

**ON TRANS-SASAKIAN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS**

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**ABSTRACT**

*The present paper deals with a study of trans-Sasakian manifolds satisfying certain curvature conditions on contact conformal curvature tensor. It is shown that the trans-Sasakian manifold satisfying the curvature conditions  $C_0(\xi, X).R = 0$ ,  $C_0(\xi, X).C_0 = 0$  and  $\tilde{C}(\xi, X).C_0 = 0$  is contact conformally semi-symmetric manifold.*

**Keywords:** *Trans-Sasakian manifold, contact conformal curvature tensor, semi-symmetric, concircular curvature tensor.*

**MSC 2010:** *53C15, 53C25, 53D15.*

**1. INTRODUCTION**

In [5], authors studied on the sixteen classes of almost Hermitian manifolds and their linear invariants. They considered unitary group  $U(n)$  on a certain space  $W$  and studied that the representation of  $U(n)$  on  $W$  has four irreducible components,  $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$ . Among these components  $W_3 \oplus W_4$  corresponds to the class of Hermitian manifolds. A new class of almost contact metric structure, called trans-Sasakian structure was studied in [9] which is an analogue of a locally conformal Kaehler structure on an almost Hermitian manifold. An almost contact metric structure  $(\varphi, \xi, \eta, g)$  (where the symbols have their usual meanings) on  $M$  is trans-Sasakian [9] if  $(M \times R, J, G)$  belongs to the class  $W_4$ , where  $J$  is the almost complex structure on  $M \times R$  defined by

$$J\left(X, f \frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X) \frac{d}{dt}\right) \tag{1.1}$$

for any vector field  $X$  on  $M$ , where  $G$  is the product metric on  $M \times R$ . Trans-Sasakian manifold is the trans-Sasakian structure of type  $(\alpha, \beta)$ , where  $\alpha$  and  $\beta$  are smooth functions on  $M$ . Trans-Sasakian manifold of type  $(0,0), (\alpha,0)$  and  $(0, \beta)$  are cosymplectic [1],  $\alpha$ -Sasakian and  $\beta$ -Kenmotsu manifold [2,7] respectively. Trans-Sasakian manifolds have been studied in [3], [10] and by others.

In [6], contact conformal curvature tensor field was introduced and defined by Jeong *et al.* in a  $(2n + 1)$ -dimensional Sasakian manifold.

**2. PRELIMINARIES**

Let  $M$  be a  $(2n + 1)$ -dimensional differentiable manifold with an almost contact metric structure  $(\varphi, \xi, \eta, g)$ , where  $\varphi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field,  $\eta$  is a 1-form and  $g$  is a compatible Riemannian metric such that [1]

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all  $X, Y \in TM$ . The fundamental 2-form  $\Phi$  of the almost contact metric structure  $(\varphi, \xi, \eta, g)$  is defined as

$$\Phi(X, Y) = g(X, \varphi Y) = -g(\varphi X, Y), \tag{2.4}$$

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Since  $\varphi$  is a skew-symmetric with respect to  $g$ .

An almost contact metric manifold  $M$  is called trans-Sasakian manifold if [9]

$$(\nabla_x \varphi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}, \tag{2.5}$$

where  $\nabla$  is Levi-Civita connection of Riemannian metric  $g$  and  $\alpha, \beta$  are smooth functions on  $M$ .

From (2.5) it follows that

$$\nabla_x \xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\}, \tag{2.6}$$

$$(\nabla_x \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{2.7}$$

In a  $(2n + 1)$ -dimensional trans-Sasakian manifold  $M$ , the following relations hold [3]:

$$R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (Y\beta)\varphi^2(X) + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (X\beta)\varphi^2(Y), \tag{2.8}$$

$$R(\xi, X)Y = (\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(Y)X] + 2\alpha\beta[g(\varphi Y, X)\xi - \eta(Y)\varphi X] + (Y\alpha)\varphi X + g(\varphi Y, X)(grad\alpha) + (Y\beta)[X - \eta(X)\xi] - g(\varphi X, \varphi Y)(grad\beta), \tag{2.9}$$

$$2\alpha\beta + (\xi\alpha) = 0, \tag{2.10}$$

$$S(X, \xi) = [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\varphi X)\alpha) - (2n - 1)(X\beta), \tag{2.11}$$

$$\eta(R(X, Y)Z) = -g(R(X, Y)\xi, Z), \tag{2.12}$$

$$\eta(R(X, Y)\xi) = \eta(R(\xi, X)\xi) = \eta(R(X, \xi)\xi) = 0, \tag{2.13}$$

$$\eta(R(\xi, X)Y) = (\alpha^2 - \beta^2 - (\xi\beta))g(\varphi X, \varphi Y). \tag{2.14}$$

In a  $(2n + 1)$ -dimensional trans-Sasakian manifold if we put  $\varphi(grad\alpha) = (2n - 1)grad\beta$ , then some of the above relations are reduced to

$$(\xi\beta) = 0, \tag{2.15}$$

$$S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X), \tag{2.16}$$

$$\eta(R(\xi, X)Y) = (\alpha^2 - \beta^2)g(\varphi X, \varphi Y), \tag{2.17}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)\{\eta(X)\xi - X\}, \tag{2.18}$$

$$R(X, \xi)\xi = -R(\xi, X)\xi. \tag{2.19}$$

We shall calculate all the results under the condition  $\varphi(grad\alpha) = (2n - 1)grad\beta$ .

In a  $(2n + 1)$ -dimensional trans-Sasakian manifold the contact conformal curvature tensor field  $C_0$  and the concircular curvature tensor  $\tilde{C}$  of type (1, 3) are defined by

$$\begin{aligned} C_0(X, Y)Z &= R(X, Y)Z + \frac{1}{2n}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY + S(X, Z)\eta(Y)\xi - S(Y, Z)\eta(X)\xi \\ &\quad + \eta(X)\eta(Z)QY - \eta(Y)\eta(Z)QX + S(\phi X, Z)\phi Y \\ &\quad - S(\phi Y, Z)\phi X + g(X, \phi Z)Q(\phi Y) - g(Y, \phi Z)Q(\phi X) \\ &\quad + 2g(X, \phi Y)Q(\phi Z) + 2S(\phi X, Y)\phi Z\} \\ &\quad + \frac{1}{2n2(n+1)}\{2n^2 - n - 2 + \frac{(n+2)r}{2n}\}\{g(Y, \phi Z)\phi X \\ &\quad - g(X, \phi Z)\phi Y - 2g(X, \phi Y)\phi Z\} + \frac{1}{2n(n+1)}\{n + 2 - \frac{(3n+2)r}{2n}\} \\ &\quad \times \{g(Y, Z)X - g(X, Z)Y\} - \frac{1}{2n(n+1)}\{4n^2 + 5n + 2 - \frac{(3n+2)r}{2n}\} \\ &\quad \times \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\} \end{aligned} \tag{2.20}$$

where  $R, S, Q$  and  $r$  denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively [6] and

$$\tilde{C}(U, V)Z = R(U, V)Z - \frac{r}{2n(2n+1)}\{g(V, Z)U - g(U, Z)V\} \tag{2.21}$$

for all  $X, Y, Z, U, V \in \chi(M)$  respectively [4].

From (2.20), we have

$$C_0(X, Y)\xi = R(X, Y)\xi + (\alpha^2 - \beta^2 - 2)\{\eta(Y)X - \eta(X)Y\}, \tag{2.22}$$

$$C_0(\xi, X)Y = R(\xi, X)Y + (\alpha^2 - \beta^2 - 2)\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.23}$$

$$\eta(C_0(X, Y)Z) = \eta(R(X, Y)Z) + (\alpha^2 - \beta^2 - 2)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{2.24}$$

$$\eta(C_0(X, Y)\xi) = 0, \tag{2.25}$$

$$\eta(C_0(\xi, X)Y) = 2(\alpha^2 - \beta^2 - 1)\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{2.26}$$

From (2.21), we also have

$$\tilde{C}(\xi, V)Z = R(\xi, V)Z - \frac{r}{2n(2n+1)}\{g(V, Z)\xi - \eta(Z)V\}. \tag{2.27}$$

**Definition:** A  $(2n + 1)$ -dimensional trans-Sasakian manifold  $M$  is said to be an Einstein manifold if its Ricci tensor  $S$  of type  $(0, 2)$  is of the form

$$S(X, Y) = \lambda g(X, Y) \tag{2.28}$$

where  $\lambda$  is a scalar function on  $M$ .

### 3. TRANS-SASAKIAN MANIFOLD SATISFYING $C_0(\xi, X).S = 0$

Now, we prove the following result

**Theorem 3.1:** If a trans-Sasakian manifold  $M$  of dimension  $(2n + 1)$  satisfies the condition  $C_0(\xi, X).S = 0$ , then the manifold is an Einstein manifold and its scalar curvature is  $r = 2n(2n + 1)(\alpha^2 - \beta^2)$ .

**Proof:** Let  $M$  be a  $(2n + 1)$ -dimensional trans-Sasakian manifold which satisfies the condition

$$(C_0(\xi, X).S)(U, V) = 0. \tag{3.1}$$

(3.1) implies that

$$S(C_0(\xi, X)U, V) + S(U, C_0(\xi, X)V) = 0. \tag{3.2}$$

Putting  $V = \xi$  in (3.2) and using (2.16) and (2.23), we obtain

$$S(X, U) = 2n(\alpha^2 - \beta^2)g(X, U). \tag{3.3}$$

Let  $\{e_i\}, i = 1, 2, \dots, 2n + 1$  be an orthonormal basis of the tangent space at any point of the manifold. Putting  $X = U = e_i$  in (3.3) and summing over  $i, 1 \leq i \leq 2n + 1$ , we get

$$r = 2n(2n + 1)(\alpha^2 - \beta^2). \tag{3.4}$$

In view of (3.3) and (3.4) it follows that the manifold  $M$  is an Einstein manifold with scalar curvature  $r = 2n(2n + 1)(\alpha^2 - \beta^2)$ . This completes the proof of the theorem.

### 4. TRANS-SASAKIAN MANIFOLD SATISFYING $\tilde{C}(\xi, X).C_0 = 0$

**Theorem 4.1:** Let  $M$  be a  $(2n + 1)$ -dimensional trans-Sasakian manifold. If  $M$  satisfies the condition  $\tilde{C}(\xi, X).C_0 = 0$ , then the manifold is contact conformally semi-symmetric.

**Proof:** By definition we have

$$\tilde{C}(\xi, X)C_0(U, V)Z - C_0(\tilde{C}(\xi, X)U, V)Z - C_0(U, \tilde{C}(\xi, X)V)Z - C_0(U, V)\tilde{C}(\xi, X)Z = 0 \tag{4.1}$$

Using (2.21) in (4.1) we get

$$\begin{aligned}
 & R(\xi, X)C_0(U, V)Z - \frac{r}{2n(2n+1)}\{g(X, C_0(U, V)Z)\xi \\
 & - \eta(C_0(U, V)Z)X\} - C_0\left(R(\xi, X)U - \frac{r}{2n(2n+1)}\{g(X, U)\xi - \eta(U)X\}, V\right)Z \\
 & - C_0\left(U, R(\xi, X)V - \frac{r}{2n(2n+1)}\{g(X, V)\xi - \eta(V)X\}\right)Z \\
 & - C_0(U, V)\left(R(\xi, X)Z - \frac{r}{2n(2n+1)}\{g(X, Z)\xi - \eta(Z)X\}\right) = 0 \\
 & [R(\xi, X)C_0(U, V)Z - C_0(R(\xi, X)U, V)Z - C_0(U, R(\xi, X)V)Z \\
 & - C_0(U, V)R(\xi, X)Z] + \frac{r}{2n(2n+1)}[\eta(C_0(U, V)Z)X
 \end{aligned}$$

Or,

$$\begin{aligned}
 & -g(X, C_0(U, V)Z)\xi + g(X, U)C_0(\xi, V)Z - \eta(U)C_0(X, V)Z \\
 & + g(X, V)C_0(U, \xi)Z - \eta(V)C_0(U, X)Z + g(X, Z)C_0(U, V)\xi \\
 & - \eta(Z)C_0(U, V)X] = 0
 \end{aligned}$$

Or,

$$\begin{aligned}
 & (R(\xi, X).C_0)(U, V)Z + \frac{r}{2n(2n+1)}[\eta(C_0(U, V)Z)X \\
 & - g(X, C_0(U, V)Z)\xi + g(X, U)C_0(\xi, V)Z - \eta(U)C_0(X, V)Z \\
 & + g(X, V)C_0(U, \xi)Z - \eta(V)C_0(U, X)Z + g(X, Z)C_0(U, V)\xi \\
 & - \eta(Z)C_0(U, V)X] = 0.
 \end{aligned} \tag{4.2}$$

Putting  $Z = \xi$  in (4.2) and taking inner product on both sides by  $\xi$  we get

$$g(R(\xi, X).C_0(U, V)\xi, \xi) + \frac{r}{2n(2n+1)}[-g(X, C_0(U, V)\xi) - \eta(C_0(U, V)X)] = 0. \tag{4.3}$$

Using (2.22) and (2.24) in (4.3), we get

$$g(R(\xi, X).C_0(U, V)\xi, \xi) + \frac{r}{2n(2n+1)}[-g(X, R(U, V)\xi) - \eta(R(U, V)X)]. \tag{4.4}$$

In view of (2.8), (2.12) and (4.4) we obtain

$$R(\xi, X).C_0(U, V)\xi = 0. \tag{4.5}$$

Equation (4.5) implies that the manifold is contact conformally semi-symmetric. This completes the proof of the theorem.

### 5. TRANS-SASAKIAN MANIFOLD SATISFYING $C_0(\xi, X).C_0 = 0$

**Theorem 5.1:** Let  $M$  be a  $(2n + 1)$ -dimensional trans-Sasakian manifold. The manifold  $M$  satisfying the condition  $C_0(\xi, X).C_0 = 0$  is contact conformally semi-symmetric if

$$(\alpha^2 - \beta^2)\{2g(V, Z)X - g(X, V)Z - g(X, Z)V\} - g(R(\xi, V)Z, X)\xi - R(X, V)Z = 0.$$

**Proof:** Let  $M$  be a trans-Sasakian manifold of dimension  $(2n + 1)$  which satisfies the condition

$C_0(\xi, X).C_0(U, V)Z = 0$ , then by definition we have

$$C_0(\xi, X)C_0(U, V)Z - C_0(C_0(\xi, X)U, V)Z - C_0(U, C_0(\xi, X)V)Z - C_0(U, V)C_0(\xi, X)Z = 0 \tag{5.1}$$

Using (2.23) in (5.1) we get

$$\begin{aligned}
 & R(\xi, X).C_0(U, V)Z + (\alpha^2 - \beta^2 - 2)[g(X, C_0(U, V)Z)\xi \\
 & - \eta(C_0(U, V)Z)X - g(X, U)C_0(\xi, V)Z + \eta(U)C_0(X, V)Z \\
 & - g(X, V)C_0(U, \xi)Z + \eta(V)C_0(U, X)Z - g(X, Z)C_0(U, V)\xi \\
 & + \eta(Z)C_0(U, V)X] = 0.
 \end{aligned} \tag{5.2}$$

Taking inner product on both sides of (5.2) by  $\xi$  and using (2.25) we obtain

$$\begin{aligned} &g(R(\xi, X).C_0(U, V)Z, \xi) + (\alpha^2 - \beta^2 - 2)[g(X, C_0(U, V)Z) \\ &- \eta(X)\eta(C_0(U, V)Z) - g(X, U)\eta(C_0(\xi, V)Z) \\ &+ \eta(U)\eta(C_0(X, V)Z) - g(X, V)\eta(C_0(U, \xi)Z) \\ &+ \eta(V)\eta(C_0(U, X)Z) - \eta(Z)\eta(C_0(U, V)X)] = 0. \end{aligned} \quad (5.3)$$

Putting  $U = \xi$  in (5.3) and using (2.23), (2.24) and (2.26), we get

$$\begin{aligned} &g(R(\xi, X).C_0(\xi, V)Z, \xi) + (\alpha^2 - \beta^2 - 2)[g(R(\xi, V)Z, X) \\ &+ \eta(R(X, V)Z) + (\alpha^2 - \beta^2)\{\eta(Z)g(X, V) - 2\eta(X)g(V, Z) \\ &+ \eta(V)g(X, Z)\}] = 0. \end{aligned} \quad (5.4)$$

Equation (5.4) implies that

$$\begin{aligned} R(\xi, X).C_0(\xi, V)Z = &(\alpha^2 - \beta^2 - 2)[(\alpha^2 - \beta^2)\{2g(V, Z)X \\ &- g(X, V)Z - g(X, Z)V\} - g(R(\xi, V)Z, X)\xi \\ &- R(X, V)Z]. \end{aligned} \quad (5.5)$$

From (5.5) it follows that the manifold  $M$  is contact conformally semi-symmetric if the right hand side vanishes i.e., if

$$(\alpha^2 - \beta^2)\{2g(V, Z)X - g(X, V)Z - g(X, Z)V\} - g(R(\xi, V)Z, X)\xi - R(X, V)Z = 0 \quad (5.6)$$

This completes the proof of the theorem.

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