defined as x-edge Steiner number of G and denoted by $s_{1x}(G)$. Any x-edge Steiner set of cardinality $s_{1x}(G)$ is called an s_{1x}-set of G. This concept is introduced in [8].

Theorem 1.1: [6] Every extreme vertex of G other than the vertex x (whether x is extreme or not) belongs to every x-edge Steiner set for any vertex x in G.

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2. THE TOTAL *x*-EDGE STEINER NUMBER OF GRAPH

Definition 2.1: Let *x* be a vertex of a connected graph *G* and $W \subset V(G)$. Then *W* is called a *total x* –edge *Steiner set* of *G* if W is an *x – edge Steiner set* of G and <W> has no isolated vertices. The minimum cardinality of a *t total x – edge Steiner* set of G is defined as *total x – edge Steiner number* of G and denoted by s_{t1x}(G). Any *total x –edge Steiner set* of cardinality s_{t1x}(G) is called a s_{t1x}– set of G.

Note 2.2: The vertex does not belongs to any minimum x – edge Steiner set of G Where as the vertex may belongs to a total x –edge Steiner set of G.

Example 2.3: For the graph *G* in Figure 2.1, the minimum total x-edge Steiner sets and the total x-edge Steiner numbers are given in Table 2.1.

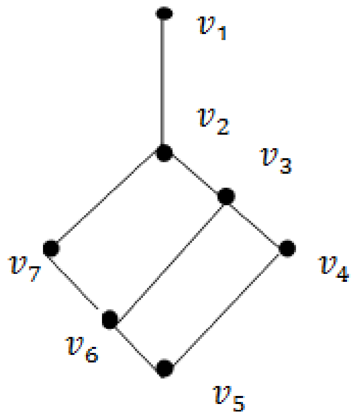

Figure2.1

Table-2.1

x-edge	s_{t1x} sets	$s_{t1x}(G)$
v_1	$\{v_2, v_4, v_5, v_7\}$, $\{v_2, v_3, v_5, v_6\}$, $\{v_2, v_3, v_4, v_5\}$	4
v_2	$\{v_1, v_2, v_4, v_5, v_6\}$	5
v_3	$\{v_1, v_2, v_5, v_6\}$	4
v_4	$\{v_1, v_2, v_5, v_6, v_7\}$	5
v_5	$\{v_1, v_2\}$	2
v_6	$\{v_1, v_2, v_3, v_5\}$	4
v_7	$\{v_1, v_2, v_4, v_5, v_6\}$	5

Theorem 2.4: Every extreme vertex of *G* other than the vertex *x* (whether *x* is extreme or not) belongs to every *total x –edge Steiner set* for any vertex *x* in *G*.

Proof: Since every *total x –edge Steiner set* of is a *x* - edge Steiner set, the result follows from Theorem 1.1.

Theorem 2.5: Let *x* be a vertex of a connected graph *G* and *v* be an extreme vertex of G such that $x \neq v$. Then every *total x –edge Steiner set* of contains at least one vertex of N(v).

Proof: Suppose there exists a *total x –edge Steiner set* of *G* such that *W* contains no element of N(v). By Theorem 2.4, $v \in W$. Then v is a isolated vertex of <W >.
Hence it follows that W is not *total x –edge Steiner set* of G.

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Corollary 2.6: For the complete graph K_p ($p\geq 2$), $s_{t1x}(K_p) = p-1$ for every vertex x.

Theorem 2.7: For the cycle $G = C_p(p \geq 3), s_{t1x}(G) = 2$ for every x in V(G).

Proof: Let $G = C_p(p \geq 4)$ be the cycle. Let p be even. Let x be any vertex of G and v be the antipodal vertex of x. Let z be an adjacent vertex of y. Let $W = \{y, z\}$ is a *total x –edge Steiner set* of G so that $s_{t1x}(G) = 2$.

Let p be odd. Let x be any vertex of G. Let y and z *be* two antipodal vertices of x. Then $W = \{y, z\}$ is a *total x-edge Steiner* of set of G so that $s_{t1x}(G) = 2$.

Theorem 2.8: For a complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$ with bipartite set $X = \{x_1, x_2, \dots, x_m\}$, $Y = \{y_1, y_2, \dots, y_n\}$, $s_{t1x}(G) = m + n - 1$.

Proof: Let $X = \{x_1, x_2, \dots, x_{m,}\}$, $Y = \{y_1, y_2, \dots, y_n\}$, be the bipartite sets of G. Let $x \in X$ and $W = X - \{x\}$. Then it is clear that W is an x-edge Steiner set of G. However it <W> has isolated vertices. Then it follows that every total x-edge Steiner set of G contains at least one vertex of $X - \{x\}$ and at least one vertex of Y. Since <W> is connected, S=G-{x} is the unique minimum total x-edge Steiner set so that $s_{t1x}(G) = m + n - 1$.

Theorem 2.9: For a star $G = K_{1,p-1}$,

$$s_{t1x}(G) = \begin{cases} p & if\ \ x\ \ is\ a\ cut\ vertex\ of\ G \\ p-1 & if\ x\ is\ an\ end\ vertex\ of\ G \end{cases}$$

Proof: If x is a cut vertex of G, then the result follows from Theorem 2.4 and 2.5. If x is an end vertex of G, then the result follows from Theorem 2.4.

Theorem 2.10: For any vertex x in G, $2 \leq s_{t1x}(G) \leq p$.

Proof: Any *total x –edge Steiner set* needs at least two vertices. Therefore $s_{t1x}(G) \geq 2$. For a vertex x, $W = V(G)$ is an *total x-edge Steiner set* of G and so $s_{t1x}(G) \leq |W| = p$.

Remark 2.11: The bounds in Theorem 2.10 are sharp. For an odd cycle G = C_{2n+1}, $s_{t1x}(G) = 2$ for every vertex x in G. For the complete graph K_p, $s_{t1x}(K_p) = p-1$ for every vertex x in G. The inequality in Theorem 2.10 can also strict. For the graph G in Figure 2.1, $s_{t1x}(G) = 4$ for $x = v_1$. Thus we have, $2 < s_{t1x}(G) < p$.

Theorem 2.12: Let G be a connected graph. Let x be a vertex of degree p -1. Then N(x) is a subset of every *total x –edge Steiner set* of G.

Proof: Let x be a vertex of degree p -1. Let $v_1, v_2, \dots, v_{p-1}$ be the neighbors of x in G. Suppose that $v_1 \in$ W. Then the edge xv_1 lies on a *Steiner W–tree* of G, say T. Since $v_1 \notin W_x$, v_1 is not an end vertex of T. Let T' be a tree obtained from T by removing the vertex v_1 in T joining all the neighbors of v_1 other than x in T to x. Then T' is a *Steiner W-tree* such that $|V(T')| = |V(T)|$ - 1 Which is a contradiction to T a *Steiner W -tree*. Therefore all N(x) is a subset of every *total x –edge Steiner set* of G.

Theorem 2.13: Let G be a connected graph and x a vertex of degree p-1. Which is not a cut vertex of G, then $s_{t1x}(G) = p$ -1.

Proof: Assume that x be a vertex of degree p -1. By Theorem 2.12, $s_{t1x}(G) \geq p$-1. Then N(x) is a *total x –edge Steiner set* of G. Since <W> is connected, <W> has no isolated vertices. Therefore W is a *total x –edge Steiner set* of G so that $s_{t1x}(G) = p$ -1.

Theorem 2.14: Let G be a connected graph and x a vertex of degree p-1 which is a cut vertex of G, then $s_{t1x}(G) = p$.

Proof: Assume that x be a vertex of degree p -1 which is a cut vertex of G. Let W be a total *x –edge Steiner set* of G. By Theorem 2.12, N(x) is a subset of every *total x –edge Steiner set* of G. Since <N(x)> contains isolated vertices, N(x) is not a *total x –edge Steiner set* of G and so $s_{t1x}(G) \geq p$. Hence it follows that N[x] is a *total x –edge Steiner set* of G and so $s_{t1x}(G) = p$.

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Theorem 2.15: For positive integers r, d and $n \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with *rad G=r*, *diam G = d* and $s_{t1x}(G) = n$ for some vertex x in G.

Proof: When $r = 1$, If $d = 1$, let $G = K_{n+1}$. Then by Corollary2.7, $s_{1x}(G) = n$, for any vertex x in G. If $d = 2$, let $G = K_{1,n}$. Then by Theorem 2.9, $s_{t1x}(G) = n$ for an end vertex x in G. Now, let $r \geq 2$. Construct a graph G with the desired properties as follows: Let C_{2r}: $u_1, u_2,\ldots u_{2r}, u_1$ be a cycle of order $2r$ and let P_{d-r+1}: $v_0, v_1, v_2,\ldots, v_{d-r}$ v_0 be a path of order $d-r+1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying u_1 in C_{2r} and v_0 in P_{d-r+1}. Let G be graph obtained from H by adding $(n-2)$ new vertices $w_1, w_2,\ldots w_{n-2}$ to H and join each vertex w_i $(1 \leq i \leq n-2)$ to the vertex v_{d-r-1}. The graph G of Figure 2.2. Then *rad G = r* and *diam G = d* and G has n-1 end vertices. Let x = u_{r+1}. Then by Theorem 2.4 and 2.5, $W = \{v_{d-r-1}, v_{d-r}, w_1, w_2,\ldots w_{n-2}\}$ is a subset of every *total x –edge Steiner set* of G. Now W is a *total x –edge Steiner set* of G. Since <W> is connected, W is a *total x –edge Steiner set* of G so that $s_{t1x}(G) =$ n-2+2 = n.

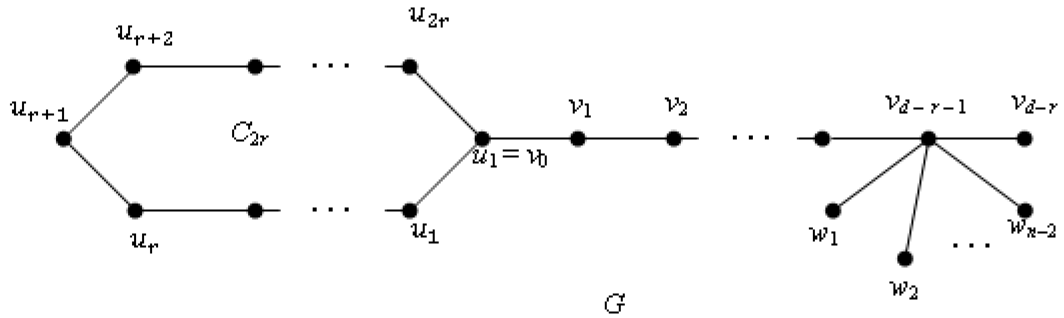

Figure-2.2

Theorem 2.16: For a connected graph G, $1 \leq s_{1x}(G) \leq s_{t1x}(G) \leq$ p.

Proof: Let x be a vertex of G. Any *x-edge Steiner se*t of G needs atleast two vertices and so $s_{1x}(G) \geq 1$. Also every *total x –edge Steiner set* of G and so $s_{1x}(G) \leq s_{t1x}(G)$. Also V(G) is a *total x –edge Steiner set* of G and so $s_{tx}(G) \leq p$. Thus $1 \leq s_x(G) \leq s_{t1x}(G) \leq$ p.

Theorem 2.17: If p, a and b are positive integers such that $4 \leq$ a $\leq b \leq p-1$, then there exists a connected graph G of order p such that $s_{1x}(G) = a$ and $s_{t1x}(G) = b$ for x be a vertex of G.

Proof: Let P: v_1, v_2, v_3, v_4, v_5 be the path on Five vertices. Let G be the graph obtained from by adding the new vertices $w_1, w_2, \ldots, w_{b-a-1}$ and $z_1, z_2, \ldots, z_{a-1}$ and join each z_i $(1 \leq i \leq a-1)$ with v_5 join each w_i $(1 \leq i \leq b-a-1)$ with v_1, v_2 and v_3. The graph G is shown in Figure 2.3.

Let $x = v_1$. Let $Z = \{z_1, z_2, \ldots, z_{a-1}\}$. By Theorem 1.1, Z is a subset of every *x-edge Steiner set* of G and so that $s_{1x}(G) \geq a$-1. It is clear that Z is not a *x-edge Steiner set* of G and so that $s_{1x}(G) \geq a$. However Z ∪ $\{v_5\}$ is a *x-edge Steiner set* of G and so $s_{1x}(G)$ = a.

Next we prove that $s_{t1x}(G) = b$. It is easily seen that each w_i $(1 \leq i \leq b-a-1)$ is a subset of every *total x-edge Steiner set* of G. Let Z_1 = Z ∪ $\{w_1, w_2, \ldots, w_{b-a-1}\}$. Then Z_1 is not a *total x-edge Steiner set* of G and so that $s_{t1x}(G) \geq a - 1 + b - a - 1 = b - 2$. Since $< Z_1 >$ has no isolated vertices, v_2 and v_5 must lie Z_2. Now $Z_2 \cup \{v_2, v_5\}$ is a *total x-edge Steiner set* of G and so that $s_{t1x}(G) =$ b.

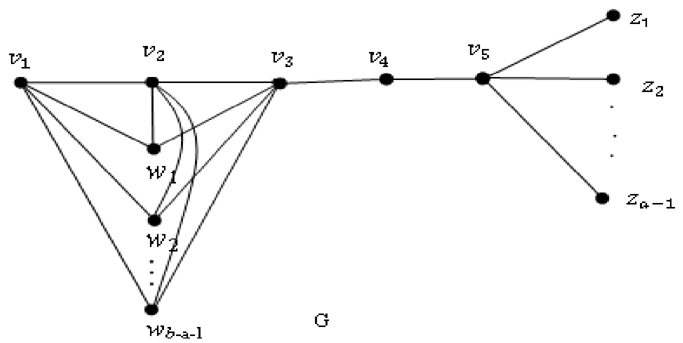

Figure-2.3

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