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COMMON FIXED POINT THEOREM IN MENGER SPACES UNDER EXPANSIVE MAPPING

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ABSTRACT

In the present paper we introduced a new concept of on common fixed point theorem in menger spaces under expansive mapping.

Key words: *fixed point, common fixed point, menger space, expansive mapping.*

1.1. INTRODUCTION

There have been lots of generalizations of metric space. One such generalization is Menger space in which, used distribution functions instead of nonnegative real numbers as value of metric.

A Menger space is a space in which the concept of distance is considered to be a probabilistic, rather than deterministic. For detail discussion of Menger spaces and their applications we refer to Schweizer and Sklar [91]. The theory of Menger space is fundamental importance in probabilistic functional analysis.

A probabilistic metric space shortly PM-Space, is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ to L, where L is the collection of all distribution functions (a distribution function F is non decreasing and left continuous mapping of reals in to $[0,1]$ with properties, $\inf F(x) = 0$ and $\sup F(x) = 1$).

The value of F at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ are assumed satisfy the following conditions;

1.1(a) $F_{u,v}(x) = 1$, for all $x > 0$, iff $u = v$;
1.1 (b) $F_{u,v}(0) = 0$, if $x = 0$;
1.1 (c) $F_{u,v}(x) = F_{v,u}(x)$;
1.1 (d) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$.

A mapping $t: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm, if it satisfies the following conditions;

1.1 (e) $t(a, 1) = a$ for every $a \in [0,1]$;
1.1 (f) $t(0, 0) = 0$,
1.1 (g) $t(a, b) = t(b, a)$ for every $a, b \in [0,1]$;
1.1 (h) $t(c, d) \geq t(a, b)$ for $c \geq a$ and $d \geq b$
1.1 (i) $t(t(a, b), c) = t(a, t(b, c))$ where $a, b, c, d \in [0,1]$.

A Menger space is a triplet (X, F, t), where (X, F) is a PM-Space, X is a non-empty set and a t-norm satisfying instead of 1.1(i) a stronger requirement.

1.1 (j) $F_{u,w}(x + y) \geq t\left(F_{u,v}(x), F_{v,w}(y)\right)$ for all $x \geq 0, y \geq 0$.

For a given metric space (X, d) with usual metric d, one can put $F_{u,v}(x) = H(x - d(u, v))$ for all $x, y \in X$ and $t > 0$. where H is defined as;

$$H(x) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$

and t-norm is defined as $t(a, b) = \min\{a, b\}$.

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Definition 1.1: A probabilistic metric space (PM- space) is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ into the collections of all distribution $F \in R$. For $x, y \in X$ we denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at u in R.

Definition 1.2: Self maps A and B of a Menger space $(X, F, *)$ are said

1. to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.
2. to be compatible if $F_{ABx_m, BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to x$, $Bx_n \to x$ for some x in X as $n \to \infty$.

Definition 1.3: Let (X, F, t) be a Menger space. If $x \in X, \varepsilon > 0$ and $\lambda \in (0, 1)$, then $(\varepsilon, \lambda) -$ neighborhood of x is called $U_x(\varepsilon, \lambda)$, is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > (1 - \lambda)\}.$$

an $(\varepsilon, \lambda) -$ topology in X is the topology induced by the family

$$\{U_x(\varepsilon, \lambda) : x \in X\ \varepsilon > 0\ and\ \lambda \in (0, 1)\}$$

of neighborhood.

Remark 1: If t is continuous, then Menger space (X, F, t) is a Housdorff space in $(\varepsilon, \lambda) -$ topology.

Let (X, F, t) be a complete Menger space and $A \subset X$. Then A is called a bounded set if

$$\lim_{u \to \infty} \inf_{x,y \in A} F_{x,y}(u) = 1.$$

Definition 1.4: A sequence $\{x_n\}$ in (X, F, t) is said

1. to be convergent to a point x in X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$such that $x_n \in U_x(\varepsilon, \lambda)$ for all $n \geq N$ or equivalently $F(x_n, x; \varepsilon) > 1 - \lambda$ for all $n \geq N$.
2. to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0, \exists$ an integer $N = N(\varepsilon, \lambda)$ such that $F(x_n, x_m, \varepsilon) > 1 - \lambda$ for all $n, m \geq N$.

Note: A Menger space (X, F, t) with

1. the continuous $t -$norm is said to be complete if every cauchy sequence
2. two mappings $f, g : X \to X$ are said to be weakly compatible if they commute at coincidence point.

Lemma 1.5: Let X be a set f, g OWC self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

Lemma 1.6: Let $\{x_n\}$ be a sequence in a Menger space (X, F, t), where t is continuous and $t(p, p) \geq p$ for all $p \in (0,1)$ and $n \in N$

$F(x_n, x_{n+1}, kp) \geq F(x_{n-1}, x_n, p)$, then $\{x_n\}$ is Cauchy sequence.

Lemma 1.7: If (X, d) is a metric space, then the metric d induces a mapping $F: X \times X \to L$ defined by

$$F(p, q) = H(x - d(p, q)), \qquad p, q \in R.$$

Further if $t: [0,1] \times [0,1] \to [0,1]$ is defined by $t(a, b) = \min\{a, b\}$, then (X, F, t) is a Menger space. It is complete if (X, d) is complete.

Lemma 2.1: Let $(X, M, *)$ be a complete Menger space.

1. Then for all x, y in X, $M(x, y, .)$ is non-decreasing.
2. If there exists $q \in (0,1)$ such that
$$M(x, y, qt) \geq M\left(x, y, \frac{t}{q^n}\right)$$
for positive integer n. Taking limit as $n \to \infty$, $M(x, y, t) \geq 1$ and hence $x = y$.
3. Menger space and let A and S be continuous mappings of X then A and S are compatible if and only if they are compatible of type (P).
4. Menger space and let A and S be compatible mappings of type (P) and Az=Sz for some $z \in X$, then
$$AAz = ASz = SAz = SSz.$$

COMMON FIXED POINT THEOREMS FOR EXPANSIVE MAPPING

In this section we prove some common fixed point theorems for expansive mapping in Menger spaces. For the proof of our next theorem we need following results

2. MAIN THEOREM

Theorem 2.1: Let $(X, M,*)$ be a complete Menger space and let A, B, S and T be self-mappings of X satisfying the following conditions:

2.1(a) $A(X) \subset T(X), B(X) \subset S(X)$;

2.1(b) S and T are continuous,

2.1(c) the pair $\{A, S\}$ and $\{B, T\}$ are expansive mappings of type (P) on X. 2.1(d) there exists $q > 1$ such that for every $x, y \in X$ and $t > 0$,

$$M_{(Ax,By)}(qt) \leq M_{(Sx,Ty)}(t) * M_{(Ax,Sx)}(t) \quad 2.1(i)$$

Then A, B, S and T have a unique common fixed point in X.

Proof: Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, for any $x_0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this $x_1 \in X$, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we define a sequence $\{y_n\}$ in X such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-1}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$, for all $n = 0,1,2, \ldots \ldots$

From2.1(d),

$$\begin{aligned} M_{(y_{2n+1},y_{2n+2})}(qt) &= M_{(Ax_{2n},Bx_{2n+1})}(qt). \\ &\leq M_{(Sx_{2n},Tx_{2n+1})}(t) * M_{(Ax_{2n},Sx_{2n})}(t) \\ &= M_{(y_{2n},y_{2n+1})}(t) * M_{(y_{2n+1},y_{2n})}(t) \\ &\leq M_{(y_{2n},y_{2n+1})}(t) * M_{(y_{2n+1},y_{2n+2})}(t) \end{aligned}$$

From lemma 1.4(a) and 1.4(c), we have

$$M_{(y_{2n+1},y_{2n+2})}(qt) \leq M_{(y_{2n},y_{2n+1})}(t) \quad 2.1(ii)$$

Similarly, we have

$$M_{(y_{2n+2},y_{2n+3})}(qt) \leq M_{(y_{2n+1},y_{2n+2})}(t) \quad 2.1(iii)$$

From 1.1(1) and 1.1(2), we have

$$M_{(y_{n+1},y_{n+2})}(qt) \leq M_{(y_n,y_{n+1})}(t) \quad 2.1(iv)$$

From 2.1(1), we have

$$\begin{aligned} M_{(y_n,y_{n+1})}(t) &\leq M_{(y_n,y_{n-1})}\left(\frac{t}{q}\right) \leq M_{(y_{n-2},y_{n-1})}\left(\frac{t}{q^2}\right) \\ &\leq \cdots \leq M_{(y_1,y_2)}\left(\frac{t}{q^n}\right) \to 1 \text{ as } n \to \infty. \end{aligned}$$

So, $M_{(y_n,y_{n+1})}(t) \to 1$ as $n \to \infty$ for any $t > 0$.

for each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$M_{(y_n,y_{n+1})}(t) < 1 - \varepsilon \text{ for all } n > n_0.$$

for $m, n \in \mathbb{N}$ we suppose $m \geq n$. Then we have that

$$\begin{aligned} M_{(y_n,y_m)}(t) &\leq M_{(y_n,y_{n+1})}\left(\frac{t}{m-n}\right) * M_{(y_{n+1},y_{n+2},)}\left(\frac{t}{m-n}\right) \\ &* \ldots \ldots * M_{(y_{m-1},y_m,)}\left(\frac{t}{m-n}\right) \\ &\leq (1-\varepsilon) * (1-\varepsilon) * \ldots \ldots (m-n) \text{ times.} \\ &\leq (1-\varepsilon) \end{aligned}$$

and hence $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, M,*)$ is complete, $\{y_n\}$ converges to some point $z \in X$, and so $\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z.

From Lemma 1.4(iii) and Lemma1.4(iv), we have

$$AAx_{2n-2} \to Sz \text{ and } SSx_{2n} \to Az \quad 2.1(v)$$

$$BBx_{2n-1} \to Tz \text{ and } TTx_{2n-1} \to Bz \quad 2.1(vi)$$

From 2.1 (d), we get

$$\begin{aligned} M_{(AAx_{2n-2},BBx_{2n-1})}(qt) \leq\ & M_{(SAx_{2n-2},TBx_{2n-1})}(t) \\ & * M_{(AAx_{2n-2},SAx_{2n-2})}(t) \\ & * M_{(BBx_{2n-1},TBx_{2n-1})}(t) \\ & * M_{(AAx_{2n-2},TBx_{2n-1})}(t) \end{aligned}$$

Taking limit as $n \to \infty$ and using 2.1(v) and 2.1(vi), we have

$$\begin{aligned} M_{(Sz,Tz)}(qt) &\le M_{(Sz,Tz)}(t) * M_{(Sz,Sz)}(t) \\ &* M_{(Tz,Tz)}(t) * M_{(Sz,Tz)}(t) \\ &\le M_{(Sz,Tz)}(t) * 1 * 1 * M_{(Sz,Tz)}(t) \\ &\le M_{(Sz,Tz)}(t). \end{aligned} \qquad 2.1(vii)$$

It follows that $Sz = Tz$.

Now, from 2.1 (d),

$$\begin{aligned} M_{(Az,BTx_{2n-1})}(qt) &\le M_{(Sz,TTx_{2n-1})}(t) * M_{(Az,Sz)}(t) \\ &* M_{(BTx_{2n-1},TTx_{2n-1})}(t) * M_{(Az,TTx_{2n-1})}(t) \end{aligned}$$

Again taking limit as $n \to \infty$ and using 2.1 (vi) and 2.1(vii), we have

$$\begin{aligned} M_{(Az,Tz)}(qt) &\le M_{(Sz,Sz)}(t) * M_{(Az,Tz)}(t) \\ &* M_{(Tz,Tz)}(t) * M_{(Az,Tz)}(t) \\ &\le M_{(Az,Tz)}(t). \end{aligned} \qquad 2.1 (viii)$$

and hence $Az = Tz$.

From 2.1 (d), 2.1 (vii) and 2.1 (viii),

$$\begin{aligned} M_{(Az,Bz)}(qt) &\le M_{(Sz,Tz)}(t) * M_{(Az,Sz)}(t) \\ &* M_{(Bz,Tz)}(t) * M_{(Az,Tz)}(t) \\ &= M_{(Az,Az)}(t) * M_{(Az,Az)}(t) \\ &* M_{(Bz,Az)}(t) * M_{(Az,Az)}(t) \\ &\le M_{(Az,Bz)}(t). \end{aligned} \qquad 2.1 (ix)$$

and hence $Az = Bz$.

From 2.1 (vii), 2.1 (viii) and 2.1 (ix), we have

$$Az = Bz = Tz = Sz. \qquad 2.1 (x)$$

Now, we show that $Bz = z$.

From 2.1 (d),

$$\begin{aligned} M_{(Ax_{2n},Bz)}(qt) &\le M_{(Sx_{2n},Tz)}(t) * M_{(Ax_{2n},Sx_{2n})}(t) \\ &* M_{(Bz,Tz)}(t) * M_{(Ax_{2n},Tz)}(t) \end{aligned}$$

and taking limit as $n \to \infty$ and using 2.1 (vii) and 2.1 (viii), we have

$$\begin{aligned} M_{(z,Bz)}(qt) &\le M_{(z,Tz)}(t) * M_{(z,z)}(t) \\ &* M_{(Bz,Tz)}(t) * M_{(z,Tz)}(t) \\ &= M_{(z,Bz)}(qt) * 1 * M_{(Az,Az)}(t) * M_{(z,Bz)}(t) \\ &\le M_{(z,Bz)}(t). \end{aligned}$$

and hence $Bz = z$. Thus from 2.1 (x), $z = Az = Bz = Tz = Sz$ and z is a common fixed point of A, B, S and T.

In order to prove the uniqueness of fixed point, let w be another common fixed point of A, B, S and T. Then

$$\begin{aligned} M_{(z,w)}(qt) &= M_{(Az,Bw)}(qt) \\ &\le M_{(Sz,Tw)}(t) * M_{(Az,Sz)}(t) \\ &* M_{(Bw,Tw)}(t) * M_{(Az,Tw)}(t) \\ &\le M_{(z,w)}(t). \end{aligned}$$

From Lemma 1.4 (ii), $z = w$. This completes the proof of theorem.

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