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HOMOMORPHISMS, STRONG REGULARITY, STRONG REDUCEDNESS AND RELATED CONCEPTS

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ABSTRACT

The aim of this paper is to generalize the homomorphism from Z_m to Z_n and give the idea to construct near-rings of low order and give examples to strongly regular and strongly reduced near-rings. We also give some examples to justify the main result in the paper "Characterizations of strongly regular near-rings"

Keywords*: Group, Near-Rings, Homomorphism, Strongly Regular, Strongly reduced.*

1. INTRODUCTION

Throught this paper we work with right near-rings.

The concept of homomorphism on near-rings are like on rings. Z_n is an abelian group under addition modulo n. Let N, N' be near-rings, $h : N \to N'$ is called near-ring homomorphism if for every $m, n \text{ in } N$ such that $h(m+n) = h(m) + h(n)$ and $h(mn) = h(m)h(n)$. Mason [1] introduced the notions of strong regularity in near-rings and characterized left regular zero-symmetric unital near-rings. Reddy and Murty [5] extended the results in [1] to arbitrary near-rings and proved that the concepts of left regularity, left strong regularity and right regularity in near-rings are equivalent and these imply right strong regularity. Yong Uk Cho and YasuyukiHirano [7] showed that the strong regularity in near-rings is equivalent to the property (*) in [5]. Narmada and Anil Kumar [3] characterize the strong regularity of near-rings.

We will use the following notations:

Given a near-ring N, $N_0 = \{n \text{ in } N : n0 = 0\}$ which is called the zero-symmetric part of N, $N_c = \{n \text{ in } N : n0 = n\} = \{n \text{ in } N : nn' = n \text{ for every } n' \text{ in } N\}$ which is called the constant part of N. Clearly N_0 and N_c are sub near-ring of N. A near-ring N is called zero-symmetric if $N = N_0$ and is called a constant near-ring if $N = N_c$

For the basic concepts and notations, we shall refer to Pilz [4]

2. PRELIMINARIES

A near-ring N is called left strongly regular if for every $a \text{ in } N$, there exist $x \text{ in } N$ such that $a = xa^2$ and left regular if for every $a \text{ in } N$, there is an $x \text{ in } N$ such that $a = xa^2 and\ a = axa$. Right strong regularity and right regularity can be defined in symmetric way. N is strongly regular if it is both left and right strongly regular .We can say that $N$is reduced if N has no non zero nilpotent elements, that is for each $a \text{ in } N$, $a^n = 0$, for some positive integer n *implies* $a = 0$. In ring theory Mc Coy[8] proved that N is reduced if and only if for each $a \text{ in } N$, $a^2 = 0$ *imlpies* $a = 0$. A near-ring N is said to be strongly reduced if for $a \text{ in } N, a^2 \text{ in } N_c$ *implies* $a \text{ in } N_c$.

Characterization of Homomorphism on Z_n:
Suppose $f : Z_m \to Z_n$ is a group homomorphism and assume $f(1) = k$, then for $m \text{ in } N$, $f(m) = mk$ and $f(-m) = -mk$. Thus for $x \text{ in } Z$, $f(x) = xf(1) = xk$

$$f(0) = 0f(1).$$

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Then for $x \ in \ Z_m, f(x) = f(\underbrace{1 + 1 + \cdots + 1}_{x}) = xf(1)$
$= xk \ for \ some \ k \ in \ Z_n.$

That is $f(x) = xk \ (mod \ n)$ is the homomorphism.

That is $f : Z_m \to Z_n$ is a homomorphism and $f(1) = k$, then the homomorphism has the form $f(x) = xk \ (mod \ n)$.

Converse of the above result is not true,

For $f(x) = xk \ (mod \ n)$,
f(x+y) ≠ (xk $+_n$ yk) (mod n).

Now assume $f(1) = k$,then $0 \equiv f(0) \equiv f(m) = f(\underbrace{1 + 1 + \cdots + 1}_{m})$

$= \underbrace{f(1) + f(1) + \cdots + f(1)}_{m} = mf(1) = mk.$

That is, k is the solution of the system $mx \equiv 0 \ (mod \ n)$.

Conversely, if k is a solution of $mx \equiv 0 \ (mod \ n)$ then $f(x) = xk \ (mod \ n)$ is a homomorphism from $Z_m \to Z_n$.

For, let $x, y \ in \ Z_m$ and suppose $mk \equiv 0 (mod \ n)$

Let$x +_m y = t, x, y \ in \ Z_m, then \ x + y = mr + t, 0 \leq t < m.$
$f(x +_m y) = f(t)kt \ (mod \ n) = k(x + y - mr)(mod \ n)$
$= f(x) +_n f(y).$

Therefore f is a homomorphism.

Theorem 2.1: The function $f: Z_m \to Z_n$ given by $f(x) = xk$ for some $k \ in \ Z_n$, fixed is homomorphism of groups if and only if $mk \equiv 0 \ (mod \ n)$

If k is the solution of the system $mk \equiv 0 \ (mod \ n)$ and $d = \gcd(m, n)$: and if $d \backslash 0$ the system has gcd (m, n) solutions.

Corollary 2.2: The function $f: Z_m \to Z_n$ is a homomorphism and $f(x) = xk$ where k is the solution of the system $mk \equiv 0 \ (mod \ n)$ and $(m, n) = 1$, then f is an on to homomorphism.

Remark [6]: Suppose $f : Z_m \to Z_n$ is a ring homomorphism and assume $f(1) = k$, since every ring homomorphism is a group homomorphism, $f(x) = xk(mod \ n)$ is a ring homomorphism if and only if k is the solution of the system$mk \equiv 0(mod \ n)$. Also $k = f(1) = f(1.1) = [f(1)]^2 = k^2$($k$ is idempotent). That is k is also a solution of the system $x^2 \equiv x(mod \ n)$

3. NEAR-RINGS ON GROUPS OF LOW ORDER

G Pilz, gives the description of near-rings of low order and from the above argument of homomorphism on near-rings, we give the idea to construct the near-rings of low order on $Z_n, n \leq 7$.

Suppose $\phi : Z_n \to Z_n$ by $\phi(x) = kx \ (mod \ n)$ is a homomorphism, we list the endomorphism α_o,α_1,…,α_{n-1} of Z_n and each $\phi_k(x)$ represent the homomorphism , every isomorphism class of near-rings of order n is determined by the n-tuple $(1,2, \dots, n-1)$.The numbers following this n-tuple denote the number of those automorphisms which yield isomorphic near-rings of Z_n.

We give multiplication table of ϕ as follows:

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$^{\circ}{}_{\phi}$	α_o	α_1	α_2	α_3	α_4	α_5	α_6
0	$\phi_o(0)$	$\phi_0(1)$	$\phi_o(2)$	$\phi_o(3)$	$\phi_o(4)$	$\phi_o(5)$	$\phi_o(6)$
1	$\phi_1(0)$	$\phi_1(1)$	$\phi_1(2)$	$\phi_1(3)$	$\phi_1(4)$	$\phi_1(5)$	$\phi_1(6)$
2	$\phi_2(0)$	$\phi_2(1)$	$\phi_2(2)$	$\phi_2(3)$	$\phi_2(4)$	$\phi_2(5)$	$\phi_2(6)$
3	$\phi_3(0)$	$\phi_3(1)$	$\phi_3(2)$	$\phi_3(3)$	$\phi_3(4)$	$\phi_3(5)$	$\phi_3(6)$
4	$\phi_4(0)$	$\phi_4(1)$	$\phi_4(2)$	$\phi_4(3)$	$\phi_4(4)$	$\phi_4(5)$	$\phi_4(6)$
5	$\phi_5(0)$	$\phi_5(1)$	$\phi_5(2)$	$\phi_5(3)$	$\phi_5(4)$	$\phi_5(5)$	$\phi_5(6)$
6	$\phi_6(0)$	$\phi_6(1)$	$\phi_6(2)$	$\phi_6(3)$	$\phi_6(4)$	$\phi_6(5)$	$\phi_6(6)$

Example:

1) $Z_1 = \{0\}, this\ case\ is\ trivial$

2) $Z_2 = \{0,1\}$ Define $\phi : Z_2 \rightarrow Z_2$ by $\phi(x) = kx$ is a group homomorphism and ϕ has gcd (2,2) = 2 homomorphism.

That is $\phi_o(x) = 0x, \phi_1(x) = 1x$ are the homomorphisms.

The multiplication table is given by

+	0	1
0	0	1
1	1	0

$^{\circ}{}_{\phi}$	α_o	α_1
0	$\phi_o(0)$	$\phi_0(1)$
1	$\phi_1(0)$	$\phi_1(1)$

$^{\circ}{}_{\phi}$	α_o	α_1
0	0	0
1	0	1

3) $Z_3 = \{0,1,2\}$

$^{\circ}{}_{\phi}$	α_o	α_1	α_2
0	$\phi_o(0)$	$\phi_0(1)$	$\phi_o(2)$
1	$\phi_1(0)$	$\phi_1(1)$	$\phi_1(2)$
2	$\phi_2(0)$	$\phi_2(1)$	$\phi_2(2)$

$^{\circ}{}_{\phi}$	α_o	α_1	α_2
0	0	0	0
1	0	1	2
2	0	2	1

4) $Z_4 = \{0,1,2,3\}$

$^{\circ}{}_{\phi}$	α_o	α_1	α_2	α_3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

5) $Z_5 = \{0,1,2,3,4\}$

$^{\circ}{}_{\phi}$	α_o	α_1	α_2	α_3	α_4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

6) $Z_6 = \{0,1,2,3,4,5\}$

$^{\circ}{}_{\phi}$	α_o	α_1	α_2	α_3	α_4	α_5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

7) $Z_7 = \{0,1,2,3,4,5,6\}$

$\circ_\phi$	α_o	α_1	α_2	α_3	α_4	α_5	α_6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

We observe that N is zero-symmetric if and only if n-tuple starts with entry $0.$ N is constant if and only if n-tuple is $(1,1,\ldots,1)$. The n-tuple $(0,0,\ldots,0)$ is the zero near-ring on Z_n.

4. STRONGLY REGULAR NEAR-RINGS

We know that every strongly regular near-ring is strongly reduced (Proposition 1 of [7]).But the following example shows that this result is not true in general.

Example: Let $N = Z$, the set of all integers with usual addition and multiplication given by $a.b = ab, for\ a, b\ in\ N, a, b \neq 0,1,-1$ .Then N is strongly reduced near-ring,but is not strongly regular,since $a\ in\ N\ with\ a \neq 0,1,-1$, there exist no $x\ in\ N\ such\ that\ a = xa^2$.

The following theorem shows that if N is strongly reduced and regular , then it becomes strongly regular.

Theorem 4.1 [3]: Let N be a near-ring. Then N is strongly regular if and only if it is strongly reduced and regular.

Now we give some examples to justify this result.

Example 1: Let $N = \{0,1,2,3,4,5\}$ be an additive group of integers modulo 6 and multiplication as follows (see Pilz [4] for near-rings of low order Z_6, no: 24, $(3,5,5,3,1,1)$).

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

Clearly, this near-ring N is non zero-symmetric and reduced and regular. The constant part of $N\ is\ \{0,3\}$. We see that this near-ring N is strongly reduced. From the above theorem, N is strongly regular.

Example 2: Z_7, no: $19, (1,1,1,1,1,1,1)$ of Pilz [4]

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6

From observation, N is strongly regular.

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Now we classify reduced and regular near-rings of order ≤ 7, which are strongly regular and strongly reduced near-rings. To do this we use $Clay$'s [2] table.

$Groups$	$Zerosymmetric,$ $reduced\ and\ regular$	$Non\ zerosymmetric,$ $reduced\ and\ regular$
Z_2		3
Z_3	3	4
Z_4	8	9
Z_5	7,8,10	9
Z_6	27,47	24,35,48,49,52,53
Z_7	18,20,21,22,23,24	19

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