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# FIXED POINT THEOREMS FOR WEAK C-CONTRACTIONS AND WEAKLY COMPATIBLE MAPPINGS IN 2-METRIC SPACE

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***ABSTRACT***

*The purpose of this paper is to study a common fixed point theorem on 2-metric space using weak C-contraction and weakly compatibility. We mainly generalize the result of Dung and Hang [6], which is unifies and generalizes many results in the literature.*

***Mathematics Subject Classification:*** *47H10, 54H25.*

***Key Wards:*** *2-metric space, weak C contraction, weak compatible maps, Fixed point.*

## INTRODUCTION

The concept of 2-metric space is a natural generalization of a metric space. It has been investigated initially by Gahler [7]. Then many researchers like Iseki [8], Rhoades [13], Simoniya [14] etc. prove many fixed points in this space. Gahler [7] introduce 2-metric space as

Let X be a non-empty set and let d: $X\times X\times X \to [0,\infty)$ be such that

(i) For every pair of distinct point x, y in X with $x \neq y$ there exists a point z in X such that $d(x, y, z) \neq 0$.
(ii) $d(x, y, z) = 0$ when at least two of the three points are equal.
(iii) For any x, y, z in X, $d(x, y, z) = d(x, z, y) = d(y, z, x)$.
(iv) For any x, y, z, w in X , $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$,
Then d is called a 2-metric [4] and (X, d) is called a 2-metric space [4].

A sequence $\{x_n\}$ in X is called a Cauchy sequence [7] when $d(x_n, x_m, a)\to 0$ as $n, m \to\infty$
A sequence $\{x_n\}$ in X is said to be converge [7] to an element x in X when $d(x_n, x, a)\to 0$ as $n\to\infty$
A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Naidu and Prasad [12] proved that every convergent sequence need not be a Cauchy sequence in 2-metric space. Chaterjea [2] introduced the notion of a C-contraction.

**Definition: [2]** Let (X, d) be a metric space and T: $X \to X$ be a map. Then T is called a C -contraction if there exists $\alpha \in (0,1/2)$ such that for all $x, y \in X$,
$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)].$$
C-contraction was generalized to weak C-contraction by Choudhury [4] as

**Definition: [4]** Let (X, d) be a metric space and T: $X \to X$ be a map. Then T is called a weak C -contraction if there exists $\psi:[0,\infty)^2\to[0,\infty)$ which is continuous and $\psi(s, t) = 0$ if and only if $s = t = 0$ such that
$$d(Tx, Ty) \leq \tfrac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty),d(y, Tx)) \text{ for all } x, y \in X.$$

Choudhury [4] proved that if X is a complete metric space, then every weak C-contraction has a fixed point. Dung and Hang [6] proved a fixed point theorem for weak C-contraction in partially ordered 2-metric space for one mapping.

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In this paper, we will prove common fixed point theorem for four mappings in 2-metric space with the help of weak C-contraction and weakly compatible mappings by using $\psi(a, b) = ½ \min\{a, b\}$.

**MAIN RESULTS**

**Theorem:** Let, (X, d) be a complete 2-metric space, and F, G, S, and T be self maps of X satisfying $S(X) \subseteq F(X)$, $T(X) \subseteq G(X)$ and weak C contraction such that

$$d(Sx, Ty, u) \leq \frac{1}{2}[d(Gx, Ty, u) + d(Fy, Sx, u)] - \psi(d(Gx, Ty, u), d(Fy, Sx, u)) \tag{1}$$

for all x, y in X and $\psi:[0,\infty)^2 \to [0,\infty)$ which is continuous and $\psi(s,t) = 0$ if and only if $s = t = 0$ and F(X) and G(X) are closed subsets of X. (T,F) and (S,G) are weakly compatible. Then, F, G, S and T have a unique common fixed point in X.

**Proof:** Let $x_0$ be any point in X and as $S(X) \subseteq F(X)$, $T(X) \subseteq G(X)$ then there exists $x_1, x_2$ in X such that $Sx_0 = Fx_1$, $Tx_1 = Gx_2$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = Sx_n = Fx_{n+1}$ and $y_{n+1} = Tx_{n+1} = Gx_{n+2}$, n=0, 1, 2,….

Now,

$$\begin{aligned} d(y_n,y_{n+1},u) = d(Sx_n,Tx_{n+1},u) &\leq \frac{1}{2}[d(Gx_n, Tx_{n+1}, u) + d(Fx_{n+1}, Sx_n, u)] - \psi(d(Gx_n, Tx_{n+1}, u), d(Fx_{n+1}, Sx_n, u)) \\ &= \frac{1}{2}[d(y_{n-1}, y_{n+1}, u) + d(y_n, y_n, u)] - \psi(d(y_{n-1}, y_{n+1}, u), d(y_n, y_n, u)) \\ &= \frac{1}{2} d(y_{n-1},y_{n+1},u) - \psi(d(y_{n-1}, y_{n+1}, u), 0) \qquad (2) \\ &\leq \frac{1}{2} d(y_{n-1}, y_{n+1}, u) \qquad (3) \end{aligned}$$

Now, if we put $u = y_{n-1}$ in (3) then we get, $d(y_n, y_{n+1}, y_{n-1}) \leq 0$. (4)

Now, from (3) and (4) we get,

$$\begin{aligned} d(y_n, y_{n+1}, u) \leq \frac{1}{2} d(y_{n-1}, y_{n+1}, u) &\leq \frac{1}{2}[d(y_{n-1}, y_{n+1}, y_n) + d(y_{n-1}, y_n, u) + d(y_n, y_{n+1}, u)] \\ &= \frac{1}{2}[d(y_{n-1}, y_n, u) + d(y_n, y_{n+1}, u)]. \qquad (5) \end{aligned}$$

Which gives that, $d(y_n, y_{n+1}, u) \leq d(y_{n-1}, y_n, u)$ (6)

So, $\{d(y_n,y_{n+1},u)\}$ is a non-negative decreasing sequence and hence it is convergent.

Let, $d(y_n,y_{n+1},u) = r$ (7)

Taking $lim_{n\to\infty}$ in (4) and using (6) we get, $r \leq \frac{1}{2} d(y_{n-1},y_{n+1},u) \leq \frac{1}{2}(r+r) = r$
i.e., $d(y_{n-1},y_{n+1},u) = 2r$ (8)

Taking $lim_{n\to\infty}$ in (2) and using (7) and (8) we get, $r \leq \frac{1}{2}. 2r - \psi(0,2r)$
i.e., $\psi(0,2r) \leq 0$ which shows that $r = 0$

So, from (7) we get, $lim_{n\to\infty} d(y_n,y_{n+1},u) = 0$ (9)

Now, we will prove that $\{y_n\}$ is a Cauchy sequence.

$$\begin{aligned} \text{Now, } d(y_n,y_{n+2},u) &\leq d(y_n,y_{n+2},y_{n+1}) + d(y_n,y_{n+1},u) + d(y_{n+1},y_{n+2},u) \\ &= d(y_n,y_{n+2},y_{n+1}) + \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1},u). \qquad (10) \end{aligned}$$

$$\begin{aligned} \text{Now, } d(y_n,y_{n+2},y_{n+1}) = d(y_{n+1},y_{n+2},y_n) &= d(Sx_{n+1}, Tx_{n+2}, y_n) \\ &\leq \frac{1}{2}[d(Gx_{n+1},Tx_{n+2},y_n) + d(Fx_{n+2},Sx_{n+1}, y_n)] - \psi(d(Gx_{n+1},Tx_{n+2},y_n), d(Fx_{n+2},Sx_{n+1},y_n)) \\ &= \frac{1}{2}[d(y_n,y_{n+2},y_n) + d(y_{n+1},y_{n+1},y_n)] - \psi(d(y_n,y_{n+2},y_n), d(y_{n+1},y_{n+1}, y_n)) \\ &= \frac{1}{2}[0+0] - \psi(0,0) = 0 \qquad (11) \end{aligned}$$

Putting the value of (11) in (10) we get, $d(y_n, y_{n+2}, u) \leq \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u)$

Similarly proceeding as above we will get,

$$d(y_n, y_{n+p}, u) \leq \sum_{r=0}^{p-1} d(y_{n+r}, y_{n+r+1}, u)$$

Taking $lim_{n\to\infty}$ on the above inequality we get, $lim_{n\to\infty}$ $d(y_n, y_{n+p}, u) = 0$ (by (9))

Which shows that $\{y_n\}$ is a Cauchy sequence in X.

Since, G(X) is complete then $\{y_n\}$ converges to a point z in G(X). i.e., $lim_{n\to\infty} y_n = z$.

Since, $T(X) \subset G(X)$, then there exists a poinjt q in X such that $Gq = z$ (12)

Now, $d(Sq,y_{n+1},u) = d(Sq,Tx_{n+1},u))$
$\leq \frac{1}{2} [d(Gq,Tx_{n+1},u) + d(Fx_{n+1}Sq, u)] - \psi(d(Gq, T x_{n+1}, u), d(Fx_{n+1}, Sq, u))$

Taking $lim_{n\to\infty}$ on the above inequality and using (12) we get,
$d( Sq, z, u) =d(Sq, z, u) ) \leq \frac{1}{2} [d(z, z, u) +d(z, Sq, u)] - \psi (d(z, z, u), d(z, Sq, u))$
$= \frac{1}{2} d(z, Sq, u)] - \psi(0, d(z, Sq, u) ) \leq \frac{1}{2} d(z, Sq, u)$
i.e, $d(z, Sq, u) \leq 0$ i.e., $d(z, Sq, u) = 0$

So, $Sq = z$. (13)

From (12) & (13) we get, $Gq = z = Sq$ (14)

Again, (S, G) are weakly compatible so $SGq = GSq$ i.e., $Sz = Gz$ (by (14)) (15)

Now, $d(Sz,y_{n+1}, u) = d(Sz,Tx_{n+1}, u))$
$\leq \frac{1}{2} [d(Gz,Tx_{n+1}, u) + d(F x_{n+1}Sz, u)] - \psi(d(Gz,Tx_{n+1}, u), d(F x_{n+1}, Sz,u) )$

Taking $lim_{n\to\infty}$ on the above inequality and using (15) we get,
$d(Sz, z, u) \leq \frac{1}{2} [d(Sz, z, u) + d(z, Sz, u)] - \psi(d(Sz, z, u), d(z, Sz, u))$ (by (15)
$= d(Sz, z, u) - \psi (d(Sz, z, u), d(z, Sz, u))$
i.e, $\psi(d(Sz, z, u), d(z, Sz, u)) \leq 0$

So, $Sz = z$

From (15) we get, $Sz = z = Gz$ (16)

Since, $S(X) \subset F(X)$, then there exists a poinjt p in X such that $Fp=z$. (17)

Now, $d(y_n, Tp, u) = d(Sx_n, Tp, u)$
$\leq \frac{1}{2} [d(Gx_n, Tp, u) + d(Fp, Sx_n, u)] - \psi(d(Gx_n, Tp, u), d(Fp, Sx_n, u))$

Taking $lim_{n\to\infty}$ on the above inequality and using (12) we get,
$d(z, Tp, u) \leq \frac{1}{2} [d(z, Tp, u) + d(z, z, u)] - \psi(d(z, Tp, u), d(z, z, u))$
$= \frac{1}{2} d(z, Tp, u) - \psi(d(z, Tp, u), 0) \leq \frac{1}{2} d(z, Tp, u)$

*i.e,* $d(z, Tp, u) \leq 0$ i.e., $d(z, Tp, u) = 0$

So, $Tp = z$ (18)

From (17) and (18) we get, $Fp = z = Tp$ (19)
As (T, F) are weakly compatible then, $TFp = FTp$ i.e., $Tz = Fz$ (by (19)) … (20)

Now, $d( y_n, Tz, u) = d(Sx_n, Tz, u) )$
$\leq \frac{1}{2} [d(Gx_n, T z, u) + d(Fz, Sx_n, u)] - \psi(d(Gx_n, T z, u), d(Fz, Sx_n, u))$

Taking $lim_{n\to\infty}$ on the above inequality and using (20) we get,
$d(z, Tz, u) \leq \frac{1}{2} [d(z, Tz, u) + d(Tz, z, u)] - \psi(d(z, Tz, u), d(Tz, z, u))$ (by (20)
$= d(z, Tz, u) - \psi(d(z, Tz, u), d(Tz, z, u))$
i.e., $\psi (d(z, Tz, u), d(Tz, z, u)) \leq 0$

By the property of ψ it is only possible when, d(Tz, z, u) = 0 *i.e.*, Tz = z
So, from (20) we get, Tz = z = Fz (21)

From (16) & (20) we get, Tz = Fz = z = Sz = Gz (22)
So, z is a common fixed point of S, F, G and T.

Now, we will prove that z is a unique fixed point.
If possible late, w(≠z) is also a fixed point of S, G, F and T.

$$\begin{aligned} \text{Now, } d(z, w, u) &= d(Sz, Tw, u) \\ &\le \tfrac{1}{2}\,[d(Gz, Tw, u) + d(Fw, Sz, u)] - \psi(d(Gz, Tw, u), d(Fw, Sz, u)) \\ &= \tfrac{1}{2}\,[d(z, w, u) + d(w, z, u)] - \psi(d(z, w, u), d(w, z, u)) \\ &= d(z, w, u) - \psi(d(z, w, u), d(w, z, u)) \end{aligned}$$

i.e., $\psi(d(z, w, u), d(w, z, u)) \le 0$

By the property of ψ it is only possible when, d(z, w, u) = 0 *i.e.*, z = w

So, z is a unique fixed point of S, G, F and T.

## CONCLUSION

In this paper we prove the main theorem for four mappings with the help of weak C-contraction and weakly comtible mappings. Dung and Hang [6] prove their main theorem for only one mapping with the help of weak C-contraction. So, this paper is a generalization of [6]. Changing the condition of ψ(a, b) we will get many generalization of this result.

## REFERENCES

1. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Pub.Math.(Debr.),57 (2000),31-37.
2. S K Chatterjea, Fixed point theorems, C.R.Acad.Bulgare Sci., 25 (1972), 727-730.
3. Y.J.Cho, B.K.Sharma & D.R.Sahu, Semi Compatiblity and Fixed Points, Math.Japon, 42(1) (1995), 91-98.
4. B.S.Choudhury, Unique Fixed Point Theorem for weakly C-contractive mappings, Katmandu Univ. J. Sci. Eng. Technol., 5 (2009),6-13.
5. B.S.Choudhury, A.Kundu, (ψ, α, β)-weak contractions in partially ordered metric spaces, Appl. Math. Lett., 25(1) (2012),6-10.
6. N.V.Dung, V.T.Hang, Fixed point theorems for weak c-contractions in partialy ordered 2-metric spaces, Fixed point theory and applications, 2013 (2013):161.
7. S.Gahler, 2-metrische Raume and Iher Topogosche Struktur, Math.nache., 26 (1963), 115-148.
8. K.Iseki, Fixed point theorems in 2-metric Spaces. Math.Sem. Notes. Kobe. Univ., 3 (1975), 133-136.
9. G.Jungck, Compatible Mappings and Common Fixed Points, Internat. J. Math. and Math. Sci., 9 (1986), 771-779.
10. G.Jungck, Common Fixed Points for Commuting and Compatible Mappings on Compacts, Proc. Amer. Math. Soc., 103 (1988), 977-985.
11. G.Jungck and B.E.Rhodes, Fixed Point Theorems for Compatible Mappings, Internat. J. Matha and Math. Sci., (3) 16 (1993), 417-428.
12. S.V.R.Naidu and J.Rajendra Prasad, Fixed Point in2-metric Spaces, Ind. J. Pure and Appl. Math., 17(8) (1986), 974-993.
13. B,E.Rhoades, Contraction Type Mapping On 2-metric Spaces, Math.Nacher,91 (1979), 151-155.
14. N.S.Simoniya, An Approach to Fixed point Theorem in a 2-metric apace, The Math. Education, 27 (1993), 8-10.
15. B.Singh & S.Jain, Semi Compatiblity and Fixed Point Theorems in Fuzzy Metric Spaces using implicit relation, Int. Math.Sci., 16 (2005), 2617-2629.
16. B.Singh & S.Jain, Semi Compatiblity and Fixed Point Theorems in Menger Spaces. Journal of the Chungcheong Mathematica Socity, 17(1) (2004), 1-17.
17. Kalishankar Tiwary, T.Basu, S.Sen, Some Common Fixed Point Theorems in Complete Metric spaces, 21(4) (1995), 451-459.

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