International Journal of Mathematical Archive-9(4), 2018, 179-182
Available online through www.ijma.info ISSN 2229 – 5046

FIXED POINT THEOREMS FOR WEAK C-CONTRACTIONS AND WEAKLY COMPATIBLE MAPPINGS IN 2-METRIC SPACE

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(Received On: 15-03-18; Revised & Accepted On: 11-04-18)

ABSTRACT

The purpose of this paper is to study a common fixed point theorem on 2-metric space using weak C-contraction and weakly compatibility. We mainly generalize the result of Dung and Hang [6], which is unifies and generalizes many results in the literature.

Mathematics Subject Classification: *47H10, 54H25.*

Key Wards: *2-metric space, weak C contraction, weak compatible maps, Fixed point.*

INTRODUCTION

The concept of 2-metric space is a natural generalization of a metric space. It has been investigated initially by Gahler [7]. Then many researchers like Iseki [8], Rhoades [13], Simoniya [14] etc. prove many fixed points in this space. Gahler [7] introduce 2-metric space as

Let X be a non-empty set and let d: $X\times X\times X \to [0,\infty)$ be such that

(i) For every pair of distinct point x, y in X with $x \neq y$ there exists a point z in X such that $d(x, y, z) \neq 0$.
(ii) $d(x, y, z) = 0$ when at least two of the three points are equal.
(iii) For any x, y, z in X, $d(x, y, z) = d(x, z, y) = d(y, z, x)$.
(iv) For any x, y, z, w in X , $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$,
Then d is called a 2-metric [4] and (X, d) is called a 2-metric space [4].

A sequence $\{x_n\}$ in X is called a Cauchy sequence [7] when $d(x_n, x_m, a)\to 0$ as $n, m \to\infty$
A sequence $\{x_n\}$ in X is said to be converge [7] to an element x in X when $d(x_n, x, a)\to 0$ as $n\to\infty$
A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Naidu and Prasad [12] proved that every convergent sequence need not be a Cauchy sequence in 2-metric space. Chaterjea [2] introduced the notion of a C-contraction.

Definition: [2] Let (X, d) be a metric space and T: X→X be a map. Then T is called a C -contraction if there exists $\alpha \in (0,1/2)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)].$$

C-contraction was generalized to weak C-contraction by Choudhury [4] as

Definition: [4] Let (X, d) be a metric space and T: X→X be a map. Then T is called a weak C -contraction if there exists $\psi:[0,\infty)^2\to[0,\infty)$ which is continuous and $\psi(s, t) = 0$ if and only if $s = t = 0$ such that

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty),d(y, Tx)) \text{ for all } x, y \in X.$$

Choudhury [4] proved that if X is a complete metric space, then every weak C-contraction has a fixed point. Dung and Hang [6] proved a fixed point theorem for weak C-contraction in partially ordered 2-metric space for one mapping.

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In this paper, we will prove common fixed point theorem for four mappings in 2-metric space with the help of weak C-contraction and weakly compatible mappings by using $\psi(a, b) = ½ \min\{a, b\}$.

MAIN RESULTS

Theorem: Let, (X, d) be a complete 2-metric space, and F, G, S, and T be self maps of X satisfying $S(X) \subseteq F(X)$, $T(X) \subseteq G(X)$ and weak C contraction such that

$$d(Sx, Ty, u) \leq \frac{1}{2}[d(Gx, Ty, u) + d(Fy, Sx, u)] - \psi(d(Gx, Ty, u), d(Fy, Sx, u)) \quad (1)$$

for all x, y in X and $\psi:[0,\infty)^2 \rightarrow [0,\infty)$ which is continuous and $\psi(s,t) = 0$ if and only if $s = t = 0$ and F(X) and G(X) are closed subsets of X. (T,F) and (S,G) are weakly compatible. Then, F, G, S and T have a unique common fixed point in X.

Proof: Let x_0 be any point in X and as $S(X) \subseteq F(X)$, $T(X) \subseteq G(X)$ then there exists x_1,x_2 in X such that $Sx_0=Fx_1$, $Tx_1=Gx_2$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n=Sx_n=Fx_{n+1}$ and $y_{n+1}=Tx_{n+1}=Gx_{n+2}$, n=0, 1, 2,….

Now,

$$\begin{aligned} d(y_n,y_{n+1},u) = d(Sx_n,Tx_{n+1},u) &\leq \frac{1}{2}[d(Gx_n, Tx_{n+1}, u) + d(Fx_{n+1}, Sx_n, u)] - \psi(d(Gx_n, Tx_{n+1}, u), d(Fx_{n+1}, Sx_n, u)) \\ &= \frac{1}{2}[d(y_{n-1}, y_{n+1}, u) + d(y_n, y_n, u)] - \psi(d(y_{n-1}, y_{n+1}, u), d(y_n, y_n, u)) \\ &= \frac{1}{2}d(y_{n-1},y_{n+1},u) - \psi(d(y_{n-1}, y_{n+1}, u), 0) \quad (2) \\ &\leq \frac{1}{2}d(y_{n-1}, y_{n+1}, u) \quad (3) \end{aligned}$$

Now, if we put $u = y_{n-1}$ in (3) then we get, $d(y_n, y_{n+1}, y_{n-1}) \leq 0$. (4)

Now, from (3) and (4) we get,

$$\begin{aligned} d(y_n, y_{n+1}, u) \leq \frac{1}{2}d(y_{n-1}, y_{n+1}, u) &\leq \frac{1}{2}[d(y_{n-1}, y_{n+1}, y_n) + d(y_{n-1}, y_n, u) + d(y_n, y_{n+1}, u)] \\ &= \frac{1}{2}[d(y_{n-1}, y_n, u) + d(y_n, y_{n+1}, u)]. \quad (5) \end{aligned}$$

Which gives that, $d(y_n, y_{n+1}, u) \leq d(y_{n-1}, y_n, u)$ (6)

So, $\{d(y_n,y_{n+1},u)\}$ is a non-negative decreasing sequence and hence it is convergent.

Let, $d(y_n,y_{n+1},u) = r$ (7)

Taking $lim_{n\rightarrow\infty}$ in (4) and using (6) we get, $r \leq \frac{1}{2}d(y_{n-1},y_{n+1},u) \leq \frac{1}{2}(r+r) = r$

i.e., $d(y_{n-1},y_{n+1},u) = 2r$ (8)

Taking $lim_{n\rightarrow\infty}$ in (2) and using (7) and (8) we get, $r \leq \frac{1}{2}. 2r - \psi(0,2r)$

i.e., $\psi(0,2r) \leq 0$ which shows that $r = 0$

So, from (7) we get, $lim_{n\rightarrow\infty} d(y_n,y_{n+1},u)= 0$ (9)

Now, we will prove that $\{y_n\}$ is a Cauchy sequence.

$$\begin{aligned} \text{Now, } d(y_n,y_{n+2},u) &\leq d(y_n,y_{n+2},y_{n+1}) + d(y_n,y_{n+1},u) + d(y_{n+1},y_{n+2},u) \\ &= d(y_n,y_{n+2},y_{n+1}) + \sum_{r=0}^{1} d(y_{n+r},y_{n+r+1},u). \quad (10) \end{aligned}$$

$$\begin{aligned} \text{Now, } d(y_n,y_{n+2},y_{n+1}) = d(y_{n+1},y_{n+2},y_n) &= d(Sx_{n+1}, Tx_{n+2}, y_n) \\ &\leq \frac{1}{2}[d(Gx_{n+1},Tx_{n+2},y_n) + d(Fx_{n+2},Sx_{n+1}, y_n)] - \psi(d(Gx_{n+1},Tx_{n+2},y_n), d(Fx_{n+2},Sx_{n+1},y_n)) \\ &= \frac{1}{2}[d(y_n,y_{n+2},y_n) + d(y_{n+1},y_{n+1},y_n)] - \psi(d(y_n,y_{n+2},y_n), d(y_{n+1},y_{n+1}, y_n)) \\ &= \frac{1}{2}[0+0] - \psi(0,0) = 0 \quad (11) \end{aligned}$$

Putting the value of (11) in (10) we get, $d(y_n, y_{n+2}, u) \leq \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u)$

Similarly proceeding as above we will get,

$$d(y_n, y_{n+p}, u) \leq \sum_{r=0}^{p-1} d(y_{n+r}, y_{n+r+1}, u)$$

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Taking $lim_{n\to\infty}$ on the above inequality we get, $lim_{n\to\infty}$ $d(y_n, y_{n+p}, u) = 0$ (by (9))

Which shows that $\{y_n\}$ is a Cauchy sequence in X.

Since, G(X) is complete then $\{y_n\}$ converges to a point z in G(X). i.e., $lim_{n\to\infty} y_n = z$.

Since, $T(X) \subset G(X)$, then there exists a poinjt q in X such that $Gq = z$ (12)

Now, $d(Sq,y_{n+1},u) = d(Sq,Tx_{n+1},u))$
$\leq \frac{1}{2}[d(Gq,Tx_{n+1},u) + d(Fx_{n+1}Sq, u)] - \psi(d(Gq, T x_{n+1}, u), d(Fx_{n+1}, Sq, u))$

Taking $lim_{n\to\infty}$ on the above inequality and using (12) we get,
$d(Sq, z, u) = d(Sq, z, u)) \leq \frac{1}{2}[d(z, z, u) + d(z, Sq, u)] - \psi (d(z, z, u), d(z, Sq, u))$
$= \frac{1}{2} d(z, Sq, u)] - \psi(0, d(z, Sq, u)) \leq \frac{1}{2} d(z, Sq, u)$
i.e, $d(z, Sq, u) \leq 0$ i.e., $d(z, Sq, u) = 0$

So, $Sq = z$. (13)

From (12) & (13) we get, $Gq = z = Sq$ (14)

Again, (S, G) are weakly compatible so $SGq = GSq$ i.e., $Sz = Gz$ (by (14)) (15)

Now, $d(Sz,y_{n+1}, u) = d(Sz,Tx_{n+1}, u))$
$\leq \frac{1}{2}[d(Gz,Tx_{n+1}, u) + d(F x_{n+1}Sz, u)] - \psi(d(Gz,Tx_{n+1}, u), d(F x_{n+1}, Sz,u))$

Taking $lim_{n\to\infty}$ on the above inequality and using (15) we get,
$d(Sz, z, u) \leq \frac{1}{2} [d(Sz, z, u) + d(z, Sz, u)] - \psi(d(Sz, z, u), d(z, Sz, u))$ (by (15)
$= d(Sz, z, u) - \psi (d(Sz, z, u), d(z, Sz, u))$
i.e, $\psi(d(Sz, z, u), d(z, Sz, u)) \leq 0$

So, $Sz = z$

From (15) we get, $Sz = z = Gz$ (16)

Since, $S(X) \subset F(X)$, then there exists a poinjt p in X such that $Fp=z$. (17)

Now, $d(y_n, Tp, u) = d(Sx_n, Tp, u)$
$\leq \frac{1}{2}[d(Gx_n, Tp, u) + d(Fp, Sx_n, u)] - \psi(d(Gx_n, Tp, u), d(Fp, Sx_n, u))$

Taking $lim_{n\to\infty}$ on the above inequality and using (12) we get,
$d(z, Tp, u) \leq \frac{1}{2}[d(z, Tp, u) + d(z, z, u)] - \psi(d(z, Tp, u), d(z, z, u))$
$= \frac{1}{2} d(z, Tp, u) - \psi(d(z, Tp, u), 0) \leq \frac{1}{2} d(z, Tp, u)$

i.e, $d(z, Tp, u) \leq 0$ i.e., $d(z, Tp, u) = 0$

So, $Tp = z$ (18)

From (17) and (18) we get, $Fp = z = Tp$ (19)
As (T, F) are weakly compatible then, $TFp = FTp$ i.e., $Tz = Fz$ (by (19)) … (20)

Now, $d(y_n, Tz, u) = d(Sx_n, Tz, u))$
$\leq \frac{1}{2}[d(Gx_n, T z, u) + d(Fz, Sx_n, u)] - \psi(d(Gx_n, T z, u), d(Fz, Sx_n, u))$

Taking $lim_{n\to\infty}$ on the above inequality and using (20) we get,
$d(z, Tz, u) \leq \frac{1}{2}[d(z, Tz, u) + d(Tz, z, u)] - \psi(d(z, Tz, u), d(Tz, z, u))$ (by (20)
$= d(z, Tz, u) - \psi(d(z, Tz, u), d(Tz, z, u))$
i.e., $\psi (d(z, Tz, u), d(Tz, z, u)) \leq 0$

By the property of ψ it is only possible when, d(Tz, z, u) = 0 *i.e.*, Tz = z
So, from (20) we get, Tz = z = Fz (21)

From (16) & (20) we get, Tz = Fz = z = Sz = Gz (22)
So, z is a common fixed point of S, F, G and T.

Now, we will prove that z is a unique fixed point.
If possible late, w(≠z) is also a fixed point of S, G, F and T.

Now, d(z, w, u) = d(Sz, Tw, u)

$$\le \frac{1}{2}\,[d(Gz, Tw, u) + d(Fw, Sz, u)] - \psi(d(Gz, Tw, u), d(Fw, Sz, u))$$

$$= \frac{1}{2}\,[d(z, w, u) + d(w, z, u)] - \psi(d(z, w, u), d(w, z, u))$$

$$= d(z, w, u) - \psi(d(z, w, u), d(w, z, u))$$

i.e., ψ(d(z, w, u), d(w, z, u)) ≤ 0

By the property of ψ it is only possible when, d(z, w, u) = 0 *i.e.*, z = w

So, z is a unique fixed point of S, G, F and T.

CONCLUSION

In this paper we prove the main theorem for four mappings with the help of weak C-contraction and weakly comtible mappings. Dung and Hang [6] prove their main theorem for only one mapping with the help of weak C-contraction. So, this paper is a generalization of [6]. Changing the condition of ψ(a, b) we will get many generalization of this result.

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Source of support: Nil, Conflict of interest: None Declared.
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