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PROPER MODULAR CHROMATIC NUMBER OF CIRCULAR HALIN GRAPHS OF LEVEL TWO

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ABSTRACT

For a connected graph G, let c: V(G)→ $\mathbb{Z}_k$ (k ≥ 2) be a vertex coloring of G. The color sum (v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N(v), that is $\sigma(v) = \sum_{u\in N(v)} c(u)$ (mod k). The coloring c is called a modular k-coloring of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs of adjacent vertices x, y ∈ G. The modular chromatic number or simply the mc-number of G is the minimum k for which G has a modular k-coloring.

For a connected graph G, let c: V(G)→ $\mathbb{Z}_k$ (k ≥ 2) be a proper vertex coloring of G. The color sum (v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N(v), that is $\sigma(v) = \sum_{u\in N(v)} c(u)$ (mod k). The coloring c is called a proper modular k-coloring of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs of adjacent vertices x, y ∈ G. The proper modular chromatic number or simply the pmc-number of G is the minimum k for which G has a proper modular k-coloring.

In this paper we determine the pmc-number of Circular Halin graphs of level two.

Keywords: *pmc-number, Circular Halin graphs.*

MSC AMS Classification 2010: *05C15.*

1. INTRODUCTION

For a connected graph G, let c: V(G)→$\mathbb{Z}_k$ (k ≥ 2) be a vertex coloring of G where adjacent vertices may be colored the same. The color sum σ(v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N(v), that is $\sigma(v) = \sum_{u\in N(v)} c(u) (\text{mod } k)$. The coloring c is called a modular sum k-coloring or simply a modular k-coloring of G, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G. A coloring c is called modular coloring if c is a modular k-coloring for some integar k ≥ 2. The modular chromatic number mc(G) is the minimum k for which G has a modular k-coloring. This concept was introduced by Okamoto, Salehi and Zhang [1, 2, 3].

Okamoto, Salehi and Zhang proved in [1] that: every nontrivial connected graph G has a modular k-coloring for some integar k ≥ 2 and mc(G) ≥ $\chi(G)$, where $\chi(G)$ denotes the chromatic number of G; for the cycle C_n of length n, mc(C_n) is 2 if n ≡ 0 (mod 4) and it is 3 otherwise; every nontrivial tree has modular chromatic number 2 or 3; for the complete multipartite graph G, mc(G) = $\chi(G)$; for the wheel $W_n = C_n$ v K_1, n ≥ 3, mc(W_n) = $\chi(W_n)$, where v denotes the join of two graphs that: for m, n ≥ 2, mc(P_m x P_n) = 2. M.Paramaguru and R.Sampath kumar proved in [4] that :every two vertex-disjoint nonempty bipartite graphs G and H, mc(G v H) = 4; for m ≥ 2 and n ≥ 2, mc(P_m v P_n) = 4; for m ≥ 2 and n ≥ 2, mc(P_m v C_{2n}) = 4; for n ≥ 2 , r, s ≥ 1, mc(P_m v $K_{r,s}$) = 4.

For a connected graph G, let c: V(G)→ $\mathbb{Z}_k$ (k ≥ 2) be a proper vertex coloring of G. The color sum (v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N(v), that is $\sigma(v) = \sum_{u\in N(v)} c(u)$ (mod k). The coloring c is called a proper modular k-coloring of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$for all pairs of adjacent vertices x, y ∈ G. A coloring c is a proper modular coloring if c is a modular k-coloring for some integer k ≥ 2. The proper modular chromatic number or

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simply the pmc-number of G is the minimum k for which G has a proper modular k-coloring. In [5], Ryan Jones dealt with modular and graceful edge coloring of graphs. He mentioned in his thesis that the results can be extended to proper modular vertex coloring. This chapter deal with certain results of pmc-number of circular halin graphs of level two. [6, 7]

We reproduce certain observations given in [1] which would be used in this article.

Observation 1.1: For every non trivial graph G, pmc(G) ≥ mc(G) ≥ χ(G).

Observation 1.2: If G has a clique of order k, then mc(G) ≥ k.

For circular halin graphs let ℓ_1 denote the vertices of the Halin graph at i[th] level where i = 0, 1 2, ... Let the vertex at level ℓ_0 be x. The vertices in ℓ_1 be v_1, $v_2, \ldots,$ v_D taken in the clockwise direction. Let the vertices emerging from v_i to level ℓ_2 be $w_{i,1}$, $w_{i,2}, \ldots,$ $w_{i,D-1}$ consecutively in the clockwise direction. We repeat a description of $H_1(2 , 6)$

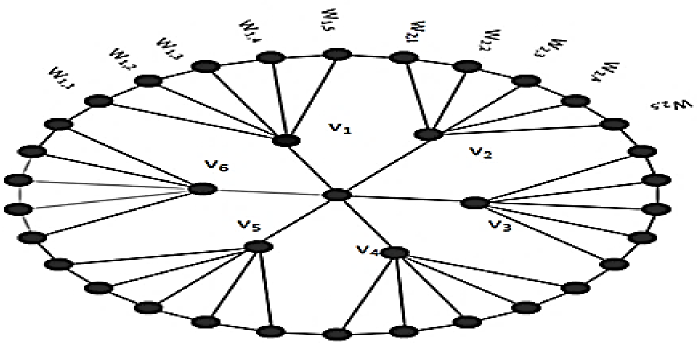

Figure-1.1: $H_1(2 , 6)$

2. MAIN RESULTS

Theorem 2.1: For a graph G = $H_1(2, D)$, pmc(G) = 4.

Proof: Consider the graph $H_1(2, D)$ where D > 2.For a graph G = $H_1(2, D)$, mc(G) = 3. Hence by Observation I. 1.pmc(G) ≥ 3.

We observe from Fig. 2. 1 when we take integer modulo 3 in $H_1(2, 3)$, the color sum of all the vertices at level ℓ = 1 is 2. Similarly the color sum for the vertices at w_{i2} for all i is 2. Hence $H_1(2, 3)$ does not follow a proper modular coloring. Therefore pmc(G) ≥ 4. (1)

Case-1: $H_1(2, D)$ where D = 3, 11, 19, 27, . . .

Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined byc(v) = $\begin{cases} 0, & \text{if } v = v_i, i = 1, 2, 3, \ldots, D, \\ 1, & \text{if } v = w_{ij}, i = 1, 2, 3, \ldots, D \text{ and j is odd}, \\ 2, & \text{if } v = x, v = w_{ij}, i = 1, 2, 3, \ldots, D \text{ and j is even}. \end{cases}$

For v = v_i, the neighboring vertices are colored as 12 2 or 1212 1212 12 2 or 1212 1212 1212 1212 12 2 and so on depending on D = 3, 11, 19, 27, . . . Here, the color sum of v under modulo 4 is 1. Hence (v) = 1.

For v = $w_{ij}, i = 1, 2, 3, \ldots, D$ and j is odd, the neighboring vertices are colored as 022 where the color sum is 4. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 0. For v = x, the neighboring vertices are colored as 0 and hence the color sum σ(v) = 0. For v = $w_{ij}, i = 1, 2, 3, \ldots, D$ and j is even the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 110. Hence (v) = 2.

Therefore (v) = $\begin{cases} 0, & \text{for } v = x, v = w_{ij}, i = 1, 2, 3, .., D \text{ and j is odd}, \\ 1, & \text{for } v_i, \ i = 1, 2, \ldots, D, \\ 2, & \text{for } w_{ij}, \ i = 1, 2, \ldots, D \text{ and j is even}. \end{cases}$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$.Then pmc(G) $\leq$ 4 (2)

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From (1) and (2), pmc(G) = 4.

Example:

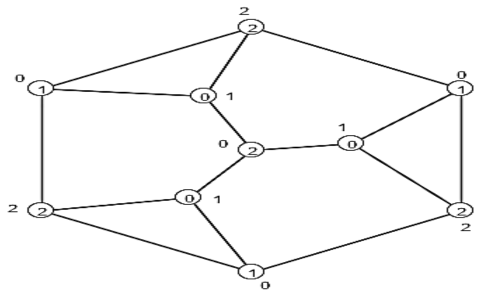

Figure-2.1: Proper modular coloring in $H_1(2, 3)$.

Case-2: $H_1(2, D)$ where D = 4, 12, 20, 28, . . .

Subcase 2.1: D = 4.

Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by $c(v) = \begin{cases} 0, & \text{if } v = v_i, i = 1, 2, 3, \ldots, D, \\ 1, & \text{if } v = w_{ij} \text{ both i, j is odd and i, j is even}, \\ 2, & \text{if } v = x, v = w_{ij}, \text{ i is even and j is odd}, \\ 3, & \text{if } v = w_{ij}, \text{ i is odd and j is even}. \end{cases}$

For v = v_i, the neighboring vertices are colored as 1312 or 2122. Here, the color sum of v under modulo 4 is 3. Hence (v) = 3. For v = x, all the neighboring vertices are colored as 0 where the color sum is 0. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 0. For v = w_{ij}, both i and j are odd the neighboring vertices are colored as 203 and hence the color sum σ(v) = 1. For v = w_{ij}, both i and j are even the color sum of the neighboring vertices is 0 where the colors assigned to the vertices are 220. Hence (v) = 0.For v = w_{ij}, i is even and j is odd, the neighboring vertices are colored as 101 and hence the color sum σ(v) = 2.

Therefore $(v) = \begin{cases} 0, & \text{for } v = x, v = w_{ij}, \text{ i and j are even}, \\ 1, & \text{for } w_{ij}, \text{ both i and j are odd}, \\ 2, & \text{for } w_{ij}, \text{ i is odd, j is even and vice versa}, \\ 3, & \text{for all } v_i, i = 1, 2, 3, \ldots, D. \end{cases}$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$.Then pmc(G) ≤ 4 (3)

From (1) and (3), pmc(G) = 4.

Subcase-2.2: D = 12, 28, 44, . . .

Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 0, & \text{if } v = v_i, i = 1, 2, 3, \ldots, D, \\ 1, & \text{if } v = w_{ij} \text{ for odd i and } j = 4k, k = 1, 2, \ldots \text{ and } w_{i,D-1} \text{ for i is even}, \\ 2, & \text{if } v = w_{ij}, \text{ i is odd and } j = 2 + 4k, k = 0, 1, 2, \ldots \\ & v = w_{ij} \text{ for i even and j odd}, j \neq D-1, \\ 3, & \text{if } v = x, v = w_{ij}, \text{ both i, j are odd and i, j are even}. \end{cases}$$

For v = v_i, i is odd, the neighboring vertices are colored as 3231 3231 323 3 or 3231 3231 3231 3231 3231 3231 323 3 and so on. Here, the color sum of v under modulo 4 is 1. Hence (v) = 1. For v = v_i, i is even, the neighboring vertices are colored as 2323 2323 231 3 or 2323 2323 2323 2323 2323 2323 231 3 and so on. Here, the color sum of v under modulo 4 is 1. Hence (v) = 1. For v = x, all the neighboring vertices are colored as 0 where the color sum is 0. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 0. For v = w_{ij}, i is odd, j = D-1, the neighboring vertices are colored as 202 and hence the color sum σ(v) = 0.

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For v = w_{ij}, both i and j are even, j ≠ D-2, the neighboring vertices are colored as 202 and hence the color sum σ(v) = 0. For v = w_{ij}, both i and j are odd≠jD -1, the neighboring vertices are colored as 201 and hence the color sum σ(v) = 3.For v = w_{ij}, i is even, j = D-2, the neighboring vertices are colored as 201 and hence the color sum σ(v) = 3.

For v = w_{ij}, i is odd and j is even the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 303. Hence (v) = 2. For v = w_{ij}, i is even and j is odd, the neighboring vertices are colored as 303 and hence the color sum σ(v) = 2.

$$\text{Therefore }(v) = \begin{cases} 0, \quad \text{for } v = x, v = w_{ij}, \text{ i and j are even}, j \neq D-2\,; \\ \qquad\qquad v = w_{iD-1} \text{ where i is odd}, \\ 1, \text{ for } v = v_i, \quad i = 1, 2, 3, \ldots. D, \\ 2, \text{for } w_{ij}, \quad \text{odd i and even j and vice versa}, \\ \qquad 3, \; v = w_{ij}, \text{ i and j are odd}, j \neq D-1; \\ \qquad\qquad v = w_{iD-2} \text{ where i is even.} \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$.Then pmc(G) ≤ 4 (4)

From (1) and (4), pmc(G) = 4.

Subcase-2.3: D = 20, 36, 52, . . .

Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 0, \text{ if } v = v_i, i = 1, 2, 3, \ldots, D, \\ 1, \text{ if } v = x, v = w_{ij} \text{ for odd i and } j = 4k, k = 1, 2, \ldots; \\ \qquad\qquad v = w_{ij} \text{ for even i and } j = 5 + 4k, k = 1, 2, , , ., j \neq D-3, \\ 2, \text{ if } v = w_{ij}, \text{i is odd and } j = 2 + 4k, k = 0, 1, 2, \ldots \\ \qquad v = w_{ij} \text{for i even}, j = 3 + 4k, k = 1, 2, . . . \text{ and } j = 1, j = D-3 \\ 3, \quad v = w_{ij}, \text{both i, j are odd and i, j are even.} \end{cases}$$

For v = v_i, i is odd, the neighboring vertices are colored as 3231 3231 3231 3231 323 1 or 3231 3231 3231 3231 3231 3231 3231 3231 3231 and so on. Here, the color sum of v under modulo 4 is 1. Hence (v) = 1. For v = v_i, i is even, the neighboring vertices are colored as 2323 1323 1323 1323 232 1 or 2323 1323 1323 1323 1323 1323 1323 1323 232 1 and so on. Here, the color sum of v under modulo 4 is 1. Hence (v) = 1. For v = x, all the neighboring vertices are colored as 0 where the color sum is 0. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 0. For v = w_{ij}, i is odd, j = 1, D-1, the neighboring vertices are colored as 202 and hence the color sum σ(v) = 0.For v = w_{ij}, i is even, j =2, D-4, D-2, the neighboring vertices are colored as 202 and hence the color sum σ(v) = 0.

For v = w_{ij}, both i and j are odd,≠j 1, D -1, the neighboring vertices are colored as 201 and hence the color sum σ(v) = 3.For v = w_{ij}, i, j are even, j ≠ 2, D-2, D-4, the neighboring vertices are colored as 201 and hence the color sum σ(v) = 3.For v = w_{ij}, i is odd and j is even the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 303. Hence (v) = 2. For v = w_{ij}, i is even and j is odd, the neighboring vertices are colored as 303 and hence the color sum σ(v) = 2.

$$\text{Therefore }(v) = \begin{cases} 0, \quad \text{for } v = x, v = w_{ij}, \text{ i and j are even}, j \neq D-2\,; \\ \qquad\qquad v = w_{iD-1} \text{ where i is odd}, \\ 1, \text{ for } v = v_i, \quad i = 1, 2, 3, \ldots. D, \\ 2, \text{for } w_{ij}, \quad \text{odd i and even j and vice versa}, \\ \qquad 3, \; v = w_{ij}, \text{ i and j are odd}, j \neq D-1; \\ \qquad\qquad v = w_{iD-2} \text{ where i is even.} \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$. Then pmc(G) ≤ 4 (5)

From (1) and (5),pmc(G) = 4.

Case-3:.$H_1(2, D)$ where D = 5, 13, 21, 29, ...

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Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 0, & \text{if } v = v_i, i = 1,2,3,\ldots, D, \\ 2, & \text{if } v = w_{ij}, i = 1,2,3,\ldots, D \text{ and j is even}, \\ 1, & \text{if } v = x, v = w_{ij}, i = 1,2,3,\ldots, D \text{ and j is odd}. \end{cases}$$

For v = v_i, the neighboring vertices are colored as 1212 1 or 1212 1212 1212 1 or 1212 1212 1212 1212 1212 1 and so on depending on D = 5, 13, 21, 29, . . . Here, the color sum of v under modulo 4 is 1. Hence (v) = 3. For v = w_{ij}, $i = 1,2,3,\ldots, D$ and j is odd, the neighboring vertices are colored as 022 where the color sum is 4. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 0. For v = x, the neighboring vertices are colored as 0 and hence the color sum (v) = 0. For v = w_{ij}, $i = 1,2,3,\ldots, D$ and j is even the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 110. Hence (v) = 2.

$$\text{Therefore } (v) = \begin{cases} 0, \text{for } v = x, v = w_{ij}, i = 1,2,3,..,D \text{ and j is odd}, \\ 3, \text{ for } v_i, \ \ i = 1,2,\ldots, D, \\ 2, \text{ for } w_{ij}, \ \ i = 1,2,\ldots, D \text{ and j is even}. \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$.Then pmc(G) ≤ 4 (6)

From (1) and (6), pmc(G) = 4.

Case-4: $H_1(2, D)$ where D = 6, 14, 22, 30, 38, ...

Subcase-4.1: D = 6, 22, 38, . . .

Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 3, \text{ if } v = v_i, i = 1,2,3,\ldots, D, \\ 2, \text{ if } v = w_{ij}, \text{i is odd and } j = 2 + 4k, k = 0,1,2,3\ldots,, \\ 1, \text{ if } v = x, v = w_{ij}, \text{i is odd and } j = 4k, k = 1,2,\ldots.,; \\ \qquad v = w_{ij}, \text{i is even and j is odd}, \\ 0, \text{ if } v = w_{ij}, \text{both i, j is odd and i, j is even} \end{cases}$$

For v = v_i, i is even, the neighboring vertices are colored as 1 0101 1 or 1 0101 0101 0101 0101 0101 1 or 1 0101 0101 0101 0101 0101 0101 0101 0101 0101 1 and so on depending on D is 6 or 22 or 38 or . . . ..Here, the color sum of v under modulo 4 is 0. Hence (v) = 0. For v = v_i, i is odd, the neighboring vertices are colored as 0 2010 1 or 0 2010 2010 2010 2010 2010 1 or 0 2010 2010 2010 2010 2010 2010 2010 2010 2010 1 and so on depending on D is 6 or 22 or 38 or . . . ..Here, the color sum of v under modulo 4 is 0. Hence (v) = 0. For v = x,all the neighboring vertices are colored as 3 and hence the color sum σ(v) = 2 which is taken under modulo 4.

For v = w_{ij}, i is odd and j is even and vice versa, the color sum of the neighboring vertices is 3 where the colors assigned to the vertices are 030. Hence (v) = 3. For v = w_{ij}, i, j is odd, j ≠ D-1, the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 123. Hence(v) = 2. For v = w_{ij}, i, j is even and i is odd and $j = D - 1$, the color sum of the neighboring vertices is 1 where the colors assigned to the vertices are 113. Hence (v) = 1.

$$\text{Therefore } (v) = \begin{cases} 0, \ \ \text{for } v = v_i, i = 1,2,3,..,D, \\ 2, \text{ for } v = x, v = w_{ij}, \text{i, j odd and } j \neq D - 1, \\ 1, \text{for } v = w_{iD-1}, \text{i is odd and } v = w_{ij}, \text{i, j are even}, \\ 3, \text{for } v = w_{ij}, \text{i is odd, j is even and vice versa}. \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$.Then pmc(G) ≤ 4 (7)

From (1) and (7), pmc(G) = 4.

Subcase-4.2: D = 14, 30, 46, ...

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Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 3, \text{ if } v = v_i, i = 1, 2, 3, \dots, D, \\ 2, \text{ if } v = w_{ij}, \text{i is odd}, j = 2 + 4k, k = 0, 1, 2, 3 \dots, ; v = w_{i1}, \text{i is even}, \\ 1, \text{ if } v = w_{ij}, \text{i is odd and } j = 4k, k = 1, 2, \dots, ; \\ \quad v = w_{ij}, \text{i is even and j is odd}, j \neq 1, \\ 0, \text{ if } v = x, v = w_{ij}, \text{both i, j is odd and i, j is even} \end{cases}$$

For v = v_i, i is even, the neighboring vertices are colored as 2 0101 0101 0101 0 or 2 0101 0101 0101 0101 0101 0101 0101 0 or 2 0101 0101 0101 0101 0101 0101 0101 0101 0101 0101 0101 0 and so on depending on D is 14 or 30 or 46 or . . . ..Here, the color sum of v under modulo 4 is 0. Hence (v) = 0. For v = v_i, i is odd, the neighboring vertices are colored as 0 2010 2010 2010 0 or 0 2010 2010 2010 2010 2010 2010 2010 0 or 0 2010 2010 2010 2010 2010 2010 2010 2010 2010 2010 2010 0 and so on depending on D is 14 or 30 or 46 or . . . ..Here, the color sum of v under modulo 4 is 1. Hence (v) = 1.

For v = x, all the neighboring vertices are colored as 3 and hence the color sum (v) = 2 which is taken under modulo 4.

For v = w_{ij}, i is odd and j is even and vice versa, the color sum of the neighboring vertices is 3 where the colors assigned to the vertices are 030. Hence (v) = 3. For v = w_{ij}, i, j is odd, the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 123. Hence (v) = 2. The same in the case of v= w_{i2} for i is even. For v = w_{ij}, i, j is even where j = 4+2k, k= 0, 1, 2, . . .the color sum of the neighboring vertices is 1 where the colors assigned to the vertices are 113. Hence (v) = 1.

$$\text{Therefore } (v) = \begin{cases} 0, \text{ for } v = v_i, \text{i is even}, \\ 2, \text{ for } v = x, v = w_{ij}, \text{i, j odd and } v = w_{i2} \text{ for i is even}, \\ 1, \text{ for } v = w_{ij}, \text{i, j are even where } j \neq 2, \\ 3, \text{ for } v = w_{ij}, \text{i is odd, j is even and vice versa.} \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$. Then pmc(G) $\leq$ 4 (8)

From (1) and (8), pmc(G) = 4.

Case-5:.$H_1(2, D)$ where D = 7, 15, 23, 31, . . .

Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by $c(v) = \begin{cases} 0, \text{ if } v = v_i, i = 1, 2, 3, \dots, D, \\ 1, \text{ if } v = w_{ij}, i = 1, 2, 3, \dots, D \text{ and j is odd}, \\ 2, \text{ if } v = x, v = w_{ij}, i = 1, 2, 3, \dots, D \text{ and j is even.} \end{cases}$

For v = v_i, the neighboring vertices are colored as 1212 12 2 or 1212 1212 1212 12 2 or 1212 1212 1212 1212 1212 12 2 and so on depending on D = 7, 15, 23, 31, . . . Here, the color sum of v under modulo 4 is 3. Hence (v) = 3. For v = w_{ij}, i = 1, 2, 3, …, D and j is odd, the neighboring vertices are colored as 022 where the color sum is 4. since coloring is defined on $\mathbb{Z}_4$,$\sigma(v)$ = 0. For v = x, the neighboring vertices are colored as 0 and hence the color sum (v) = 0. For v = w_{ij}, i = 1, 2, 3, …, D and j is even the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 110. Hence (v) = 2.

$$\text{Therefore } (v) = \begin{cases} 0, \text{ for } v = x, v = w_{ij}, i = 1, 2, 3, .., D \text{ and j is odd}, \\ 3, \text{ for } v = v_i, \ i = 1, 2, \dots, D, \\ 2, \text{ for } v = w_{ij}, \ i = 1, 2, \dots, D \text{ and j is even.} \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$. Then pmc(G) $\leq$ 4 (9)

From (1) and (9), pmc(G) = 4.

Case-6: $H_1(2, D)$ where D = 8, 16, 24, 32, . . .

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Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 0, & \text{if } v = v_i, \text{ i is odd; } v = w_{ij}, \text{ i, j are even,} \\ 2, & \text{if } v = w_{ij}, \text{ i is odd and j is even,} \\ 1, & \text{if } v = x, v = w_{ij}, \text{ i, j are odd,} \\ 3, & \text{if } v = w_{ij}, \text{ i is even and j is odd,} \end{cases}$$

For v = v_i, i is odd, the neighboring vertices are colored as 1212 121 1 or 1212 1212 1212 121 1 or 1212 1212 1212 1212 1212 121 1 and so on depending on D =8, 16, 24, 32, . . . Here, the color sum of v under modulo 4 is 3. Hence (v) = 3. For v = v_i, i is even, the neighboring vertices are colored as 3030 303 1 or 3030 3030 3030 303 1 or 3030 3030 3030 3030 3030 303 1 and so on depending on D =8, 16, 24, 32, . . . Here, the color sum of v under modulo 4 is 1. Hence (v) = 1.For v = w_{ij}, i is odd, j = 1, j = D-1, the neighboring vertices are colored as 023 where the color sum is 5. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 1.

For v = w_{ij}, i, j is odd,j ≠ 1, j ≠ D-1, the neighboring vertices are colored as 022 where the color sum is 4. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 0. For v = w_{ij}, i, j is even, the neighboring vertices are colored as 323 where the color sum is 8. since coloring is defined on $\mathbb{Z}_4$,σ(v) = 0. For v = x, the neighboring vertices are colored as 0202 0202 or 0202 0202 0202 0202 or 0202 0202 0202 0202 0202 0202 and hence the color sum (v) = 0. For v = w_{ij}, i is odd and j is even the color sum of the neighboring vertices is 2 where the colors assigned to the vertices are 110. Hence (v) = 2. For v = w_{ij}, i is even and j is odd the color sum of the neighboring vertices is 3 where the colors assigned to the vertices are 210. Hence (v) = 3.

$$\text{That is, } \sigma(v) = \begin{cases} 0, & \text{for } v = x, v = w_{ij}, \text{ i, j are odd, } j \neq 1, j \neq D-1; \\ & v = w_{ij}, \text{ i, j are even,} \\ 1, & \text{for } v = v_i, \text{ i is even, } v = w_{i1} = w_{iD-1} \text{ where i is odd,} \\ 3, & \text{for } v = v_i, \text{ i is odd, } v = w_{i1}, v = w_{i7}, \text{ i is even,} \\ 2, & \text{for } v = w_{ij}, \text{ i is odd and j is even;} \\ & v = w_{ij}, \text{ i is even and j is odd, } j \neq 1, 7. \end{cases}$$

Hence σ(x) ≠ σ(y) for all adjacent vertices x and y in $H_1(2, D)$. Then pmc(G) ≤ 4 (10)

From (1) and (10), pmc(G) = 4.

Case-7: $H_1(2, D)$ where D = 9, 17, 25, 33, . . .

$$\text{Consider the coloring c: V(G)} \to \mathbb{Z}_4 \text{ defined by } c(v) = \begin{cases} 0, & \text{if } v = v_i, i = 1, 2, 3, \dots, D, \\ 1, & \text{if } v = w_{ij}, i = 1, 2, 3, \dots, D \text{ and j is odd,} \\ 2, & \text{if } v = w_{ij}, i = 1, 2, 3, \dots, D \text{ and j is even,} \\ 3, & \text{if } v = x. \end{cases}$$

$$\text{Therefore } (v) = \begin{cases} 0, & \text{for } v = x, v = w_{ij}, i = 1, 2, 3, .., D \text{ and j is odd,} \\ 3, & \text{for } v = v_i, \; i = 1, 2, \dots, D, \\ 2, & \text{for } v = w_{ij}, \; i = 1, 2, \dots, D \text{ and j is even.} \end{cases}$$

Hence σ(x) ≠ σ(y) for all adjacent vertices x and y in $H_1(2, D)$.Then pmc(G) ≤ 4 (11)

From (1) and (11), pmc(G) = 4.

Case-8: $H_1(2, D)$ where D = 10, 18, 26, 34, 42, 50, . . .

Subcase-8.1: D = 10, 26, 42, . ..

Consider the coloring c: V(G) → $\mathbb{Z}_4$ defined by

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$$c(v) = \begin{cases} 1, & \text{if } v = v_i, i = 1, 2, 3, \ldots, D, \\ 2, & \text{if } v = w_{ij}, i \text{ is odd and } j = 1 + 4k, k = 0, 1, 2, 3 \ldots,, \\ 1, & \text{if } v = x, v = w_{ij}, i \text{ is odd and } j \text{ is even;} \\ & v = w_{ij}, i \text{ is even and } j \text{ is odd,} \\ 0, & \text{if } v = w_{ij}, \text{both } i, j \text{ is odd and } i, j \text{ is even.} \end{cases}$$

$$\text{Therefore } (v) = \begin{cases} 0, & \text{for } v = v_i, i = 1, 2, 3, \ldots, D, \\ 2, & \text{for } v = x, v = w_{ij}, i \text{ is odd and } j \text{ is even;} \\ & v = w_{i1}, \; v = w_{iD-1} \text{ for } i \text{ is even,} \\ 1, & \text{for } v = w_{ij}, \text{both } i, j \text{ are odd and } i, j \text{ are even,} \\ 3, & \text{for } v = w_{ij}, i \text{ is even}, j \text{ is odd }, j \neq 1, D-1. \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$. Then $pmc(G) \leq 4$ (12).

From (1) and (12), pmc(G) = 4.

Subcase-8.2: D = 18, 34, 50, ...

Consider the coloring c: $V(G) \rightarrow \mathbb{Z}_4$ defined by

$$c(v) = \begin{cases} 1, & \text{if } v = v_i, i = 1, 2, 3, \ldots, D, \\ 2, & \text{if } v = w_{ij}, i \text{ is odd}, j = 2 + 4k, k = 0, 1, 2, 3 \ldots, ; \\ & v = w_{i4}, w_{iD-4} \text{where } i \text{ is even, and } v = x, \\ 0, & \text{if } v = w_{ij}, i \text{ is odd and } j \text{ is even and vice versa,} \\ 3, & \text{if } v = w_{ij}, i \text{ is odd and } j = 3 + 4k, k = 0, 1, 2, \ldots, \\ & v = w_{ij}, \text{both } i \text{ and } j \text{ are even}, j \neq 4 \text{ and } j \neq D-4. \end{cases}$$

$$\text{Therefore } \sigma(v) = \begin{cases} 0, & \text{for } v = v_i, \\ 2, & \text{for } v = x, v = w_{ij}, i \text{ is odd}, j \text{ is even ;} \\ & v = w_{ij} \text{ for } i \text{ is even}, j = 1, 3, 5, D-1, D-3, D-5, \\ 1, & \text{for } v = w_{ij}, i, j \text{ are even and } ij \text{ are odd,} \\ 3, & \text{for } v = w_{ij}, i \text{ is even}, j \text{ is odd }, j \neq 1, 3, 5, D-1, D-3, D-5. \end{cases}$$

Hence $\sigma(x) \neq \sigma(y)$ for all adjacent vertices x and y in $H_1(2, D)$. Then $pmc(G) \leq 4$ (13)

From (1) and (13), pmc(G) = 4.

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