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FIXED POINT THEOREM FOR GENERALIZED R - WEAKLY COMMUTING MAPPING

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ABSTRACT

A number of authors have defined various contractive type self-mapping of metric spaces which are generalizations of well-known Banach contraction principle and have used the same technique. The contractive condition on maps produce suitable iterations, which give Cauchy sequence and a hypothesis of completeness in the range containing these sequences. The concept of generalized R- weakly commuting mappings has been used in probabilistic metric space. Using this we have obtained a number of fixed point theorems in the light of some new contractive conditions.

1. INTRODUCTION

Banach contraction principle had many applications but it is suffering from a drawback of assuming *f* to be continuous. In 1968, kannan surpassed this hurdle. He took a number *a* with 0 < *a* < ½, such that for all $x, y \in X$, $d(fx, fy) \leq a[d(x, fx) + d(y, fy)]$. Then *f* has a unique fixed point. This result has been extended by Chattetrjee [1] and others. In 1969, Meir and Keeler obtained a remarkable generalization of the Banach contraction principle. They proved that if for every $\varepsilon > 0$ there exist $\delta > 0$ such that, $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon$. Then *f* has a fixed point. Many authors generalized the above result and gave very useful results in process of the study. Bonsall initiated the study of fixed points of sequence of mappings. Let *F* be a family of self-mappings on a set *X*. An element $x \in X$ is called a fixed point of *F* if $fx = x$, for every $f \in F$. It is obvious that if a map $f \in F$ commutes with any $s \in F$ and has unique fixed point, then so is true for *F*. After studing the idea of Bonsall, Singh [7], Stojakovic [10] has also obtained some results on sequence of contraction mappings. Most of the papers dealing with fixed points for sequence of maps fall into one of three categories. The first category assume that each pair f_i, f_j satisfies the same contractive condition and concludes that $\{f_n\}$ has a common fixed point. The second category assume that each f_n satisfies the same contractive condition and $\{f_n\}$ converges point wise to a limit function *f*. The conclusion is that *f* has a fixed point *p*, which is the limit of each of the fixed points p_n of f_n. The third type assume that each f_n has a fixed point p_n, and that $\{f_n\}$ converges uniformly to a function which satisfies a particular contractive condition, with *p* is a fixed point of *f*. The conclusion is that $f_n \to p$. Rhoades extended a fourth class of theorems for a sequence of maps. In this the function f_i, f_j satisfy pair wise contraction principle, but with different contractive constants. The conclusion is that the sequence has a common unique fixed point. The sequences produce a limit point, which becomes a fixed point of the mapping. The contractive condition on mapping has two roles; first they assure that certain iterations are Cauchy, and second, they assure the uniqueness of fixed point.

Some common fixed point theorems using sequence which are not necessarily obtained as a sequence of iterates of certain mappings are motivated by a result of Jungck [3]. He proved that a continuous self-mapping *f* of a complete metric space (*X*,*d*) has a fixed point provided there exists $q \in (0,1)$ and a mapping $g : X \to X$ which commute with *f* and satisfies

(a) $g(X) \subseteq f(X)$
(b) $d(gx, gy) \leq q d(fx, fy)$, for all $x, y \in X$. Then *g* and *f* have unique common fixed point.

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The above theorem promoted the commutative maps as a tool for generalizing some of the results. Subsequently, using commuting- map concept, a variety of variations and generalizations of the above theorem were obtained by Yeh [12], Park [5], Singh and Pant [8]. In 1982, Sessa [6] introduce a generalization of the commuting map-concept by saying that maps $f, g:(X,d)\to(X,d)$ are weakly commuting if $d(fgx, gfx)\le d(fx,gx)$ for $x\in X$.

This property is strictly weaker than the commutativity. In 1986, Jungck [4] introduced the concept of compatibility, which is weaker than weakly commutativity.

The above concept of S. Sessa [6] was generalized by Singh and Pant [8] by introducing the definition of weakly commuting mappings in probabilistic metric space. In the sequel Dimri and Gairola [2] defined R-weakly commuting mappings in probabilistic metric space and proved some common fixed point theorems.

The above developments motivated Piyush Kumar Tripathi [11] to introduce the definition of generalized R- weakly commuting mappings in probabilistic metric space, generalizing the definition of R- weakly commuting mappings defined by Dimmri and Gairola [2] and Singh and Pant [8]. As a consequence of this definition, Piyush Kumar Tripathi [11] proved some common fixed point theorems using a lemma of Singh and Pant [8].

Definition [8]: Two self mappings *f* and *g* on a probabilistic metric space *X* will be called weakly commuting if

$$F_{fgp,gfp}(x)\ge F_{fp,gp}(x)\ \forall\ p\in X \text{ and } x>0\,.$$

In 1998 R.C. Dimiri and U.C. Gairola [2] revised the above definition of S.L. Singh and B.D. Pant and called R-weakly commuting mappings.

Definition [2]: Two self mappings *f* and *g* on a probabilistic metric space *X* is said to R- weakly commuting if there exist a real number $R>0$ such that $F_{fgp,gfp}(Rx)\ge F_{fp,gp}(x)\ \forall\ p\in X$ and $x>0$.

Piyush Kumar Tripathi [11] defined generalized R-weakly commuting mappings as,

Definition[11]: Two self mappings *f* and *g* on a probabilistic metric space *X* be called generalized R- weakly commuting if there exist a real number $R>0$ such that

$$F_{fgp,gfq}(Rx)\ge F_{fp,gq}(x)\ \forall\ p,q\in X \text{ and } x>0\,.$$

From the definition of generalized R- weakly commuting mappings we notice that generalized R- weakly commuting mappings implies R- weakly commuting mappings without holding converse.

Following is the useful lemma proved by S.L. Singh and B.D. Pant [8].

1.1 Lemma: Suppose $\{p_n\}$ is a sequence in Menger probabilistic metric space (*X*, *F*, *t*), where *t* is continuous and $t(x,x)\ge x\ \forall\ x\in[0,1]$. If $\exists\ k\in(0,1)$ such that $\forall x\ >0$ and positive integer *n*, $F_{p_n,p_{n+1}}(kx)\ge F_{p_{n-1},p_n}(x)$. Then $\{p_n\}$ is a Cauchy sequence.

2. MAIN RESULTS

2.1 Theorem: Suppose (*X*, *F*, *t*) be a complete Menger probabilistic metric space, where $F_{p,q}$ is strictly increasing distribution function in (0, 1), $\forall p\ne q$. Let *f* and *g* are two pair of self mappings on *X*, satisfying.

(a) if $\exists\ k\in(0,1)$ s.t $F_{fp,fq}(kx)\ge F_{gp,gq}(x)\ \forall\ p,q\in X$ and $x>0$

(b) $f(X)\subset g(X)$ and f is continuous.

(c) f and g are generalized R - weakly commuting mappings.

Then *f* and *g* have unique common fixed point.

Proof: Let $p_0\in X$. Choose $p_1\in X$ such that $f(p_0)=g(p_1)$, this can be done because $f(X)\subset g(X)$, so we can construct a sequence $\{p_n\}$ such that $f(p_n)=g(p_{n+1})$ .Then

$$F_{fp_n,fp_{n+1}}(kx)\ge F_{gp_n,gp_{n+1}}(x)=F_{fp_{n-1},fp_n}(x)$$

i.e. $F_{fp_n, fp_{n+1}}(kx) \geq F_{fp_{n-1}, fp_n}$, so by lemma 2.1.1 [S 12] $\{fp_n\}$ is a Cauchy sequence. Since (X,F,t) is complete so $fp_n \to z \in X$, $gp_n \to z$. Since f is continuous so $ffp_n \to fz \in X$ and $fgp_n \to fz$.

Again since f and g are generalized R- weakly commuting mappings so $F_{fgp_n, gfp_n}(Rx) \geq F_{fp_n, gp_n}(x)$. Taking $n \to \infty$, we get $F_{fz, gfp_n}(Rx) \geq F_{z,z}(x) = 1$,

Therefore $F_{fz, gfp_n}(Rx) = 1$, so $gfp_n \to fz$.

Claim: z is a common fixed point of f and g. First we prove that $z = f(z)$ otherwise $F_{fp_n, ffp_n}(kx) \geq F_{gp_n, gfp_n}$, taking $n \to \infty$ we get $F_{z, fz}(kx) \geq F_{z, fz}(x)$ which is not possible because $F_{p,q}$ is strictly increasing distribution function in (0,1), $\forall p \neq q$,but in our case $kx < x$.

Therefore $z = f(z)$. Since $f(X) \subset g(X)$ so $\exists\, z_1 \in X$ such that $z = fz = gz_1$.

Now,

$F_{ffp_n, fz_1}(kx) \geq F_{gfp_n, gz_1}(x)$. Taking $n \to \infty$, we get $F_{fz, fz_1}(kx) \geq F_{fz, fz}(x) = 1$, that is

$F_{fz, fz_1}(kx) = 1$, i.e. we get $z = fz = fz_1 = gz_1$.

Again,

$$F_{fz, gz}(Rx) = F_{fgz_1, gfz_1}(Rx) \geq F_{fz_1, gz_1}(x) = 1, \text{ so } F_{fz, gz}(Rx) = 1 \Rightarrow fz = gz = z.$$

Therefore z is common fixed point of f and g. For uniqueness suppose z and z' are two common fixed point of f and g.
Then $F_{z,z'}(kx) = F_{fz, fz'}(kx) \geq F_{gz, gz'}(x) = F_{z,z'}(x) \Rightarrow F_{z,z'}(kx) \geq F_{z,z'}(x)$.

Which is not possible because $F_{p,q}$ is strictly increasing distribution function in (0,1) $\forall p \neq q$, but in our case $kx < x$.
Therefore z is unique common fixed point of f and g.

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