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BEST SIMULTANEOUS APPROXIMATION IN FUZZY ANTI-2-NORMED LINEAR SPACES

B. Surender Reddy*

Department of Mathematics, PGCS, Saifabad, Osmania University, Hyderabad-500004, AP, INDIA

Email: bsrmathou@yahoo.com

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ABSTRACT

The main aim of this paper is to consider the t-best simultaneous approximation in fuzzy anti-2-normed linear spaces. We develop the theory of t-best simultaneous approximation in its quotient spaces. Then we discuss the relationship in t-proximinality and t-Chebyshevity of a given space and its quotient space.

Key Words: *Fuzzy anti-2-norms, simultaneous approximation, simultaneous t-proximinality, simultaneous t-chebyshe - vity, Quotient space.*

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1. INTRODUCTION:

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The theory of fuzzy sets was introduced by Zadeh [22] in 1965, since then many mathematicians have studied from several angles [15,5]. The idea of fuzzy norm was initiated by Katsaras in [13]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [12]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [14].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [13], Felbin [6], and Bag and Samanta [1]. Veeramani [21] introduced the concept of t-best approximations in fuzzy metric spaces. The concept of 2-norm on a linear space has been introduced and developed by Gahler in [7,8] and Gunawan and Mashadi [10]. Recently, Vaezpour and Karimi [20], studied on the set of all t-best approximations on fuzzy normed spaces and proved several theorems pertaining to this set.

In [11] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [3] and investigated their important properties. In [16] Surender Reddy introduced the notion of convergent sequence and Cauchy sequence in fuzzy anti-2-normed linear space. In [17] Surender Reddy introduced the set of all t-best approximations on fuzzy anti-normed linear spaces. In [18] Surender Reddy introduced the set of all t-best approximations on fuzzy anti-2-normed linear spaces. Recently Goudarzi and Vaezpour [9] considered the set of all t-best simultaneous approximation in fuzzy normed spaces and used the concept of simultaneous t-proximinality and simultaneous t-Chebyshevity to introduce the theory of t-best simultaneous approximation in quotient spaces.

In [19] Surender Reddy considered the set of all t-best simultaneous approximation in fuzzy anti-normed linear spaces and used the concept of simultaneous t-proximinality and simultaneous t-Chebyshevity to introduce the theory of t-best simultaneous approximation in quotient spaces.

In this paper, we consider the set of all t-best simultaneous approximation in fuzzy anti-2-normed linear spaces and use the concept of simultaneous t-proximinality and simultaneous t-Chebyshevity to introduce the theory of t-best simultaneous approximation in quotient spaces.

***Corresponding author:** *B. Surender Reddy*, ***E-mail:** bsrmathou@yahoo.com*

2. PRELIMINARIES:

Definition 2.1: Let X be a real linear space of dimension greater than one and let $\|\bullet,\bullet\|$ be a real valued function on $X \times X$ satisfying the following conditions

$2N_1$: $\|x, y\| = 0$ if and only if x and y are linearly dependent,

$2N_2$: $\|x, y\| = \|y, x\|$

$2N_3$: $\|\alpha x, y\| = |\alpha|\ \|x, y\|$, for every $\alpha \in R$,

$2N_4$: $\|x, y+z\| \le \|x, y\| + \|x, z\|$

then the function $\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet,\bullet\|)$ is called a 2-normed linear space.

Example 2.2: Let $X = R^3$ be a real linear space. Define $\|\bullet,\bullet\| : X \times X \to R$ by $\|x, y\| = \max\{|x_1 y_2 - x_2 y_1|, |x_2 y_3 - x_3 y_2|, |x_3 y_1 - x_1 y_3|\}$, where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ are in R^3. Then $(X, \|\bullet,\bullet\|)$ is a 2-normed linear space.

Definition 2.3: Let X be a linear space over a real field F. A fuzzy subset N of $X \times X \times R$ is called a fuzzy 2-norm on X if the following conditions are satisfied for all $x, y, z \in X$.

$(2-N_1)$ For all $t \in R$ with $t \le 0$, $N(x, y, t) = 0$,

$(2-N_2)$: For all $t \in R$ with $t > 0$, $N(x, y, t) = 1$ if and only if x, y are linearly dependent

$(2-N_3)$: $N(x, y, t)$ is invariant under any permutation of x, y

$(2-N_4)$: For all $t \in R$ with $t > 0$, $N(x, cy, t) = N(x, y, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(2-N_5)$: For all $s, t \in R$, $N(x, y+z, s+t) \ge \min\{N(x, y, s), N(x, z, t)\}$,

$(2-N_6)$: $N(x, y, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x, y, t) = 1$.

Then the pair (X, N) is called a fuzzy 2-normed linear space (briefly F-2-NLS).

Example 2.4: Let $(X, \|\bullet,\bullet\|)$ be a 2-normed linear space. Define

$$N(x, y, t) = \frac{t}{t + \|x, y\|}, \text{ if } t > 0,\ t \in R,\ x, y \in X$$
$$= 0, \text{ if } t \le 0,\ t \in R,\ x, y \in X.$$

Then (X, N) is a fuzzy 2-normed linear space.

Definition 2.5: Let X be a linear space over a real field F. A fuzzy subset N of $X \times X \times R$ is called a fuzzy anti-2-norm on X if the following conditions are satisfied for all $x, y, z \in X$.

$(a-2-N_1)$ For all $t \in R$ with $t \le 0$, $N(x, y, t) = 1$,

$(a-2-N_2)$: For all $t \in R$ with $t > 0$, $N(x, y, t) = 0$ if and only if x, y are linearly dependent

$(a-2-N_3)$: $N(x, y, t)$ is invariant under any permutation of x, y

$(a-2-N_4)$: For all $t \in R$ with $t > 0$, $N(x, cy, t) = N(x, y, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(a-2-N_5)$: For all $s, t \in R$, $N(x, y+z, s+t) \le \max\{N(x, y, s), N(x, z, t)\}$,

$(a-2-N_6)$: $N(x, y, t)$ is a non-increasing function of $t \in R$ and $\lim_{t \to \infty} N(x, y, t) = 0$.

Then the pair (X, N) is called a fuzzy anti-2-normed linear space (briefly Fa-2-NLS).

Remark 2.6: From $(a-2-N_3)$, it follows that in Fa-2-NLS,

$(a-2-N_4)$: For all $t \in R$ with $t > 0$, $N(cx, y, t) = N(x, y, \frac{t}{|c|})$, if $c \neq 0$, $c \in F$,

$(a-2-N_5)$: For all $s, t \in R$, $N(x+z, y, s+t) \leq \max\{N(x, y, s), N(z, y, t)\}$.

Example 2.7: Let $(X, \|\bullet,\bullet\|)$ be a 2-normed linear space. Define

$$N(x, y, t) = \frac{k\|x, y\|}{kt^n + m\|x, y\|}, \text{ if } t > 0,\ t \in R,\ k, m, n \in R^+,\ x, y \in X$$
$$= 1, \text{ if } t \leq 0,\ t \in R,\ x, y \in X.$$

Then (X, N) is a fuzzy anti-2-normed linear space. In particular if $k = m = n = 1$ we have

$$N(x, y, t) = \frac{\|x, y\|}{t + \|x, y\|}, \text{ if } t > 0,\ t \in R,\ x, y \in X$$
$$= 1, \text{ if } t \leq 0,\ t \in R,\ x, y \in X,$$

which is called the standard fuzzy anti-2-norm induced by the 2-norm $\|\bullet,\bullet\|$.

Definition 2.8: A sequence $\{x_k\}$ in a fuzzy anti-2-normed linear space (X, N) is said to be converges to $x \in X$ if given $t > 0, 0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_1, x_k - x, t) < r$, $\forall\ k \geq n_0$.

Theorem 2.9: In a fuzzy anti-2-normed linear space (X, N), a sequence $\{x_k\}$ converges to $x \in X$ if and only $\lim_{k\to\infty} N(x_1, x_k - x, t) = 0$, $\forall\ t > 0$.

Definition 2.10: Let (X, N) be a fuzzy anti-2-normed linear space. Let $\{x_k\}$ be a sequence in *X* then $\{x_k\}$ is said to be a Cauchy sequence if $\lim_{k\to\infty} N(x_1, x_{k+p} - x_k, t) = 0$, $\forall\ t > 0$ and $p = 1,2,3,\ldots$.

Definition 2.11: A fuzzy anti-2-normed linear space (X, N) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.12: A complete fuzzy anti-2-normed linear space (X, N) is called a fuzzy anti-2-Banach space.

Definition 2.13: Let (X, N) be a fuzzy anti-2-normed linear space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x, r, t) = \{y \in X : N(x_1, x - y, t) < r\}$$
$$B[x, r, t] = \{y \in X : N(x_1, x - y, t) \leq r\}$$

Definition 2.14: Let (X, N) be a fuzzy anti-2-normed linear space. A subset *A* of *X* is said to be open if there exists $r \in (0,1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 2.15: Let (X, N) be a fuzzy anti-2-normed linear space. A subset *A* of X is said to be closed if for any sequence $\{x_k\}$ in *A* converges to $x \in A$.

i.e., $\lim_{k\to\infty} N(x_1, x_k - x, t) = 0$, for all $t > 0$ implies that $x \in A$.

Corollary 2.16: Let (X, N) be a fuzzy anti-2-normed linear space. Then *N* is a continuous function on $X \times X \times R$.

3. *t*-BEST SIMULTANEOUS APPROXIMATION:

Definition 3.1: Let (X,N) be a fuzzy anti-2-normed linear space. A subset *A* of *X* is called *F*-bounded if there exists $t>0$ and $0<r<1$ such that $N(x_1,x,t)<r$, $\forall\ x\in A$.

Definition 3.2: Let (X,N) be a fuzzy anti-2-normed linear space, *W* be a subset of *X* and *M* be a *F*-bounded subset in *X*. For $t>0$, we define

$$d(M,W,t)=\inf_{w\in W}\sup_{m\in M}N(x_1,m-w,t).$$

An element $w_0\in W$ is called a *t*-best simultaneous approximation to *M* from *W* if for $t>0$,

$$d(M,W,t)=\sup_{m\in M}N(x_1,m-w_0,t).$$

The set of all *t*-best simultaneous approximations to *M* from *W* will be denoted by $S_W^t(M)$ and we have,

$$S_W^t(M)=\{w\in W:\sup_{m\in M}N(x_1,m-w,t)=d(M,W,t)\}$$

Definition 3.3: Let *W* be a subset of a fuzzy anti-2-normed linear space (X,N) then *W* is called a simultaneous *t*-proximinal subset of *X* if for each *F*-bounded set *M* in *X*, there exists at least one *t*-best simultaneous approximation from *W* to *M*. Also *W* is called a simultaneous *t*-Chebyshev subset of *X* if for each *F*-bounded set *M* in *X*, there exists a unique *t*-best simultaneous approximation from *W* to *M*.

Definition 3.4: Let (X,N) be a fuzzy anti-2-normed linear space. A subset *E* of X is said to be convex if $(1-\lambda)x+\lambda y\in E$ whenever $x,y\in E$ and $0<\lambda<1$.

Lemma 3.5: Every open ball in a fuzzy anti-2-normed linear space (X,N) is convex.

Theorem 3.6: Suppose that *W* is a subset of a fuzzy anti-2-normed linear space (X,N) and *M* is *F*-bounded in *X*. Then $S_W^t(M)$ is a *F*-bounded subset of *X* and if *W* is convex and is a closed subset of X then $S_W^t(M)$ is closed and is convex for each *F*-bounded subspace *M* of *X*.

Proof: Since *M* is *F*-bounded, there exists $t>0$ and $0<r<1$ such that $N(x_1,x,t)<r$, for all $x\in M$. If $w\in S_W^t(M)$, then $\sup_{m\in M}N(x_1,m-w,t)=d(M,W,t)$.

Now, for all $m\in M$ and $w\in S_W^t(M)$,

$$\begin{aligned}N(x_1,w,2t)&=N(x_1,w-m+m,2t)\\&\le\max\{N(x_1,w-m,t),N(x_1,m,t)\}\\&\le\sup_{m\in M}\max\{N(x_1,w-m,t),N(x_1,m,t)\}\\&\le\max\{\sup_{m\in M}N(x_1,w-m,t),\sup_{m\in M}N(x_1,m,t)\}\\&\le\max\{d(M,W,t),r\}\le r_0,\text{ for some }0<r_0<1.\end{aligned}$$

Then $S_W^t(M)$ is *F*-bounded. Suppose that *W* is convex and is a closed subset of *X*. We show that $S_W^t(M)$ is convex and closed. Let $x,y\in S_W^t(M)$ and $0<\lambda<1$. Since *W* is convex, there exists $z_\lambda\in W$ such that $z_\lambda=\lambda x+(1-\lambda)y$, for each $0<\lambda<1$. Now for $t>0$ we have,

$$\sup_{m\in M}N(x_1,(\lambda x+(1-\lambda)y)-m,t)=\sup_{m\in M}N(x_1,z_\lambda-m,t)\ge d(M,W,t).$$

On the other hand, for a given $t>0$, take the natural number *n* such that $t>\frac{1}{n}$, we have

$$\sup_{m\in M}N(x_1,(\lambda x+(1-\lambda)y)-m,t)=\sup_{m\in M}N(x_1,\lambda(x-y)+y-m,t)$$

$$\leq \sup_{m\in M} \max\{N(x_1, x-y, \frac{1}{\lambda n}), N(x_1, y-m, t-\frac{1}{n})\}$$

$$\leq \max\{N(x_1, x-y, \frac{1}{\lambda n}), \sup_{m\in M} N(x_1, y-m, t-\frac{1}{n})\}$$

$$\leq \lim_{n\to\infty} \max\left(\sup_{m\in M} N(x_1, y-m, t-\frac{1}{n})\right) = d(M, W, t).$$

So $S_W^t(M)$ is convex. Finally let $\{w_n\} \subset S_W^t(M)$ and suppose $\{w_n\}$ converges to some w in X. Since $\{w_n\} \subset W$ and W is closed so $w \in W$. Therefore by Corollary 2.16, for $t > 0$ we have

$$\sup_{m\in M} N(x_1, m-w, t) = \sup_{m\in M} N(x_1, \lim_{n\to\infty} w_n - m, t)$$

$$= \lim_{n\to\infty} \sup_{m\in M} N(x_1, w_n - m, t) = d(M, W, t).$$

Theorem 3.7: The following assertions are hold for $t > 0$,

(i) $d(M + x, W + x, t) = d(M, W, t), \quad \forall \quad x \in X$,

(ii) $d(\lambda M, \lambda W, t) = d(M, W, \frac{t}{|\lambda|}), \quad \forall \quad \lambda \in C$,

(iii) $S_{W+x}^t(M + x) = S_W^t(M) + x, \quad \forall \quad x \in X$,

(iv) $S_{\lambda W}^{|\lambda| t}(\lambda M) = \lambda S_W^t(M) + x, \quad \forall \quad \lambda \in C$,

Proof: (i) $d(M + x, W + x, t) = \inf_{w\in W} \sup_{m\in M} N(x_1, (m + x) - (w + x), t)$

$$= \inf_{w\in W} \sup_{m\in M} N(x_1, m - w, t) = d(M, W, t)$$

(ii) Clearly equality holds for $\lambda = 0$, so suppose that $\lambda \neq 0$. Then,

$$d(\lambda M, \lambda W, t) = \inf_{w\in W} \sup_{m\in M} N(x_1, \lambda(m-w), t)$$

$$= \inf_{w\in W} \sup_{m\in M} N(x_1, m-w, \frac{t}{|\lambda|}) = d(M, W, \frac{t}{|\lambda|})$$

(iii) $x + W \in S_{W+x}^t(M + x)$ if and only if,

$\sup_{m+x\in M+x} N(x_1, m + x - w - x, t) = d(M + x, W + x, t)$ and by (i), the above equality holds if and only if,

$\sup_{m\in M} N(x_1, m - w, t) = d(M, W, t)$ for all $w \in W$ and this shows that $w \in S_W^t(M)$. So $x + w \in S_W^t(M) + x$.

(iv) $y_0 \in S_{\lambda W}^{|\lambda| t}(\lambda M)$ if and only if $y_0 \in \lambda W$ and,

$$d(\lambda W, \lambda M, |\lambda| t) = \sup_{\lambda m\in \lambda M} N(x_1, y_0 - \lambda m, |\lambda| t) = \sup_{m\in M} N(x_1, \frac{y_0}{\lambda} - m, t)$$

But by (ii), we have $d(\lambda M, \lambda W, |\lambda| t) = d(W, M, t)$. So we have $\frac{y_0}{\lambda} \in W$ and $d(M,W,t) = \sup_{m\in M} N(x_1, \frac{y_0}{\lambda} - m, t)$

or equivalently $\frac{y_0}{\lambda} \in S_W^t(M)$ and the proof is completed.

Corollary 3.8: Let *A* be a nonempty subset of a fuzzy anti-2-normed linear space (X, N) then the following statements are hold.

(i) *A* is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev) if and only if *A+y* is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev), for each $y \in X$,

(ii) A is simultaneous t-proximinal (respectively simultaneous t-Chebyshev) if and only if αA is simultaneous $|\alpha|t$-proximinal (respectively simultaneous $|\alpha|t$-Chebyshev), for each $\alpha \in C$.

Corollary 3.9: Let A be a nonempty subspace of a fuzzy anti-2-normed linear space X and M be a F-bounded subset of X. Then for $t > 0$,
(i) $d(A, M + y, t) = d(A, M, t)$, $\forall\ y \in A$,
(ii) $S_A^t(M + y) = S_A^t(M) + y$, $\forall\ y \in A$,
(iii) $d(A, \alpha M, |\alpha| t) = d(A, M, t)$, for $0 \neq \alpha \in C$,
(iv) $S_A^{|\alpha| t}(\alpha M) = \alpha S_A^t(M)$, for $0 \neq \alpha \in C$.

4. SIMULTANEOUS t-PROXIMINALITY AND SIMULTANEOUS t-CHEBYSHEVITY IN QUOTIENT SPACES:

In this section we give characterization of simultaneous t-proximinality and simultaneous t-Chebyshevity in quotient spaces.

Definition 4.1: Let (X, N) be a fuzzy anti-2-normed linear space, M be a linear manifold in X and let $Q: X \to X/M$ be the natural map $Qx = x + M$. We define

$$N(x_1, x + M, t) = \inf\{N(x_1, x + y, t) : y \in M\}, \quad t > 0$$

Theorem 4.2: If M is a closed subspace of a fuzzy anti-2-normed linear space (X, N) and $N(x_1, x + M, t)$ is defined as above then
(a) N is a fuzzy anti-2-norm on X/M.
(b) $N(x_1, Qx, t) \leq N(x_1, x, t)$.
(c) If (X, N) is a fuzzy anti-2-Banach space then so is $(X/M, N)$.

Proof: (a) It is clear that $N(x_1, x + M, t) = 1$ for $t \leq 0$. Let $N(x_1, x + M, t) = 0$ for $t > 0$. By definition there is a sequence $\{x_k\}$ in M such that $N(x_1, x + x_k, t) \to 0$. So $x + x_k \to 0$ or equivalently $x_k \to (-x)$ and since M is closed so $x \in M$ and $x + M = M$, the zero element of X/M. On the other hand we have,

$$\begin{aligned} N(x_1, (x + M) + (y + M), t) &= N(x_1, (x + y) + M, t) \\ &\leq N(x_1, (x + m) + (y + n), t) \\ &\leq \max\{N(x_1, x + m, t_1), N(x_1, y + n, t_2)\} \end{aligned}$$

for $m, n \in M$, $x_1, x, y \in X$ and $t_1 + t_2 = t$. Now if we take infimum on both sides, we have, $N(x_1, (x + M) + (y + M), t) \leq \max\{N(x_1, x + M, t_1), N(x_1, y + M, t_2)\}$.

Also we have,
$$\begin{aligned} N(x_1, \alpha(x + M), t) &= N(x_1, \alpha x + M, t) \\ &= \inf\{N(x_1, \alpha x + \alpha y, t) : y \in M\} \\ &= \inf\{N(x_1, x + y, \tfrac{t}{|\alpha|}) : y \in M\} = N(x_1, x + M, \tfrac{t}{|\alpha|}) \end{aligned}$$
and the remaining properties are obviously true. Therefore N is a fuzzy anti-2-norm on X/M.

(b) We have, $N(x_1, Qx, t) = N(x_1, x + M, t) = \inf\{N(x_1, x + y, t) : y \in M\} \leq N(x_1, x, t)$

(c) Let $\{y_k + M\}$ be a Cauchy sequence in X/M. Then there exists $\varepsilon_k > 0$ such that $\varepsilon_k \to 0$ and, $N(x_1, (y_k + M) - (y_{k+1} + M), t) \leq \varepsilon_k$. Let $z_1 = 0$. We choose $z_2 \in M$ such that, $N(x_1, y_1 - (y_2 - z_2), t) \leq \max\{N(x_1, (y_1 - y_2) + M, t), \varepsilon_1\}$.

But $N(x_1,(y_1-y_2)+M,t)\le \varepsilon_1$. Therefore, $N(x_1,y_1-(y_2-z_2),t)\le \max\{\varepsilon_1,\varepsilon_1\}=\varepsilon_1$.

Now suppose z_{k-1} has been chosen, $z_k\in M$ can be chosen such that

$N(x_1,(y_{k-1}+z_{k-1})-(y_k+z_k),t)\le \max\{N(x_1,(y_{k-1}-y_k)+M,t),\varepsilon_{k-1}\}$ and therefore,
$N(x_1,(y_{k-1}+z_{k-1})-(y_k+z_k),t)\le \max\{\varepsilon_{k-1},\varepsilon_{k-1}\}=\varepsilon_{k-1}$.

Thus, $\{y_k+z_k\}$ is Cauchy sequence in *X*. Since *X* is complete, there is an y_0 in *X* such that $y_k+z_k\to y_0$ in *X*. On the other hand $y_k+M=Q(y_k+z_k)\to Q(y_0)=y_0+M$. Therefore every Cauchy sequence $\{y_k+M\}$ is convergent in X/M and so X/M is complete and $(X/M,N)$ is a fuzzy anti-2-Banach space.

Definition 4.3: Let *A* be a nonempty set in a fuzzy anti-2-normed linear space (X,N). For $x\in X$ and $t>0$, we shall denote the set of all elements of *t*-best approximation to *x* from *A* by $P_A^t(x)$;

i.e., $P_A^t(x)=\{y\in A: d(A,x,t)=N(x_1,y-x,t)\}$.

where, $d(A,x,t)=\inf\{N(x_1,y-x,t): y\in A\}=\inf\limits_{y\in A} N(x_1,y-x,t)$.

If each $x\in X$ has at least (respectively exactly) one *t*-best approximation in *A* then *A* is called a *t*-proximinal (respectively *t*-chebyshev) set.

Lemma 4.4: Let (X,N) be a fuzzy anti-2-normed linear space and *M* be a *t*-proximinal subspace of *X*. For each nonempty *F*-bounded set *S* in X and $t>0$,

$$d(S,M,t)=\sup_{s\in S}\inf_{m\in M} N(x_1,s-m,t)$$

Proof: Since *M* is *t*-proximinal it follows that for each $s\in S$ there exists $m_s\in P_M^t(S)$ such that for $t>0$, $N(x_1,s-m_s,t)=\inf\limits_{m\in M} N(x_1,s-m,t)$.

So, $d(S,M,t)=\inf\limits_{m\in M}\sup\limits_{s\in S} N(x_1,s-m,t)\le \sup\limits_{s\in S} N(x_1,s-m_s,t)\le \sup\limits_{s\in S}\inf\limits_{m\in M} N(x_1,s-m,t)$

$$\le \inf_{m\in M}\sup_{s\in S} N(x_1,s-m,t)=d(S,M,t)$$

This implies that, $d(S,M,t)=\sup\limits_{s\in S}\inf\limits_{m\in M} N(x_1,s-m,t)$.

Example 4.5: Let $(X=R^2,\|\bullet,\bullet\|)$ be a anti-2-normed linear space and consider (X,N) as its standard induced fuzzy anti-2-normed linear space (Example 2.7). A nonempty subset *S* of *X* is *F*-bounded if and only if *S* is bounded in $(X,\|\bullet,\bullet\|)$. If we take $M=R$ we can easily prove that *M* is proximinal in $(X,\|\bullet,\bullet\|)$.

Lemma 4.6: Let (X,N) be a fuzzy anti-2-normed linear space, *M* be a *t*-proximinal subspace of *X* and *S* be an arbitrary subset of *X* then the following assertions are equivalent:
(i) *S* is a *F*-bounded subset of *X*.
(ii) S/M is a *F*-bounded subset of X/M.

Proof: Suppose that *S* be a *F*-bounded subset of *X*. Then there exist $t>0$, $0<r<1$ such that, $N(x_1,x,t)<r$, for all $x\in S$. But,

$$N(x_1,x+M,t)=\inf_{y\in M} N(x_1,x+y,t)\le N(x_1,x,t)\le r.$$

So, $(i)\Rightarrow(ii)$ is proved. Now to prove that $(ii)\Rightarrow(i)$. Let S/M be a *F*-bounded subset of X/M. Since *M* is *t*-proximinal, then for each $s\in S$ there exists $m_s\in M$ such that $m_s\in P_M^t(S)$. So for each $s\in S$,

$$N(x_1,s-m_s,t)=\inf_{m\in M} N(x_1,s-m,t) \tag{1}$$

Now from Lemma 4.4, we conclude that for $t > 0$,

$$\sup_{s\in S} N(x_1, s-m_s, t) = \sup_{s\in S}\inf_{m\in M} N(x_1, s-m, t) = \inf_{m\in M}\sup_{s\in S} N(x_1, s-m, t)$$

Then for $0 < r < 1$ such that $\sup_{s\in S} N(x_1, s-m_s, t) \le r$ and $t > 0$ there exists $m_r \in M$ such that,
$\sup_{s\in S} N(x_1, s-m_r, t) \le \sup_{s\in S} N(x_1, s-m_s, t) - r \le 0$.

So by (1), for all $s \in S$ we have,

$$\begin{aligned} N(x_1, s, t) &= N(x_1, s-m_r+m_r, t) \\ &\le \max\{N(x_1, s-m_r, \tfrac{t}{2}), N(x_1, m_r, \tfrac{t}{2})\} \\ &\le \sup_{s\in S}\max\{N(x_1, s-m_r, \tfrac{t}{2}), N(x_1, m_r, \tfrac{t}{2})\} \\ &\le \max\{(\sup_{s\in S} N(x_1, s-m_s, \tfrac{t}{2}) - r), N(x_1, m_r, \tfrac{t}{2})\} \\ &= \max\{(\sup_{s\in S}\inf_{m\in M} N(x_1, s-m, \tfrac{t}{2}) - r), N(x_1, m_r, \tfrac{t}{2})\} \end{aligned}$$

$$N(x_1, s, t) \le \max\{(\sup_{s\in S} N(x_1, s+M, \tfrac{t}{2}) - r), N(x_1, m_r, \tfrac{t}{2})\}. \tag{2}$$

Since S/M is *F*-bounded, by its definition we can find $0 < r_0 < 1$ such that in the right hand side of (2) be less than or equal to r_0 and this completes the proof.

Lemma 4.7: Let *M* be a *t*-proximinal subspace of a fuzzy anti-2-normed linear space (X, N) and $W \supseteq M$ a subspace of *X*. Let *K* be *F*-bounded in *X*. If $w_0 \in S^t_W(K)$, then $w_0 + M \in S^t_{W/M}(K/M)$.

Proof: Since *K* is *F*-bounded by Lemma 4.6, K/M is F-bounded in X/M. Assume that $w_0 \in S^t_W(K)$ and $w_0 + M \notin S^t_{W/M}(K/M)$. Thus there exists $w' \in M$ such that for $t > 0$,

$$\begin{aligned} \sup_{k\in K} N(x_1, k-(w'+M), t) &< \sup_{k\in K} N(x_1, k-(w_0+M), t) \\ &\le \sup_{k\in K} N(x_1, k-w_0, t) = d(K, W, t) \end{aligned} \tag{3}$$

On the other hand for each $k \in K$ and for $t > 0$,

$$N(x_1, k-(w'+M), t) = \inf_{m\in M} N(x_1, k-(w'+m), t)$$

Then for each $0 < \varepsilon < 1$ and $k \in K$ there exists $m_k \in M$ such that for $t > 0$,

$$N(x_1, k-(w'+m_k), t) \le N(x_1, k-(w'+M), t) + \varepsilon.$$

Since $w' + m_k \in M$ we conclude that

$$d(K, W, t) \le \sup_{k\in K} N(x_1, k-(w'+m_k), t) \le \sup_{k\in K} N(x_1, k-(w'+M), t) + \varepsilon$$

Thus,
$$d(K, W, t) \le \sup_{k\in K} N(x_1, k-(w'+M), t) \tag{4}$$

By (3) and (4) we get, $d(K, W, t) \le \sup_{k\in K} N(x_1, k-(w'+M), t) < d(K, W, t)$, and this is a contradiction. Therefore $w_0 + M \in S^t_{W/M}(K/M)$ and the proof is completed.

Corollary 4.8: Let M be a t-proximinal subspace of a fuzzy anti-2-normed linear space (X,N) and $W \supseteq M$ a subspace X. If W is simultaneous t-proximinal then W/M is a simultaneous t-proximinal subspace of X/M.

Corollary 4.9: Let M be a t-proximinal subspace of a fuzzy anti-2-normed linear space (X,N) and $W \supseteq M$ a subspace X. If W is simultaneous t-proximinal then then for each F-bounded set K in X,

$$Q(S_W^t(K)) \subseteq S_{W/M}^t(K/M).$$

Theorem 4.10: Let M be a t-proximinal subspace of a fuzzy anti-2-normed linear space (X,N) and $W \supseteq M$ subspace of X. If K is F-bounded set in X such that $w_0 + M \in S_{W/M}^t(K/M)$ and $m_0 \in S_M^t(K - w_0)$, then $w_0 + m_0 \in S_W^t(K)$.

Proof: In view of Lemma 4.4, for $t > 0$ we have,

$$\begin{aligned}
\sup_{k\in K} N(x_1,(k-w_0)-m_0,t) &= \inf_{m\in M}\sup_{k\in K} N(x_1,(k-w_0)-m,t) \\
&= \sup_{k\in K}\inf_{m\in M} N(x_1,k-(w_0+m),t) \\
&= \sup_{k\in K} N(x_1,k-(w_0+M),t) \\
&\le \sup_{k\in K} N(x_1,k-(w+M),t) \quad \forall\ w\in W \\
&\le \sup_{k\in K} N(x_1,k-w,t) \quad \forall\ w\in W.
\end{aligned}$$

Hence, $\sup_{k\in K} N(x_1,k-(w_0+m_0),t) \le \sup_{k\in K} N(x_1,k-w,t) \ \ \forall\ w\in W$.

But $w_0 + m_0 \in W$. Then $w_0 + m_0 \in S_W^t(K)$ and so the proof is completed.

Theorem 4.11: Let M be a t-proximinal subspace of a fuzzy anti-2-normed linear space (X,N) and $W \supseteq M$ a simultaneous t-proximinal subspace of X. Then for each F-bounded set K in X,

$$Q(S_W^t(K)) = S_{W/M}^t(K/M)$$

Proof: By Corollary 4.9, we obtain $Q(S_W^t(K)) \subseteq S_{W/M}^t(K/M)$. Also by Lemma 4.6, W/M is simultaneous t-proximinal in X/M. Now let, $w_0 + M \in S_{W/M}^t(K/M)$, where $w_0 \in W$. By simultaneous t-proximinality of M there exists $m_0 \in M$ such that $m_0 \in S_M^t(K - w_0)$. Then in view of Theorem 4.10, we conclude that $w_0 + m_0 \in S_W^t(K)$. Therefore $w_0 + M \in Q(S_W^t(K))$ and the proof is completed.

Corollary 4.12: Let W and M be subspaces of a fuzzy anti-2-normed linear space (X,N). If M is simultaneous t-proximinal then the following assertions are equivalent:

(i) W/M is simultaneous t-proximinal in X/M.
(ii) $W + M$ is simultaneous t-proximinal in X.

Proof: $(i) \Rightarrow (ii)$. Let K be an arbitrary F-bounded set in X. Then by Lemma 4.6, K/M is a F-bounded set in X/M. Since $(W+M)/M = W/M$ and M are simultaneous t-proximinal it follows that there exists $w_0 + M \in (W+M)/M$ and $m_0 \in M$ such that $w_0 + M \in S_{(W+M)/M}^t(K/M)$ and $m_0 \in S_M^t(K - w_0)$. By Theorem 4.10, we have $w_0 + m_0 \in S_{W+M}^t(K)$. This shows that $W + M$ is simultaneous t-proximinal in X.

$(ii) \Rightarrow (i)$. Since $W+M$ is simultaneous *t*-proximinal and $W+M \supseteq M$, by Corollary 4.8, we have $(W+M)/M = W/M$ is simultaneous *t*-proximinal.

Theorem 4.13: Let *W* and *M* be subspaces of a fuzzy anti-2-normed linear space (X, N). If *M* is simultaneous *t*-Chebyshev then the following assertions are equivalent:
(i) W/M is simultaneous *t*-Chebyshev in X/M.

(ii) $W+M$ is simultaneous *t*-Chebyshev in *X*.

Proof: $(i) \Rightarrow (ii)$, By hypothesis $(W+M)/M = W/M$ is simultaneous *t*-Chebyshev. Assume that (ii) is false. Then some *F*-bounded subset *K* of *X* has two distinct simultaneous *t*-best approximations such as l_0 and l_1 in $W+M$. Thus we have,

$$l_0,\ l_1 \in S^t_{W+M}(K). \tag{5}$$

Since $W+M \supseteq M$ by lemma 4.6, l_0+M, $l_1+M \in S^t_{(W+M)/M}(K/M) = S^t_{W/M}(K/M)$.

Since W/M is simultaneous *t*-Chebyshev, $l_0+M = l_1+M$. So there exists $0 \neq m_0 \in M$ such that $l_1 = l_0 + m_0$.

By (5) for all $t > 0$,

$$\begin{aligned}\sup_{k\in K} N(x_1, (k-l_0)-m_0, t) &= \sup_{k\in K} N(x_1, k-l_1, t)\\ &= \sup_{k\in K} N(x_1, k-l_0, t)\\ &= d(K, W+M, t)\\ &= d(K-l_0, W+M, t) \le d(K-l_0, M, t)\end{aligned}$$

This shows that both *m* and zero are simultaneous *t*-best approximations to $S-l_0$ from *M* and this is a contradiction.

$(ii) \Rightarrow (i)$. Assume that (i) does not hold. Then for some *F*-bounded subset *K* of *X*, K/M has two distinct simultaneous *t*-best approximations such as $w+M$ and $w'+M$ in W/M. Thus $w-w' \notin M$. Since *M* is simultaneous *t*-proximinal there exists simultaneous *t*-best approximations *m* and m' to $K-w$ and $K-w'$ from *M* respectively. Therefore $m \in S^t_M(K-w)$ and $m' \in S^t_M(K-w')$. Since $W+M \supseteq M$, $w+M$ and $w'+M$ are in $S^t_{W/M}(K/M) = S^t_{(K+M)/M}(K/M)$, by Theorem 4.10, $w+m$ and $w'+m' \in S^t_{W+M}(K)$. But $W+M$ is simultaneous *t*-Chebyshev. Thus $w+m = w'+m'$ and so $w-w' \in M$, which is a contradiction.

Corollary 4.14: Let *M* be simultaneous *t*-Chebyshev subspace of a fuzzy anti-2-normed linear space (X, N). If $W \supseteq M$ is a simultaneous *t*-Chebyshev subspace in *X*, then the following assertions are equivalent:
(i) *W* is simultaneous *t*-Chebyshev in *X*.
(ii) W/M is simultaneous *t*-Chebyshev in X/M.

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