International Journal of Mathematical Archive-9(4), 2018, 98-104
Available online through www.ijma.info ISSN 2229 – 5046

LEFT STAR AND RIGHT STAR PARTIAL ORDERING OF CONJUGATE UNITARY MATRICES

A. GOVINDARASU# AND S. SASSICALA#1

PG and Research Department of Mathematics,
A. V. C. College (Autonomous) Mannampandal, Mayiladuthurai-609 305, India.

(Received On: 14-03-18; Revised & Accepted On: 02-04-18)

ABSTRACT

The concept of Left star and Right star partial ordering of conjugate unitary matrices is introduced. Characterizations of Left star and Right star partial ordering of conjugate unitary matrices are obtained and derived some theorems.

Key Words: *Unitary matrix, Conjugate normal matrix, conjugate unitary matrix and Left Star and Right Star partial ordering.*

Mathematics Subject Classification: *15B10, 15B99.*

INTRODUCTION

Two partial orderings in the set of complex matrices are introduced by combining each of the conditions $A^*A = A^*B$ and $AA^* = BA^*$, which we define the star partial ordering, with one of the conditions Range(A)⊆Range $(B)$and Range(A^*)⊆Range (B^*), defined by Baksalary J.K and Mitra S.K in [2].

Some characterizations of the star partial ordering and rank subtractivity for matrices was discussed by R.E.Hartwig and G.P.H.Styan in [6]. Also some characterizations of left star, right star and partial ordering between matrices of the same size are obtained Hongxing Wang and Jin Xu in [7].

Jorma.K.Merikoski and Xiaoji Liu found several characterizations of $A \leq * B$, [8] in the case of normal matrices. A relationship between star and minus partial ordering was settled by J.K.Baksalary in [1]. In this paper we introduce the concept of left star and right star partial ordering of conjugate unitary matrices.

1.1 NOTATIONS

Let $C_{n\times n}$ be the space of nxn complex matrices of order n. For $A \in C_{n\times n}$. Let $\bar{A}$, A^*denote conjugate, conjugate transpose of a matrix A respectively. Also the permutation matrix Vsatisfies $V^T = \bar{V} = V^* = V$ and $V^2 = I$

A matrix $A \in C_{n\times n}$ is called unitary if $AA^* = A^*A = I$ [9]

A matrix $A \in C_{n\times n}$ is called normal if $AA^* = A^*A$ [5]

A matrix $A \in C_{n\times n}$ is called conjugate normal if $AA^* = \overline{A^*A}$ [3]

A matrix $A \in C_{n\times n}$ is called is called conjugate unitary if $AA^* = \overline{A^*A} = I$ [4]

2. LEFT STAR AND RIGHT STAR PARTIAL ORDERING OF CONJUGATE UNITARY MATRICES

Definition 2.1: Star Partial Ordering:- For $A, B \in C_{n\times n}$, $A \underset{*}{\leq} B$ iff $A^*A = A^*B$ and $AA^* = BA^*$.

Definition 2.2: Left Star Partial Ordering: Let $A, B \in C_{n\times n}$, $A * \leq B$ iff $A^*A = A^*B$ and Range(A)⊆Range (B).

Definition 2.3: Right Star Partial Ordering:- Let $A, B \in C_{n\times n}$, $A \leq * B$ if $AA^* = BA^*$ and Range(A^*)⊆Range (B^*).

Corresponding Author: S. Sassicala#1
 # PG and Research Department of Mathematics,
A. V. C. College (Autonomous) Mannampandal, Mayiladuthurai-609 305, India.

Theorem 2.4: Let $AV *\leq VA$. If $A \in C_{n\times n}$ be conjugate unitary matrix and $VA^* = VA, AV = A^*V$ then $\overline{VA}$ is involutary.

Proof:
Let $AV *\leq VA$.
$AV *\leq VA$
$\Longrightarrow (AV)^*(AV) = (AV)^*(VA)$ and Range $(AV) \subseteq$ Range(VA)
$\Longrightarrow V^*A^*AV = V^*A^*VA$
$\Longrightarrow VA^*AV = VA^*VA \quad (\because V^* = V)$
$\Longrightarrow VA^*AV = VA.VA \quad (\because VA^* = VA)$
$\Longrightarrow VA^*AV = (VA)^2$

Taking conjugate on both sides, we have,
$\overline{VA^*AV} = \overline{(VA)^2}$
$\overline{V^2} = \overline{(VA)^2} \quad (\because \overline{A^*A} = I)$
$\bar{I} = \overline{(VA)^2} \quad (\because V^2 = I)$
$I = (\overline{VA})^2 \quad (\because \bar{I} = I)$
$i.e., (\overline{VA})^2 = I$

Again let $AV *\leq VA$
$AV *\leq VA \Longrightarrow (AV)^*(AV) = (AV)^*(VA)$ and Range $(AV) \subseteq$ Range(VA)
$\Longrightarrow V^*A^*AV = V^*A^*VA$
$\Longrightarrow VA^*AV = VA^*VA \quad (\because V^* = V)$
$\Longrightarrow (VA^*)(AV) = (VA^*)(VA)$
$\Longrightarrow VAA^*V = VA.VA \ (\because VA^* = VA, AV = A^*V)$
$\Longrightarrow V(AA^*)V = (VA)^2$
$\Longrightarrow V^2 = (VA)^2 \quad (\because AA^* = I)$
$\Longrightarrow I = (VA)^2 \quad (\because V^2 = I)$

Taking conjugate on both sides, we have,
$\bar{I} = \overline{(VA)^2}$
$\Longrightarrow I = (\overline{VA})^2 \quad (\because \bar{I} = I)$
$i.e., (\overline{VA})^2 = I$
$\Longrightarrow \overline{VA}$ is involutary.

Theorem 2.5: Let $VA *\leq AV$. If $A \in C_{n\times n}$ be conjugate unitary matrix and $A^*V = AV, VA^* = VA$ then $\overline{VA}$ is involutary.

Proof: Similar proof of Theorem 2.4.

Theorem 2.6: Let $AV \leq* VA$. If $A \in C_{n\times n}$ be conjugate unitary matrix and $A^*V = AV, VA^* = VA,$ then (VA) is involutary.

Proof:
Let $AV \leq* VA$.
$AV \leq* VA$
$\Longrightarrow (AV)(AV)^* = (VA)(AV)^*$ and Range $(AV)^* \subseteq$Range$(VA)^*$
$\Longrightarrow AVV^*A^* = VAV^*A^*$
$\Longrightarrow AVVA^* = VAVA^* \quad (\because V^* = V)$
$\Longrightarrow AV^2A^* = VAVA \ (\because VA^* = VA)$
$\Longrightarrow AA^* = (VA)^2 \quad (\because V^2 = I)$
$\Longrightarrow I = (VA)^2 \quad (\because AA^* = I)$
$ie., \ (VA)^2 = I$

Again let $AV \leq* VA$.
$AV \leq* VA \Longrightarrow (AV)(AV)^* = (VA)(AV)^*$ and Range $(AV)^* \subseteq (VA)^*$
$\Longrightarrow AVV^*A^* = VAV^*A^*$
$\Longrightarrow AVVA^* = VAVA^* \quad (\because V^* = V)$
$(A^*V)(VA) = (VA)(VA) \quad (\because A^*V = AV, VA^* = VA)$
$\Longrightarrow A^*A = (VA)^2 \quad (\because V^2 = I)$

Taking conjugate on both sides, we have,

$\overline{A^*A} = \overline{(VA)^2}$

$\Rightarrow I = \overline{(VA)^2} \quad (\because \overline{A^*A} = I)$

Again taking conjugate on both sides, we have,

$\bar{I} = (VA)^2$

$\Rightarrow I = (VA)^2 \quad (\because \bar{I} = I)$

$ie.,$ $(VA)^2 = I$

$\Rightarrow (VA)$is involutary.

Theorem 2.7: Let $(VA) \leq * (AV)$.If $AV * \leq VA$. If $A \in C_{n\times n}$ be conjugate unitary matrix and $A^*V = AV, VA^* = VA$ then (AV) is involutary.

Proof: Similar proof of Theorem 2.6

Theorem 2.8: Let $A, B \in C_{n\times n}$ are conjugate unitary matrices and if Range(A) =Range(A^*),Range (B) =Range(B^*) then A$*\leq B \Rightarrow A^{-1} \leq * B^{-1}$

Proof: Let $A * \leq B$ if $A^*A = A^*B$ and Range$(A)\subseteq$Range (B)

$\Rightarrow$Pre and post multiply by V on both sides, we have,

$VA^*AV = VA^*BV$

Taking conjugate on both sides, we have,

$\overline{VA^*AV} = \overline{VA^*BV}$

$\Rightarrow \bar{V}\overline{A^*A}\bar{V} = \overline{VA^*BV}$

$\Rightarrow \bar{V}\bar{V} = \overline{VA^*BV} \quad (\because \overline{A^*A} = I)$

$\Rightarrow \overline{V^2} = \overline{VA^*BV}$

$\Rightarrow \bar{I} = \overline{VA^*BV} \quad (\because V^2 = I)$

Taking conjugate on both sides, we have,

$I = VA^*BV$

$\Rightarrow$Pre and post multiplying by V on both sides, we have,

$V^2 = A^*B$

$\Rightarrow I = A^*B \quad (\because V^2 = I)$

$\Rightarrow B^{-1} = A^*$

$\Rightarrow B^{-1} = A^{-1} \quad (\because AA^* = \overline{A^*A} = I \Rightarrow A^* = A^{-1})$

Post multiplying by $(A^{-1})^*$ on both sides, we have,

$(B^{-1})(A^{-1})^* = (A^{-1})(A^{-1})^*$

$ie.,$ $(A^{-1})(A^{-1})^* = (B^{-1})(A^{-1})^*$ (1)

Given Range(A) =Range(A^*), Range (B) =Range(B^*)

$\Rightarrow$ Range$(A^*)\subseteq$Range (B^*) (2)

Again let $A * \leq B$ if $A^*A = A^*B$ and Range$(A)\subseteq$Range (B)

$A^*A = A^*B$

Pre multiplying by $(A^*)^{-1}$on both sides, we have,

$(A^*)^{-1}A^*A = (A^*)^{-1}A^*B$

$\Rightarrow A = B$

Post multiplying by A^* on both sides, we have,

$AA^* = BA^*$

$\Rightarrow I = BA^* \quad (\because AA^* = I)$

$\Rightarrow B^{-1} = A^*$

$\Rightarrow B^{-1} = A^{-1} \quad (\because AA^* = \overline{A^*A} = I \Rightarrow A^* = A^{-1})$

Post multiplying $(A^{-1})^*$by on both sides, we have,

$(B^{-1})(A^{-1})^* = (A^{-1})(A^{-1})^*$

$ie.,$ $(A^{-1})(A^{-1})^* = (B^{-1})(A^{-1})^*$ (3)

Given Range(A) =Range(A^*), Range (B) =Range(B^*)
$\Longrightarrow$ Range$(A^*)\subseteq$Range (B^*) (4)

Therefore from equations,(1) ,(2),(3) &(4) ,we have,
$A{*}\leq B \Longrightarrow A^{-|} \leq{*}\, B^{-|}$

Theorem 2.9: If $A, B \in C_{n\times n}$ are conjugate unitary matrices and Range(A) =Range(A^*), Range (B) =Range(B^*) then $A\leq{*}\, B \Longrightarrow A^{-|}\, {*}\leq B^{-|}$

Proof: Similar proof of Theorem 2.8

Theorem 2.10: If $A, B \in C_{n\times n}$ are conjugate unitary matrices, then $A{*}\leq B$, $B{*}\leq C$
$\Longrightarrow A{*}\leq C$

Proof:
$A{*}\leq B \Longrightarrow A^*A = A^*B$ and Range$(A)\subseteq$Range (B)
$B{*}\leq C \Longrightarrow B^*B = B^*C$ and Range$(B)\subseteq$Range (C)
$B{*}\leq C \Longrightarrow B^*B = B^*C$
$\Longrightarrow B^*AA^*B = B^*C \quad (\because AA^* = I)$
$\Longrightarrow (B^*A)(A^*A) = (B^*C) \quad (\because A^*B = A^*A)$
$\Longrightarrow B^*AA^*A = B^*C$

Pre multiplying by $(B^*)^{-|}$ on both sides,we have,
$(B^*)^{-|}B^*AA^*A = (B^*)^{-|}B^*C$
$\Longrightarrow AA^*A = C$

Taking conjugate on both side, we have,
$\bar{A}\ \overline{A^*A} = \bar{C}$
$\Longrightarrow \bar{A} = \bar{C} \quad (\because \overline{A^*A} = I)$

Again taking conjugate, on both side, we have,
$A = C$

Pre multiplying by A^*on both sides we have,
$A^*A = A^*$ C (1)

Given Range$(A)\subseteq$Range $(B)$and Range$(B)\subseteq$Range (C)
$\Longrightarrow$Range$(A)\subseteq$Range (C) (2)

Again let $B{*}\leq C \Longrightarrow B^*B = B^*C$ and Range$(B)\subseteq$Range (C)
$B{*}\leq C \Longrightarrow B^*B = B^*C$
$\Longrightarrow B^*AA^*B = B^*C \quad (\because AA^* = I)$
$\Longrightarrow (B^*A)(A^*A) = (B^*C) \quad (\because A^*B = A^*A)$
$\Longrightarrow B^*AA^*A = B^*C$
$\Longrightarrow B^*A = B^*C \quad (\because AA^* = I)$

Pre multiplying by $(B^*)^{-|}$ on both sides, we have,
$A = C$

Pre multiplying by A^*on both sides we have,
$A^*A = A^*$ C (3)

Given Range$(A)\subseteq$Range (B) and Range$(B)\subseteq$Range (C)
$\Longrightarrow$Range$(A)\subseteq$Range (C) (4)
Therefore from eqns.(1) ,(2),(3) &(4) ,we have,
$A{*}\leq C$
That is $A{*}\leq B$, $B{*}\leq C \Longrightarrow A{*}\leq C$

Theorem 2.11: If $A, B \in C_{n\times n}$ are conjugate unitary matrices, then $A\leq{*}\, B$, $B\leq{*}\, C$
$\Longrightarrow A\leq{*}\, C$

Proof: Similar proof of Theorem 2.10

Theorem 2.12: If $A, B \in C_{n\times n}$ are conjugate unitary matrices, then A$\leq * B \Longrightarrow A^* * \leq B^*$

Proof:
A$\leq * B \Longrightarrow AA^* = BA^*$ and Range$(A^*)\subseteq$Range(B^*)
$AA^* = BA^*$

Post multiplying by A on both sides, we have,
$AA^*A = BA^*A$
$\Longrightarrow A = BA^*A \quad (\because AA^* = I)$

Taking conjugate on both sides, we have,
$\Longrightarrow \bar{A} = \bar{B}\, \overline{A^*A}$
$\Longrightarrow \bar{A} = \bar{B} \quad (\because \overline{A^*A} = I)$

Again taking conjugate on both sides, we have,
$\Longrightarrow A = B$

Pre multiplying by B^* on both sides, we have,
$\Longrightarrow B^*A = B^*B$

Taking conjugate on both sides, we have,
$\Longrightarrow \overline{B^*A} = I \ (\because \ \overline{B^*B} = I\ )$

Again taking conjugate on both sides, we have,
$\Longrightarrow B^*A = \bar{I}$
$\Longrightarrow B^*A = I \ \ (\because \bar{I} = I)$

Post multiplying by A^* on both sides, we have,
$\Longrightarrow B^*AA^* = A^*$
$\Longrightarrow B^* = A^* \quad (\because AA^* = I)$

Pre multiplying by B on both sides, we have,
$\Longrightarrow BB^* = BA^*$
$\Longrightarrow I = BA^* \quad (\because BB^* = I\ )$
$\Longrightarrow AA^* = BA^* \quad (\because AA^* = I)$

Taking conjugate transpose on both sides, we have,
$\Longrightarrow (AA^*)^* = (BA^*)^*$
$\Longrightarrow (A^*)^*A^* = (A^*)^*B^*$ (1)
$\Longrightarrow$Range$(A^*)\subseteq$Range (B^*) (2)

Therefore from eqns. (1) & (2) ,we have,
A$\leq * B \Longrightarrow A^* * \leq B^*$

Theorem 2.13: If $A, B \in C_{n\times n}$ are conjugate unitary matrices, then A$* \leq B \Longrightarrow A^* \leq * B^*$

Proof: Similar proof of Theorem 2.12

Theorem2.14: If $A, B \in C_{n\times n}$ are conjugate unitary matrices and Range$(A)\subseteq$Range (A^*A), Range$(B)\subseteq$Range (B^*B) then A $* \leq B \Longrightarrow A^*A * \leq B^*B$

Proof: Let A$* \leq B \Longrightarrow A^*A = A^*B$ and Range$(A)\subseteq$ Range(B)
$A^*A = A^*B$

Taking conjugate transpose on both sides, we have,
$(A^*A)^* = (A^*B)^*$
$\Longrightarrow A^*A = B^*A$
$\Longrightarrow A^*AA^*A = B^*BB^*A \quad (\because AA^* = I, BB^* = I\ )$
$\Longrightarrow (A^*A)(A^*A) = (B^*B)(B^*A)$

Taking conjugate transpose on both sides, we have,

$\Rightarrow\left((A^*A)(A^*A)\right)^* = \left((B^*B)(B^*A)\right)^*$
$\Rightarrow(A^*A)^*(A^*A)^* = (B^*A)^*(B^*B)^*$
$\Rightarrow(A^*A)^*(A^*A)^* = (A^*B)(B^*B)$
$\Rightarrow(A^*A)^*(A^*A) = (A^*A)(B^*B) \quad (\because (A^*A)^* = (A^*A))$
$\Rightarrow(A^*A)^*(A^*A) = (A^*A)^*(B^*B) \quad (\because (A^*A)^* = (A^*A))$ (1)

Again taking $A^*A = A^*B$

Taking conjugate on both sides, we have,

$\overline{A^*A} = \overline{A^*B}$
$I\,\overline{A^*A} = I\overline{A^*B}$
$\Rightarrow\overline{A^*A}\overline{A^*A} = \overline{B^*B}\overline{A^*B} \quad (\because \overline{A^*A} = I, \overline{B^*B} = I)$

Again taking conjugate on both sides, we have,

$\Rightarrow (A^*A)(A^*A) = (B^*B)(A^*B)$
$\Rightarrow (A^*A)(A^*A) = (B^*B)(A^*A) \quad (\because A^*B = A^*A)$

Taking conjugate transpose on both sides, we have,

$\Rightarrow\left((A^*A)(A^*A)\right)^* = \left((B^*B)(A^*A)\right)^*$
$\Rightarrow(A^*A)^*(A^*A)^* = (A^*A)^*(B^*B)^*$
$\Rightarrow(A^*A)^*(A^*A) = (A^*A)^*(B^*B) \quad (\because (A^*A)^* = (A^*A), (B^*B)^* = (B^*B))$ (2)

Given Range$(A)\subseteq$Range (A^*A), Range$(B)\subseteq$Range (B^*B)
$\Rightarrow$ Range $(A^*A)\subseteq$Range (B^*B) (3)

Therefore, from equations (1), (2) and (3), we have,
$A *\leq B \Rightarrow A^*A *\leq B^*B$

Theorem 2.15: If $A, B \in C_{n\times n}$ are conjugate unitary matrices, and Range$(A)\subseteq$ Range (A^2), Range$(B)\subseteq$Range (B^2) then $A *\leq B \Rightarrow A^2 *\leq B^2$

Proof: Let $A *\leq B$
$A *\leq B \Rightarrow A^*A = A^*B$ and Range$(A)\subseteq$Range (B)
$A^*A = A^*B$

Pre multiplying by A on both sides, we have,

$AA^*A = AA^*B$
$A = B \quad (\because AA^* = I)$
$A.I = B.I$
$AAA^* = BBB^* \quad (\because AA^* = I, BB^* = I)$
$A^2A^* = B^2B^*$

Post multiplying by B on both sides, we have,

$A^2A^*B = B^2B^*B$
$A^2A^*A = B^2B^*B \quad (\because A^*B = A^*A)$

Taking conjugate on both sides, we have,

$\overline{A^2}\,\overline{A^*A} = \overline{B^2}\overline{B^*B}$
$\overline{A^2}=\overline{B^2} \quad (\because \overline{A^*A} = I, \overline{B^*B} = I)$

Again taking conjugate on both sides, we have,

$A^2 = B^2$

Pre multiplying by $(A^2)^*$on both sides we have,

$(A^2)^*A^2 = (A^2)^*B^2$ (1)

Given Range$(A)\subseteq$Range (A^2) and Range$(B)\subseteq$Range (B^2)
$\Rightarrow$ Range $(A^2)\subseteq$Range (B^2) (2)

Therefore, from equations (1) and (2) we have

$A^2 *\leq B^2$

That is $A *\leq B \Longrightarrow A^2 *\leq B^2$

Theorem 2.16: If $A, B \in C_{n\times n}$ are conjugate unitary matrices, and Range$(A^*)\subseteq$ Range$(A^*)^2$, Range$(B^*)\subseteq$Range $(B^*)^2$ then $A \leq* B \Longrightarrow A^2 \leq* B^2$

Proof: Similar proof of Theorem 2.16.

CONCLUSION

Left star and Right star partial ordering of conjugate unitary matrix was defined and theorems related to left star and right star partial ordering of conjugate unitary matrices are derived.

REFERENCES

1. Baksalary J.K. "Relationship between the star and minus ordering", Lin.Alg.Appl., 82, 163-167, (1986).
2. Baksalary J.K and Mitra S.K. Left-star and right-star partial orderings. *Linear Algebra Appl.*, 149:73–89, (1991).
3. Faßbender A, H.Kh.D.Ikramov B., 'Conjugate-normal matrices: a survey' Lin.Alg. Appl. Vol., 429, pp.1425-1441, (2008).
4. Govindarasu.A and Sassicala.S 'Characterizations of conjugate unitary matrices' Journal of ultra scientist of physical science, JUSPS-A Vol.29 (1), 33 – 39 (2017).
5. Grone. R., Johnson. C.R,.Sa. E.M., Wolkowicz. H., 'Normal matrices' *Lin. Alg. Appl. 87, pp.213*-225, (1987).
6. Hartwig.R.E and Styan.G.P.H, "On some characterization of the star partial ordering for matrices and rank subtractivity."Lin. Alg. Appl., Vol82, pp145-161, (1986).
7. Hongxing Wang and Jin Xu "Some Results on Characterizations of Matrix Partial Orderings", Journal of Applied Mathematics., Article 11(2014).
8. Jorma.K.Merikoski and Xiaoji Liu "On the star partial ordering of normal matrices." Journal of inequalities in Pure and Applied Mathematics, Vol.7, issue 7, Art 17, (2006).
9. Sharma.A.K., 'Text book of matrix theory 'DPH Mathematics series, (2004).