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GAUSSIAN PRIME LABELLING OF UNICYCLIC GRAPHS

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ABSTRACT

A graph G on n vertices is said to have a prime labelling if there exists a labelling from the vertices of G to the first n natural numbers such that any two adjacent vertices have relatively prime labels. Gaussian integers are the complex numbers whose real and imaginary parts are both integers. A Gaussian prime labelling on G is a bijection f: V (G) → [γ_n], the set of the first n Gaussian integers in the spiral ordering such that if uv ∈ E(G),then $\gamma(u)$ and $\gamma(v)$ are relatively prime. Using the order on the Gaussian integers, we discuss the Gaussian prime labelling of unicyclic graphs.

Keywords: *Gaussian integers, Gaussian prime labelling, unicyclic graphs.*

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INTRODUCTION

In this paper we discuss the Gaussian prime labelling of unicyclic graphs. In the first section we introduce Gaussian integers and their properties. Steven klee [7] extend the prime labelling to Gaussian prime labelling by using the first n Gaussian integers. We take the spiral ordering of the Gaussian integers defined by Hunter Lehmann and Andrew Park [6].The spiral ordering allows us to linearly order the Gaussian integers. In the second section we discuss Gaussian prime labelling of unicyclic graphs.

Graph labelling where the vertices are assigned values subject to certain conditions have many applications in Engineering and Science [1]. For all terminology and notations in Graph theory, we follow [2] and for all terminology regarding graph labelling, we follow [3].The Prime labelling concept was introduced by Roger Entriger. A graph on n vertices is said to have a prime labelling if its vertices can be labelled with the first n natural numbers in such a way that any two adjacent vertices have relatively prime labels. In [6] Hunter Lehmann and Andrew Park extend the notion of prime labelling to Gaussian integers. They define a spiral ordering on the Gaussian integers that allow us to linearly order the Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime. Steven Klee [7] proved that the path graph, star graph, spider graph, n-centipede tree, double star tree and firecracker admits Gaussian prime labelling.

All graphs in this paper are finite undirected graphs without loops or multiple edges. We follow [6] for definition and information on the Gaussian integers.

1. GAUSSIAN INTEGERS

The Gaussian integers, denoted Z[i], are the complex numbers of the form a+bi, where a, b ϵZ and $i^2 = -1$. The norm of a Gaussian integer a+bi, denoted by N(a+bi), is given by a^2+b^2. A Gaussian integer is even if it is divisible by 1+i and odd otherwise. A unit in the Gaussian integers is one of ±1, ±i. An associate of a Gaussian integer α is u.α where u is a Gaussian unit. A Gaussia n integer ρ is prime if its only divisors are ±1, ±i, ±ρ or ±ρi. The Gaussian integers α and β are relatively prime if their only common divisors are units in Z[i].

The Gaussian integers are not totally ordered. So to define the first n Gaussian integers, we use the spiral ordering of the Gaussian integers introduced in [6].

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Definition 1 ([6]): The spiral ordering of the Gaussian integers is a recursively defined ordering of the Gaussian integers. We denote the n[th] Gaussian integer in the spiral ordering by γ_n. The ordering is defined beginning with $\gamma_1=1$ and continuing as:

$$\gamma_{n+1} = \begin{cases} \gamma_n + i, & if\ Re(\gamma_n) \equiv 1(mod2), Re(\gamma_n) > Im(\gamma_n) + 1 \\ \gamma_n - i, & if\ Im(\gamma_n) \equiv 0(mod2), Re(\gamma_n) \leq Im(\gamma_n) + 1, Re(\gamma_n) > 1 \\ \gamma_n + 1, & if\ Im(\gamma_n) \equiv 1(mod2), Re(\gamma_n) < Im(\gamma_n) + 1 \\ \gamma_n + i, & if\ Im(\gamma_n) \equiv 0(mod2), Re(\gamma_n) = 1 \\ \gamma_n = i, & if\ Re(\gamma_n) \equiv 0(mod2), Re(\gamma_n) \geq Im(\gamma_n) + 1, Im(\gamma_n) > 0 \\ \gamma_n + 1, & if\ Re(\gamma_n) \equiv 0(mod2), Im(\gamma_n) = 0. \end{cases}$$

The first 10 Gaussian integers under this ordering are 1,1+i,2+i,2,3,3+i,3+2i,2+2i,1+2i,1+3i,…and $[\gamma_n]$ denote the set of the first n Gaussian integers in the spiral ordering. Here we exclude the imaginary axis to ensure that the spiral ordering excludes associates. Consecutive Gaussian integers in the spiral ordering are separated by a unit and therefore alternate parity, as in the usual ordering of N. Furthermore, odd integers with indices separated by a power of two are not guaranteed to be relatively prime to each other.

In [6] Hunter Lehmann and Andrew Park proved the following properties of Gaussian integers in spiral ordering.

(1) Let α be a Gaussian integer and u be a unit. Then α and α+u are relatively prime.
(2) Consecutive Gaussian integers in the spiral ordering are relatively prime.
(3) Let α be an odd Gaussian integer, let c be a positive integer, and let u be a unit. Then α and $\alpha + u.(1+i)^c$ are relatively prime.
(4) Consecutive odd Gaussian integers in the spiral ordering are relatively prime.
(5) Let α be a Gaussian integer and let p be a prime Gaussian integer. Then α and α + p are relatively prime if and only if p does not divides α.

Definition 2([7]): Let G be a graph on n vertices. A Gaussian prime labelling of G is a bijection $f : V(G) \to [\gamma_n]$, the set of the first n Gaussian integers in the spiral ordering such that if uv ∈ E(G), then γ(u) and γ(v) are relatively prime..

2. GAUSSIAN PRIME LABELLING OF UNICYCLIC GRAPHS

A graph is called an unicyclic graph if it has a unique sub graph isomorphic to a cycle. The vertex lying on the cycle of a unicyclic graph is called a cycle vertex. A pendant is a path on two vertices with exactly one vertex being a cycle vertex. The non-cycle vertex of a pendant is called a pendant vertex. For all m, n ∈ N with n≥ 3, the graph $C_n \otimes K_{1,m}$ is the graph obtained by attaching m pendants to each cycle vertex of C_n. Attach the copy of the star $K_{1,m}$ at its vertex of degree m to each cycle vertex of C_n.

Theorem 1: All $C_n \otimes K_{1,3}$ are Gaussian prime for n∈N.

Proof: Let G be the graph $C_n \otimes K_{1,3}$ and there are 4n vertices in G. Let $c_0, c_1, c_2, \ldots c_{n-1}$ denote the cycle vertices and let the pendant vertices adjacent to c_i is denoted by v_i^j, $1 \leq j \leq 3$.

Define the labelling $f : V(G) \to \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{4n}\}$ as

$$f(c_0) = \gamma_1$$

$$f(c_i) = \begin{cases} \gamma_{4i-1}, & 1 \leq i \leq 7 \\ \gamma_{4i}, & i \geq 8\ and\ i\ even \\ \gamma_{4i-1}, & i \geq 9\ and\ i\ odd \end{cases}$$

For $1 \leq i \leq 7$

$$f(v_i^j) = \begin{cases} \gamma_{4i-2}, & j = 1 \\ \gamma_{4i}, & j = 2 \\ \gamma_{4i+1}, & j = 3 \end{cases}$$

For i ≥ 8 and i even

$$f(v_i^j) = \begin{cases} \gamma_{4i-2}, & j = 1 \\ \gamma_{4i-1}, & j = 2 \\ \gamma_{4i+1}, & j = 3 \end{cases}$$

For i≥ 9 and i odd

$$f(v_i^j) = \begin{cases} \gamma_{4i-2}, & j = 1 \\ \gamma_{4i}, & j = 2 \\ \gamma_{4i+1}, & j = 3 \end{cases}$$

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For i =0

$$f(v_i^j) = \gamma_{4n-3+j}, 1 \leq i \leq 3$$

It is to be seen that all adjacent cycle vertices are assigned Gaussian prime labels.

gcd(f(c_i), f(c_{i+1})) = 1 ,since consecutive Gaussian integers are prime. Also gcd (f(c_1), f(c_n)) = 1.

Now, we will show the Gaussian prime labelling for the remaining pendant vertices for each value of j.

For 1 ≤ i ≤ 7,

If j=1, then gcd (f(c_i), f(v_i^1)) = gcd ($\gamma_{4i-1}, \gamma_{4i-2}$) = 1.
If j=2, then gcd (f(c_i), f(v_i^2)) = gcd ($\gamma_{4i-1}, \gamma_{4i}$) = 1.
If j=3, then gcd (f(c_i), f(v_i^3)) = gcd ($\gamma_{4i-1}, \gamma_{4i+1}$) = 1.

For i ≥ 8 and i even

If j=1, then gcd (f(c_i), f(v_i^1)) = gcd ($\gamma_{4i}, \gamma_{4i-2}$) = 1.
If j=2, then gcd (f(c_i), f(v_i^2)) = gcd ($\gamma_{4i}, \gamma_{4i-1}$) = 1.
If j=3, then gcd (f(c_i), f(v_i^3)) = gcd ($\gamma_{4i}, \gamma_{4i+1}$) = 1.

For i ≥ 9 and i even

If j=1, then gcd (f(c_i), f(v_i^1)) = gcd ($\gamma_{4i-1}, \gamma_{4i-2}$) = 1.
If j=2, then gcd (f(c_i), f(v_i^2)) = gcd ($\gamma_{4i-1}, \gamma_{4i}$) = 1.
If j=3, then gcd (f(c_i), f(v_i^3)) = gcd ($\gamma_{4i-1}, \gamma_{4i+1}$) = 1.

For i = 0 and 1 ≤ j ≤ 3

gcd (f(c_i), f(v_i^j)) = gcd ($\gamma_1, \gamma_{4n-3+j}$) = 1.

Therefore all $C_n \otimes K_{1,3}$ are Gaussian prime.

Theorem 2: All $C_n \otimes K_{1,5}$ are Gaussian prime.

Proof: Let G be the graph $C_n \otimes K_{1,5}$ and there are 6n vertices in G. Let $c_1, c_2, ..., c_n$ denote the cycle vertices and let the pendant vertices adjacent to c_i is denoted by v_i^j , 1 ≤ j ≤ 5.

Define the labelling f : V (G) → {$\gamma_1, \gamma_2, \gamma_3$, ..., γ_{6n}} as

$$f(c_1) = \gamma_1$$

$$f(c_i) = \begin{cases} \gamma_{6(i-1)+4}, & i = 2,4,6, \dots \\ \gamma_{6(i-1)+3}, & i = 3,5,7, \dots \end{cases}$$

$$f(v_i^j) = \begin{cases} \gamma_{j+1}, & i = 1, 1 \leq j \leq 5 \\ \gamma_{6(i-1)+j}, & i = 2,4,6, \dots, 1 \leq j \leq 6, j \neq 4 \\ \gamma_{6(i-1)+j}, & i = 3,5,7, \dots, 1 \leq j \leq 6, j \neq 3 \end{cases}$$

For i = 1 and 1 ≤ j ≤ 5

It is clear that gcd (f(c_i), f(v_i^j)) = gcd (γ_1, γ_{j+1}) = 1.

For i =2, 4, 6,...

gcd (f(c_i), f(v_i^j)) = gcd ($\gamma_{6(i-1)+4}, \gamma_{6(i-1)+j}$).

If j=1, then gcd (f(c_i), f(v_i^1)) = gcd ($\gamma_{6(i-1)+4}, \gamma_{6(i-1)+1}$) .

Let δ = $\gamma_{6(i-1)+4} - \gamma_{6(i-1)+1}$, then δ are 2+i,1+2i,3i,3 and their associates. They are Gaussian prime numbers.
So, using property 5 of Gaussian integers $\gamma_{6(i-1)+4}$ *and* $\gamma_{6(i-1)+1}$ are relatively prime. Therefore (f(c_i), f(v_i^1)) = 1.

If j=2, then gcd (f(c_i), f(v_i^2)) = gcd ($\gamma_{6(i-1)+4}, \gamma_{6(i-1)+2}$) = 1, since consecutive even Gaussian integers in the spiral ordering are relatively prime.

If j=3, then gcd (f(c_i), f(v_i^3)) = gcd ($\gamma_{6(i-1)+4}, \gamma_{6(i-1)+3}$) = 1, since consecutive Gaussian integers in the spiral ordering are relatively prime.

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If j=5, then gcd ($f(c_i)$, $f(v_i^5)$) = gcd ($\gamma_{6(i-1)+4}, \gamma_{6(i-1)+5}$) = 1, since consecutive Gaussian integers in the spiral ordering are relatively prime.

If j=6, then gcd ($f(c_i)$, $f(v_i^6)$) = gcd ($\gamma_{6(i-1)+4}, \gamma_{6(i-1)+6}$) = 1, since consecutive even Gaussian integers in the spiral ordering are relatively prime.

For i =3, 5, 7,...

gcd ($f(c_i)$, $f(v_i^j)$) = gcd ($\gamma_{6(i-1)+3}, \gamma_{6(i-1)+j}$)

If j=1, then gcd ($f(c_i)$, $f(v_i^1)$) = gcd ($\gamma_{6(i-1)+3}, \gamma_{6(i-1)+1}$) = 1, since consecutive odd Gaussian integers in the spiral ordering are relatively prime.

If j=2, then gcd ($f(c_i)$, $f(v_i^2)$) = gcd ($\gamma_{6(i-1)+3}, \gamma_{6(i-1)+2}$) = 1, since consecutive Gaussian integers in the spiral ordering are relatively prime.

If j=4, then gcd ($f(c_i)$, $f(v_i^4)$) = gcd ($\gamma_{6(i-1)+3}, \gamma_{6(i-1)+4}$) = 1, since consecutive Gaussian integers in the spiral ordering are relatively prime.

If j=5, then gcd ($f(c_i)$, $f(v_i^5)$) = gcd ($\gamma_{6(i-1)+3}, \gamma_{6(i-1)+5}$) = 1, since consecutive odd Gaussian integers in the spiral ordering are relatively prime.

If j=6, then gcd ($f(c_i)$, $f(v_i^6)$) = gcd ($\gamma_{6(i-1)+3}, \gamma_{6(i-1)+6}$)

Let $\delta = \gamma_{6(i-1)+6}$ - $\gamma_{6(i-1)+3}$, then δ are 1,3 and their associates. They are Gaussian prime numbers. So, using property 5, $\gamma_{6(i-1)+3}$, *and* $\gamma_{6(i-1)+6}$ are relatively prime. Therefore ($f(c_i)$, $f(v_i^6)$) = 1.

Also, all adjacent cyclic vertices are assigned Gaussian prime labelling.

gcd ($f(c_1)$, $f(c_2)$) = gcd (γ_1, γ_{10}) = 1
gcd ($f(c_1)$, $f(c_n)$) = gcd ($\gamma_1, \gamma_{6(n-1)+4}$) = 1,if n is even and
gcd ($f(c_1)$, $f(c_n)$) = gcd ($\gamma_1, \gamma_{6(n-1)+3}$) = 1,if n is odd.
gcd ($f(c_i)$, $f(c_{i+1})$) = gcd ($\gamma_{6(i-1)+4}, \gamma_{6i+3}$) = gcd ($\gamma_{6i-2}, \gamma_{6i+3}$) = 1.

The difference of the labels γ_{6i-2} and γ_{6i+3} are Gaussian prime numbers, using property 5 they are relatively prime numbers. All $C_n \otimes K_{1,5}$ are Gaussian prime.

We prove the Gaussian prime labelling of $C_n \otimes K_{1,3}$ and $C_n \otimes K_{1,5}$. Now we form a conjecture to the Gaussian prime labelling of star graph attached to cycle.

The conjecture is

Conjecture 1: All $C_n \otimes K_{1,m}$ are Gaussian prime for all n∊N and odd m∊N.

Now we consider the Gaussian prime labelling of one level and two level complete ternary trees attached to crown graph $C_n \otimes K_1$. We denote $C_n \otimes P_2 \otimes K_{1,3}$ to be the graph that result by attaching one level complete ternary tree to each pendant vertex of $C_n \otimes K_1$ .We denote $C_n \otimes P_2 \otimes K_{1,3} \otimes K_{1,3}$ to be the graph that result by attaching two level complete ternary tree to each pendant vertex of $C_n \otimes K_1$.

Definition 3: A complete ternary tree is a directed rooted tree with every internal vertex having three children.

An n–cycle pendant with one level ternary tree is obtained by attaching a single pendant to each cycle vertex of C_n followed by attaching a one level complete ternary tree to each pendant vertex and is denoted by $C_n \otimes P_2 \otimes K_{1,3}$.

Theorem 3: All $C_n \otimes P_2 \otimes K_{1,3}$ are Gaussian prime for all n∊N.

Proof: Let G be the graph $C_n \otimes P_2 \otimes K_{1,3}$ and there are 5n vertices in G. Let $c_1, c_2, ..., c_n$ denote the cycle vertices, let $v_1, v_2, ..., v_n$ denote the pendant vertices adjacent to c_i and let the non cyclic vertices adjacent to v_i is denoted by v_i^j, $1 \leq j \leq 3$.

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Define the labelling f : V (G) → $\{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_{5n}\}$ as

$$f(c_i) = \gamma_{5i-4,} 1 \le i \le n$$
$$f(v_i) = \gamma_{5i-2,} 1 \le i \le n$$
$$f(v_i^j) = \begin{cases} \gamma_{5i-2+j}, & j \ne 3 \\ \gamma_{5i-3}, & j = 3 \end{cases}$$

gcd (f(c_i), f(c_{i+1})) = gcd ($\gamma_{5i-4}, \gamma_{5i+1}$)

The difference of the labels γ_{5i+1} and γ_{5i-4} are Gaussian prime numbers, using property 5 they are relatively prime numbers. Consecutive Gaussian integers in the spiral ordering are relatively prime and consecutive odd Gaussian integers in the spiral ordering are relatively prime. The labeling shows that all $C_n \otimes P_2 \otimes K_{1,3}$ are Gaussian prime.

Theorem 4: All $C_n \otimes P_2 \otimes K_{1,3} \otimes K_{1,3}$ are Gaussian prime for all n∊N.

Proof: Let G be the graph $C_n \otimes P_2 \otimes K_{1,3} \otimes K_{1,3}$ and there are 14n vertices in G. Let $c_1, c_2, ..., c_n$ denote the cycle vertices, let $v_1, v_2, ..., v_n$ denote the pendant vertex adjacent to c_i , let the non cyclic vertices adjacent to v_i is denoted by v_i^j , $1 \le j \le 3$ and let the remaining vertices adjacent to v_i^j be denoted by v_i^j (k) for $1 \le k \le 3$.

Define the labelling f : V (G) → $\{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_{14n}\}$ as

$$f(c_1) = \gamma_{1,}$$
$$f(c_i) = \gamma_{14i-13,} \text{ for i even}$$
$$f(c_i) = \gamma_{14i-12,} \text{ for i odd}$$
$$f(v_i) = \begin{cases} \gamma_{14i-12}, & i = 1 \text{ and } i \text{ even} \\ \gamma_{14i-10}, & i \text{ odd} \end{cases}$$

For i =1 and i even,

$$f(v_i^j) = \begin{cases} \gamma_{14i-9}, & j = 1 \\ \gamma_{14i-5}, & j = 2 \\ \gamma_{14i-3}, & j = 3 \end{cases}$$

$$f\left(v_i^j(k)\right) = \begin{cases} \gamma_{14i-11}, & j = 1, k = 1 \\ \gamma_{14i-10}, & j = 1, k = 2 \\ \gamma_{14i-8}, & j = 1, k = 3 \\ \gamma_{14i-7}, & j = 2, k = 1 \\ \gamma_{14i-6}, & j = 2, k = 2 \\ \gamma_{14i-4}, & j=2, k=3 \\ \gamma_{14i-2}, & j=3, k=1 \\ \gamma_{14i-1}, & j=3, k=2 \\ \gamma_{14i}, & j=3, k=3 \end{cases}$$

For i odd,

$$f(v_i^j) = \begin{cases} \gamma_{14i-11}, & j = 1 \\ \gamma_{14i-7}, & j = 2 \\ \gamma_{14i-1}, & j = 3 \end{cases}$$

$$f\left(v_i^j(k)\right) = \begin{cases} \gamma_{14i-13}, & j = 1, k = 1 \\ \gamma_{14i-9}, & j = 1, k = 2 \\ \gamma_{14i-8}, & j = 1, k = 3 \\ \gamma_{14i-6}, & j = 2, k = 1 \\ \gamma_{14i-5}, & j = 2, k = 2 \\ \gamma_{14i-4}, & j=2, k=3 \\ \gamma_{14i-3}, & j=3, k=1 \\ \gamma_{14i-2}, & j=3, k=2 \\ \gamma_{14i}, & j=3, k=3 \end{cases}$$

The labelling shows $C_n \otimes P_2 \otimes K_{1,3} \otimes K_{1,3}$ is Gaussian prime.

Now we propose the conjecture

Conjecture 2: All unicyclic graphs are Gaussian prime.

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