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SOME ASPECTS OF MODULES OVER A RING AND ITS COLORINGS

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ABSTRACT

In this paper we investigate some graph theoretic aspects of modules, in particular coloring of its vertices. In our discussion especially we have dealt here the cases of finite dimensional modules and in some cases some sort of finiteness conditions on its annihilators. Results obtained here are basically on its finite cliques and chromatic numbers.

Key words: *Annihilators, Nil-radical, Goldie ring, Goldie module, f-clique, f-chromatic number.*

AMS Subject Classification2000: *16D10, 16D80, 16Y30, 16Y60, 15N40, 16P60, 05C15.*

1. INTRODUCTION

Here we investigate some graph theoretic aspects related to coloring of elements of modules over a ring introducing the notion of adjacency in between its elements. In an additive algebraic structure, the notion of a product is not well acceptable. In-stead, of course, there may arise such an operation with the help of another structure equipped with two operations, one of them bearing the nature of so-called product. And the same may be obtained when we be able to give an external composition to the preceding algebraic structure with the help of the later one. If M is with such an additive structure and A is the other one with operation '•', at least bearing the semi-group character such that we have the map : $M \times A \to M$ with $(m,a) \to m \cdot a$ (we will write it as just ma), then for a map $f : M \to A$, it is possible to define a product '$\bullet_f$' such that for $a,b \in A, a \bullet_f b = a \cdot f(b)$ (or just $af(b)$). The system $(M,+,\bullet_f)$ is now our sought structure for having a system with the chosen M with a pseudo product $\bullet_f$. And this motivates us to define the notion of so-called adjacency between two elements in the chosen algebraic structure, in spite of absence of such an explicit product operation beforehand. And such an arrangement may help one to make his stage ready for investigation of the adjacency character of elements in the said algebraic paired structure (with one operation only). For such an algebraic structure our expected graph is the triplet (V,E,f) with V as the set of vertices-the elements of M, E -the set of edges where for $a,b (\in M)$, "*a is f-adjacent to b*" if $a \bullet_f b = 0$ (or '*a and b are f-adjacent*') .The pseudo product described here may coincide with the product available in R when M coincides with the ring R with respect to some definite map f (in particular the identity map) . The technique explored here elegantly appears as an effective one for introducing the notion of coloring in case of the module in discussion.

It is observed here some interesting and elegant connections between finiteness of the chromatic number of such a graph and the notion of a so-called Goldie module [4] which reads as a module M over a ring R where,

(1) M is finite dimensional
(2) R Satisfies the ascending chain condition (acc.) for annihilators of subsets of M in R.

i.e. any collection of annihilators of subsets of M in R satisfies the maximal condition.

First we give some interesting facts revealing the interrelation between such a module and the corresponding f-algebraic triplet, when the latter structure appears as a result of an attachment of respective order preserving map f.

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In this context we would like to refer Chartrand et al [3] and Chowdhury [5], [9] for graph and near-ring theoretic pre-requisites. Moreover we refer the works of Beck [1], Bruijn et-al [2] for coloring of rings.

It is to be mentioned that the results due to A.W. Goldie on the structure of semi-prime rings [7], [8] seem to be still relevant due to its elegancy.

If M is a right module over R (denoted M_R) and if $f : M \to R$ is an R-homomorphism, then the triple $(M,+,\bullet_f)$ where $m_i \bullet_f m_j = m_i f(m_j)$ is a ring. We call it an *h-ring* denote M_f .Unless otherwise specified, through out this paper M (or M_R) will mean a right module over R and f is an R-homomorphism from M to R. A module M_R is an *f- color- module* if the *f-chromatic number* [§ 3][denoted $\chi(M_f)$] of the corresponding *h-ring* M_f is finite .The main results of this paper are dealt here from two main aspects viz. *h-ring* and that of coloring of module such as:

A sub-module of the module appears as a right ideal of the corresponding *h-ring* structure of the same. The *h-ring* structure of a module M_R gives rise to the *h-ring* structure of the quotient of M modulo an ideal contained in the kernel of the map induced by the preceding map f . M_R is a right Goldie module if and only if the corresponding *h-ring* is right Goldie. Prime character of a sub-module modulo the kernel of the homomorphism leads to the prime character of the image sub-module of the ring as a module over itself .Pseudo product defined with the help of just a map f from the module M_R to the ring R gives rise to an abelian near-ring structure. A group structure is sufficient to make it into a right near ring with respect to the *pseudo- product* defined. Occurrence of infinite number of f *-right finite* elements in a module M_R is a sufficient condition for having an infinite f *-clique* in it. An infinite f *-clique* in an *h-ring* helps to get such an induced clique in the quotient modulo the kernel and conversely .The existence of an infinite f *-clique* in a module is a necessary condition for containing a nilpotent element in the *h-ring* M_f .A finite dimensional module turns out to be a right Goldie *h- ring* if it is with only finite f *-cliques*. An *h-ring* originated from a finite dimensional f *-color module* is a right Goldie ring .The f *-chromatic number* of the *h-ring* never exceeds the i *-chromatic number* of the attached ring R_i i.e. R (In the sense of Beck [1]). In case of an f *-color-module* M_R , the *h-ring* M_f modulo right annihilator of any subset of M is a ψ *-color-module*, (ψ is induced by f).

2. EXAMPLES AND OBSERVATIONS

The following examples justify how different types of f-maps helps one to get various types of algebraic structures such as right near ring ,left nearing, semi ring, semi-near ring etc from some given once.

Example: 1 Consider the module $Z[x]$ over the ring Z . Define $f : Z[x] \to Z$ by $f\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i$, $a_i \in Z$, then $(Z[x],+,\bullet_f)$ is a ring. Hence $Z[x]$ is an *h-ring.*

Example: 2 $Z_{30}[x]$ is an additive abelian group of polynomials over Z_{30} .Then $(Z_{30}[x],+,\circ)$ is a right nearing where' $\circ$ 'is defined as the composition of mappings .Then $I_6[x]$ is an additive subgroup of $Z_{30}[x]$ where $I_6 = \{0,6,12,18,24\}$.Let $N = (Z_{30}[x],+,\circ)$ and $G = (I_6[x],+)$ .Now G is an N-subgroup of N . Define $\phi : G \to N$ by $\phi(g) = 2g$ .Then $(G,+,\bullet_\phi)$ is a right near-ring .Thus G is a ϕ *-right near-ring*.

Example: 3 Consider the semi group $(G,\cdot)$ where $G = \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}\right\}$.

Let $A_0 = \begin{pmatrix}0&0\\0&0\end{pmatrix}$, $A_1 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A_2 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $A_3 = \begin{pmatrix}1&1\\1&1\end{pmatrix}$.

Here

$\cdot$	A_0	A_1	A_2	A_3
A_0	A_0	A_0	A_0	A_0
A_1	A_0	A_1	A_2	A_3
A_2	A_0	A_2	A_1	A_3
A_3	A_0	A_3	A_3	A_0

And $(Z_2,+,\cdot)$ is a ring. We define $f: G \to Z_2$ by $f(A) = (a_{11})$ where $A = (a_{ij})$.

Here $\bullet_f$ is right distributive but not left distributive. Hence $(G,\cdot,\bullet_f)$ is a *semi near-ring* (right)
Thus G is f *-semi near-ring* (right).

Note:
(1) If we define $f: G \to Z_2$ by $f(A) = \det(A)$ where $A = (a_{ij})$ then $(G,\cdot,\bullet_f)$ is a *semi near-ring* (right).
(2) If we define $f: G \to Z_2$ by $f(A) = traceA$ or $f(A) = \sum a_{ij}$ where $A = (a_{ij})$.Then $(G,\cdot,\bullet_f)$ is a *semi-ring*.[

Here $traceA$ =0 and $\sum a_{ij}$ =0 as $a_{ij} \in Z_2$]

Example: 4 Let R be a ring and $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in R \right\}$ .Then M is a module over R .Define $f: M \to R$ by $f(A) = traceA$ .Then '$\bullet_f$' is both left and right distributive .Thus $(M,+,\bullet_f)$ is a ring. Thus M is an *h-ring*.

Example: 5 Let R be a ring .Then R is a module over itself. Consider a map $f: R \to R$ defined as $f(x) = 2x$. Then '$\bullet_f$' is both left and right distributive. Thus $(R,+,\bullet_f)$ is a ring. Hence R is an *h-ring*.

Note: In a given ring we can insert a number of rings.

Example: 6 Consider the group $G = \{0,a,b\}$ under the addition defined by the following table.

+	0	a	b
0	0	a	b
a	a	b	0
b	b	0	a

And let the set $N = M(G) = \left\{ \alpha \mid \alpha : G \xrightarrow{map} G \right\}$ .We define $\phi: G \to N$ by

.	0	a	b
$\phi(0) = \alpha_1$	0	0	a
$\phi(a) = \alpha_2$	0	a	0
$\phi(b) = \alpha_3$	a	0	0

Then only two colors are needed to color the vertices of $(G,+,\bullet_\phi)$. Thus $\chi(G,+,\bullet_\phi)$ is 2. If $\phi: G \to N$, defined by

.	0	a	b
$\phi(0) = \alpha_1$	0	0	0
$\phi(a) = \alpha_2$	a	a	a
$\phi(b) = \alpha_3$	b	b	b

Then $(G,+,\bullet_\phi)$ is a left near-ring .Only one color is needed to color the vertices of $(G,+,\bullet_\phi)$.Thus $\chi(G,+,\bullet_\phi)$ is 1.

Example: 7 We consider $R = Z_6 = \{0,1,2,3,4,5\}$. $f : R \to R$ by $f(x) = 2x$ .Then $(R,+,\bullet_f)$ is a ring. We need only three colors to color the ring $(R,+,\bullet_f)$.Thus $\chi(R_f) = 3$.If we define $g : R \to R$ by $g(x) = 3x$, then only four colors are needed to color the ring $(R,+,\bullet_g)$.Thus $\chi(R_g) = 4$ .If we consider the identity map $i : R \to R$, we need three colors to color the ring $(R,+,\bullet_i)$.Thus $\chi(R_i) = 3$.

3. PRELIMINARIES

Definitions: The graph of the module M is the triplet (V, E, f) where the elements of the module M is the set V of vertices, E is the set of edges and f is an R-homomorphism from a module M_R to the module R_R .An element $x(\in M)$ is f *- adjacent* to y if $x \bullet_f y = 0$ or we call it "x and y are *f-adjacent* ''(clearly such a graph is a directed one) .Two elements "x and y are f *- adjacent* to each other" if $x \bullet_f y = 0 = y \bullet_f x$. If x and y are not distinct then (x, x) is an f *- loop* .A subset C (finite / infinite) of M is an f *-clique* if any two elements of C are f *-adjacent* to each other. Thus pseudo product of any two distinct element of C vanishes. The minimum numbers of colors which can be assigned to the elements of M so that no two *f- adjacent* elements have same color is the f *-chromatic* (denoted $\chi(M_f)$) number of M . A module M is an *f-color module* if the f *- chromatic* number of the corresponding *h- ring* M_f is finite.

If $\chi(M_f) = n$, whatever be the choice of the R-homomorphism $f(\neq 0)$, then n is the chromatic number of the module M denoted $\chi(M) = n$

One may interestingly note here that chromatic number of a ring, what Beck has insisted [1] need not be unique. The $\chi(R)$ *in the sense what Beck has meant is nothing but our* $\chi(R_i)$ *where* $i : R \to R$ *is the identity homomorphism.*

A module M_R is said to be *prime* if for any sub-module N_R of M_R $Ann(M_R) = Ann(N_R)$. The M *divisor of* R is $Z(M) = \{r \in R \mid mr = 0, for\quad some\quad non\quad zero\quad m \in M\}$.*Associated prime* of M Is $Ass(M) = \{P \mid P = Ann(S_i), S_i \subseteq M\}$ Where $Ann(S_i)$ is a prime ideal of R. An element x in M is *f-left-finite* (*f-right- finite*) if $M \bullet_f x$ ($x \bullet_f M$) is finite. Nil radical of a ring R is the intersection of all prime ideals of R .A ring R is *reduced* if *nil-radical* $J = (0)$. It is to be noted that in case of a non-commutative ring R, for every $r \in R, rR = R$ then R is a division ring. On the other hand, one can say in case of a division ring R, $aR = R$ for all $a \in R$. More generally one would like to note as Dheena [6] has introduced the notion of a *Duo-Near-ring* (sub-commutative) which may be considered as analogous to what has been mentioned above. This justifiably motivates us as it is seen in a case of a near-ring [6] to give the notion of the *left* (*right*) *sub-commutativity* in a ring R is as follows. A ring R is *left-sub commutative* if for $a,b \in R$ we have $d \in R$ such that $ab = da$ (*right sub- commutative* if for $a,b \in R$ we have $c \in R$ such that $ab = bc$) . The ring R is *sub-commutative*, if for $a,b \in R$ we have $c,d \in R$ such that $ab = bc = da$. The number of elements in a set is denoted by # here.

Lemma: 3.1 If M is a module over R and $f : M \to R$ is an R-homomorphism, then $(M,+,\bullet_f)$ where $m_i \bullet_f m_j = m_i f(m_j)$ is a ring.

Note:

(1) This ring is our sought *h-ring*.

(2) Normally such an *h-ring* is not commutative. It is easily observed that in case of a PID D if $f : D \to D$ is a D-homomorphism with (say) $f(x) = mx, (m \in D)$, then D_f is commutative. It happens to be true for any such f.

(3) If even the ring R does not contain unity 1, yet when M is an R-module, M_f may contain its unity. Following explanation justifies what we have meant.

Let $R = \{(2m,2n) / m,n \in Z\}$ .Then $(R,+,\cdot)$ is a commutative ring and is without unity such that for

$(2m,2n),(2a,2b) \in R$, $(2m,2n) + (2a,2b) = (2(m+a),2(n+b))$ and

$(2m,2n) \cdot (2a,2b) = (2ma,4nb)$. Now $(Z,+)$ is a module over R with the map $Z \times R \to Z$ where $m(2\alpha,2\beta) = m\alpha \in Z$ It is easy to see that for $m,n \in Z$ and $(2,2n) \in R$, $m(2,2n) = m$.

i.e. R has no unity, but $(2,2n) = e \in R$ is such that $me = m \forall m \in Z$ .Thus e acts as a right unity for the module Z over the ring R .For any $n \in Z$ it is true. Now $f : Z \to R$ with $f(m) = (2m,0)$ is an R-homomorphism. And for 1∈Z, we have $m \bullet_f 1 = mf(1) = m(2,0) = m \forall m \in Z$ .Thus 1∈Z acts as the unity in $(Z)_f$ //

(4) When $f = I$ -the identity map and the attached ring is commutative then the *h-ring* M_f is commutative.

Lemma 3.2: Let M be a module over a ring R and $f : M \to R$ is an R-homomorphism. If N_R is a sub-module of M_R then G is a right ideal of M_f .

Proof: Here for $n_1, n_2 \in N$, $n_1 + n_2 \in N$ .And for, $n \in N$ and $m \in M$, $n \bullet_f m = nf(m) \in N$ .Thus N is a right ideal of M_f .

Note: It is not safe to claim that for N - a right ideal of M_f , we should get the corresponding additive group $(N,+)$ as a sub-module of M_R However the following result reveals that our claim become true obviously if f is onto. For, $n_1, n_2 \in N_R, n_1 + n_2 \in N$ And, $n \in N, r \in R$ gives $nr = nf(m) = n \bullet_f m \in N$ .Thus $(N,+)$ is a sub-module of M_R .

Lemma 3.3: Let M be a module over a ring R and $f : M \to R$ is an R-homomorphism. Then for any subset $S(\neq \phi) \subseteq M$, $S \bullet_f M$ is a sub-module M_R

The following lemma follows immediately from the ring structure of M_f making $\frac{M}{N}$ is a (right) M_f module.

Lemma 3.4: Let N_R be a sub-module of M_R .Then $\frac{M}{N}$ is a (right) M_f -module.

Lemma 3.5 (a): Let N_R be a finite sub-module of the module M_R .Then $\frac{N : x}{Annx}$ is a finite R-module.
[$N : x = \{r \in R \mid xr \in N\}$].

Lemma 3.5(b): Let N_R be a finite sub-module of M_R .Then $\frac{N : x}{r_{M_f}(x)}$ is finite with respect to the ring N_f . [$N_f : x = \{m \in M \mid x \bullet_f m \in N\}$].

Proof: Here N is a right ideal of M_f [Lemma 3:2].Now $r_{M_f}(x + N) = \{m \in M \mid (x+N) \bullet_f m = N\}$

$\{m\in M \mid (x+N)f(m)=N\}=\{m\in M \mid (xf(m)+N)=N\}=\{m\in M \mid xf(m)\in N\}=$
$\{m\in M \mid x\bullet_f m\in N\}=N:x$ .Now the map $\phi: r_{M_f}(x+N)\to x\bullet_f r_{M_f}(x+N)$
Defined by $\phi(\alpha)=x\bullet_f \alpha$ is an epimorphism and $\ker(\phi)=r_{M_f}(x)$.

Therefore $\dfrac{r_{M_f}(x+N)}{r_{M_f}(x)}\cong x\bullet_f r_{M_f}(x+N)$ .Now $x\bullet_f r_{M_f}(x+N)$
$=\{x\bullet_f m \mid m\in r_{M_f}(x+N)\}=\{xf(m) \mid m\in r_{M_f}(x+N)\}\subseteq N$
[Since gives $m\in N:x$ .Thus $x\bullet_f m\in N$ i.e. $xf(m)\in N$] Therefore the result follows since N is finite //

Lemma 3.6: Let the module M_R be with an infinite numbers of f *-right-finite* elements. Then M contains an infinite f *-clique.*

Proof: Let $x_1, x_2, \ldots$ be different f *-right-finite* elements in M . Then the elements $x_1\bullet_f x_2, \ldots, x_1\bullet_f x_n$ belong to the finite right ideal $x_1\bullet_f M$ .Now for some infinite subsequence a_n of {2, 3… n…} we have $x_1\cdot_f x_{a_1} = x_1\cdot_f x_{a_2}$ =……..Consider the sequence $x_{a_1}, x_{a_2}, \ldots$ and repeat the procedure. Continuing in this way we construct a subsequence $y_1, y_2, \ldots, y_n, \ldots$ of the sequence $x_1, x_2, \ldots$ such that $y_i\bullet_f y_j = y_i\bullet_f y_k$ where $i, j>k$ .In this subsequence $y_1=x_1, y_2=x_{a_1}$ …..etc. Put $z_{ij}=y_i-y_j$ .Now $z_{ij}\bullet_f z_{kr}=z_{ij}f(z_{kr})=(y_i-y_j)f(y_k-y_r)$ $=(y_i-y_j)(f(y_k)-f(y_r)) \quad = y_if(y_k)-y_if(y_r)-y_jf(y_k)+y_jf(y_r)$ =0.We shall now construct an infinite f *-clique* of M

We consider $z_{1,2}\bullet_f z_{3,4}=z_{1,2}\bullet_f z_{3,5}$ =0.Since, $z_{1,2}\neq z_{3,5}$, at least one of $z_{3,4}$ and $z_{3,5}$ is different From $z_{1,2}$. If for instance $z_{1,2}\neq z_{3,5}$ then $\{z_{1,2}, z_{3,5}\}$ is an f *-clique* with two elements. We observe that $z_{6,7}, z_{6,8}, z_{6,9}$ are different and if for instance $z_{6,9}\notin\{z_{1,2}, z_{3,5}\}$ then $\{z_{1,2}, z_{3,5}, z_{6,9}\}$ is an *f- clique* with three elements. Continuing in this way we get an infinite f *-clique* of M .

Note: The above result holds in case of a module M with an infinite number of f *-left -finite* elements too. The following lemma is also easily follows from what have been described in itself.

Lemma: 3.7 Let I be a left ideal of R . And $(Ann_M I=)N(\subseteq Ker(f))$ is a sub-module of M . Then, $\left(\dfrac{M}{N}\right)_\psi$ is an *h-ring* for the ψ induced by f where $\psi:\dfrac{M}{N}\to R$ is such that $\psi(m+N)=f(m)$.

Lemma: 3.8 Let $N=Ker(f)$ be a finite sub-module of the module M_R . Then the module M contains an infinite f *-clique* if and only if $\frac{M}{N}$ contains an infinite ψ –clique where ψ is induced by f with $\psi(m+N)=f(m)$.

Proof: Let C be an infinite f *-clique* of M .Then we will show that $\overline{C}=\{c+N \mid c\in C\}$ is a *ψ-clique* of $\frac{M}{N}$ .Now for $c_i+N, c_j+N\in C$, $(c_i+N)\bullet_\psi (c_j+N) = (c_i+N)\psi(c_j+N)=(c_i+N)f(c_j)= c_if(c_j)+N$ =0+N $=\overline{0}$. Similarly $(c_j+N)\bullet_\psi (c_i+N)=\overline{0}\forall i\neq j$. Thus $\overline{C}=\{c+N \mid c\in C\}$ is a *ψ-clique* of $\dfrac{M}{N}$ .And $(c_i+N)\neq(c_j+N)$, $(i\neq j)$ as $N=Ker(f)$. Conversely, let $\{\overline{c_i}\}_{i=1}^\infty$ be an infinite *ψ-clique* of $\dfrac{M}{N}$ .Hence, $c_i\bullet_f c_j\in N$ for $i\neq j$ .Therefore the set of products $\{c_i\bullet_f c_j\}_{i\neq j}$ is a finite set. Now we can construct an infinite f *-clique* of M as it is done in the Lemma 3.6.

4. MAIN RESULTS

The main results are here presented as follows.

4.1: Characterizations of f-chromatic number

A module M_R (**or** $_R M$) is a *zero module* if $M = 0$

Theorem: 4.1.1 $\chi(M_f) = 1$ if and only if M is a zero module over R.

Proof: Assume $\chi(M_f) = 1$.

Case I: Suppose, $M \neq 0$ and $f = 0$ .Then there is an $m_i \neq 0 \in M$ .Let $m_j \in M$ be another element of M . Then $m_i \bullet_f m_j = m_i f(m_j) = m_i \cdot 0 = 0$, $i \neq j$. Thus m_i, m_j are *f-adjacent*, a contradiction to $\chi(M_f) = 1$.

Case II: Assume that $M = 0$ and $f \neq 0$. This case does not arise [since f is an R-homomorphism]

Case III: Assume that $M \neq 0$ and $f \neq 0$. Let $m_i \neq 0 \in M$. Now $m_i \bullet_f 0 = m_i f(0) = m_i .0 = 0$ gives $m_i, 0$ are f *-adjacent* - a contradiction to $\chi(M_f) = 1$.Thus, $M = 0$ and $f = 0$. Hence M is a zero module. Conversely assume M is a zero module .Then $(0,0)$ is an f-loop. Thus only one color is sufficient to color the vertex of the module M .Thus $\chi(M_f) = 1$.

Note:

(1) In case of a graph $(R, E, 0)$, $\chi(R_0)$ is the order of the ring i.e. $0(R)$

(2) $M(\neq 0)$ is a module over a ring R with unity. And $f : M \to R$ is any R-homomorphism. Then $2 \leq \chi(M_f) \leq \#$ M.

(3) If $\chi(R_f) = k$, for all $f(\neq \hat{0})$, then we say that k is the chromatic number of R denoted as $\chi(R)$. For a field F, $\chi(F_f) = 2$, for all $f(\neq \hat{0})$ as we have only one such f viz: the identity map.

(4) It is to be noted that the chromatic number of a ring R as discussed in [1] appears as a very restricted one. It is not difficult to see that in case of the ring of integer modulo n, when n is other than prime, the f *-chromatic number* varies for different R-homomorphism f .

The problem may be considered as an open one to investigate under what condition one may expect the existence of $\chi(R)$

Proposition: 4.1.2 Let $p_1, p_2, ..., p_k, q_1, ..., q_r$ be different primes and $N = p_1^{2n_1} ... p_k^{2n_k} . q_1^{2m_1+1} ...... q_r^{2m_r+1}$ and let $f : Z_N \to Z$ be a homomorphism such that $f(\overline{m})$= k, where k is the remainder when m is divided by N .

Then $\chi(Z_N)_f$ =clique$(Z_N)_f$ = $p_1^{n_1} \cdots p_k^{n_k} . q_1^{m_1+1} ...... q_r^{m_r+1}$ + r

Proof: Let y$_0$ = $p_1^{n_1} ... p_k^{n_k} . q_1^{m_1+1} ..... q_r^{m_r+1}$ .Then $f(\overline{y_0}) = y_0$ .Consider the ideal

$Z_N \bullet_f \overline{\alpha} = \{\overline{z_i} \bullet_f \overline{\alpha} \mid \overline{z_i} \in Z_N\}$[$y_0 = f(\overline{y_0}) = f(\overline{\alpha})$, say].

Now

$(\overline{z_i} \bullet_f \overline{\alpha}) \bullet_f (\overline{z_j} \bullet_f \overline{\alpha}) = (\overline{z_i} f(\overline{\alpha})) \bullet_f (\overline{z_j} f(\overline{\alpha})) = \overline{z_i} f(\overline{\alpha}) f(\overline{z_j}) f(\overline{\alpha}) = \overline{z_i} f(\overline{z_j}) f(\overline{\alpha}) f(\overline{\alpha}) ==$ $\overline{z_i} f(\overline{z_j}) f(\overline{y_0}) y_0 = \overline{z_i} f(\overline{z_j}) f(\overline{y_0} y_0) = \overline{z_i} f(\overline{z_j}) f((\overline{y_0})^2) = \overline{z_i} f(\overline{z_j}) f(0) = 0$ .Similarly,

$\left(\overline{z_j} \bullet_f \overline{\alpha}\right) \bullet_f \left(\overline{z_i} \bullet_f \overline{\alpha}\right)$=0 .Thus $Z_N \bullet_f \overline{\alpha}$ is an f *-clique*, whose cardinality is $p_1^{n_1} \ldots p_k^{n_k}.q_1^{m_1} \ldots\ldots q_r^{m_r}$ Then the set $\overline{C} = Z_N \bullet_f \overline{\alpha} \cup \left\{\overline{y_1} \bullet_f \overline{\alpha}, \ldots, \overline{y_n} \bullet_f \overline{\alpha}\right\}$is an f *-clique* of $(Z_N)_f$ containing $s = p_1^{n_1} \ldots p_k^{n_k}.q_1^{m_1} \ldots\ldots q_r^{m_r} + r$ numbers of elements. Hence clique$(Z_N)_f \geq s$. In order to show$\chi(Z_N)_f \leq s$, we attach a distinct color to each of the elements of the f *-clique* $\overline{C}$. Furthermore let $\overline{x_i} \bullet_f \overline{\alpha} = \overline{x_i} f\left(\overline{\alpha}\right) = \dfrac{\overline{N}}{p_i^{n_i}} f\left(\overline{\alpha}\right)$, where $\overline{x_i} = \dfrac{\overline{N}}{p_i^{n_i}}, 1 \leq i \leq k$ .Then $\overline{x_1} \bullet_f \overline{\alpha}, \overline{x_2} \bullet_f \alpha, \ldots., \overline{x_k} \bullet_f \overline{\alpha}$ are also elements of $\overline{C}$.[Since $\overline{x_i} f\left(\overline{\alpha}\right) \bullet_f \overline{x_j} f\left(\overline{\alpha}\right)$=0. And hence have been equipped with a color. Let $C\left(\overline{y}\right)$ denote the color of an element $\overline{y}$ and color the remaining elements of$(Z_N)_f$ as follows. Pick $\overline{x} \notin Z_N \bullet_f \overline{\alpha}$. If $p_1^{n_1} \ldots p_k^{n_k}$ divides x define, $C\left(\overline{x}\right) = C\left(\overline{y_j} \bullet_f \overline{\alpha}\right)$, where $j - mini\{i \mid q_i^{m_i+1} \nmid x\}$. If $p_1^{n_1} \ldots p_k^{n_k}$ does not divides x define, $C\left(\overline{x}\right) = C\left(\overline{x_j} \bullet_f \overline{\alpha}\right)$, where j = min{i | $p_i^{n_i} \nmid$ x}.It is easily seen that this coloring attaches different colors to *f-adjacent* vertices. Thus $\chi\left((Z_N)_f\right) \leq s$ Hence $\chi\left((Z_N)_f\right) = s$.

This is as an example of an *h-ring* where f *-chromatic number* coincides with clique of the *h-ring*.

Theorem: 4.1.3 The f *-chromatic* number of an *h-ring* M_f with respect to the simple graph of a *left sub-commutative* ring R never exceeds i *-chromatic number* of R_i where i is the identity homomorphism.

Proof: It can be shown that for every f *-clique* of the module M there corresponds an i *-clique* in R. Next we show if *clique* R_i is finite then clique M_f is also finite. Suppose that $\overline{C} = \{m_1, m_2, \ldots\ldots\} \subseteq M$ be an infinite f *-clique* of M . Consider the subset $C = \{f(m_1), f(m_2), \ldots\ldots\ldots\}$ of R.

Now $f(m_i)f(m_j) = f(m_i f(m_j)) = f(m_i \bullet_f m_j) = f(0) = 0$ gives C is an i *-clique* of R which is infinite, a contradiction. Thus clique $M_f < \infty$ .Now let $C = \{r_1, r_2, \ldots\ldots\} \subseteq R$ is an infinite i *-clique* of R . Then consider $\overline{C} = \{mr_1, mr_2, \ldots..\} \subseteq M$ .Since $mr_i \bullet_f mr_j = mr_i f(mr_j) = mr_i f(m) r_j = mrr_i r_j = mr0 = 0$ [$\forall i \neq j,$ for some $r \in R$].Therefore $\overline{C}$ is an f *-clique* of M .Thus we get $\chi(M_f) \leq \chi(R_i)$.

Note: However if M is a module over a zero ring R then M itself is an f *-clique* here, and since $\chi(R) = 1$ for zero ring R , therefore $\chi(R) \leq \chi(M_f)$ .Thus this inequality does not hold if M is a module over a zero ring R .

Theorem: 4.1.4 Let M be a module over a *right- sub commutative ring* R . If the *h-ring* M_f contains a nilpotent element, which is not f *-right- finite*, then M_f contains an infinite f *-clique.*

Theorem: 4.1.5 Let M be an f *-color module* over a *right sub- commutative ring* R and S be any subset of M **.** Then $\dfrac{M}{r_{M_f}(S)}$ is a ψ *- color module* where ψ is induced by f .

Proof: Let $\left\{\overline{m_1}, \ldots., \overline{m_n}\right\} \subseteq \dfrac{M}{r_{M_f}(S)}$ is a ψ *-clique* of $\dfrac{M}{r_{M_f}(S)}$

Now $\overline{m_i} \bullet_\psi \overline{m_j} = \overline{0}$ gives that $m_i \bullet_f m_j \in r_{M_f}(S)$ .Thus $S \bullet_f (m_i \bullet_f m_j) = 0$

Now, $\left\{ \sum_{finite} s_i \bullet_f m_j \mid s_i \in S, m_j \in M \right\}$ is an f - *clique* of $S \bullet_f M$. Clearly, clique $\frac{M}{r_{M_f}(S)} \leq$ clique $S \bullet_f M$.

Now every f - *clique* of $S \bullet_f M$ is also an f -*clique* of M_f as $S \bullet_f M$ is a sub- ring of M_f .Since M is f -*color module* therefore clique $M_f < \infty$ which gives clique $S \bullet_f M$ is finite .Thus clique $\frac{M}{r_{M_f}(S)}$ is finite .Thus, $\frac{M}{r_{M_f}(S)}$ is a ψ -*color module*.//

4.2: Inheritance of Goldie Characters

Theorem: 4.2.1 Let M be a module over R **.**If M satisfies the acc. on annihilators of subsets of M in R then M_f has the acc. on right annihilators of its subsets.

Proof: We consider an ascending chain of right annihilators say,
$r_{M_f}(S_1) \subseteq r_{M_f}(S_2) \subseteq r_{M_f}(S_3) \subseteq \ldots$ where $S_i \subseteq M, i = 1,2,\ldots\ldots$ we claim $r_{M_f}(S_k) = r_{M_f}(S_{k+1}) = \ldots$ for some $k \in Z^+$. Now for every ascending chain of right annihilators say, $r_{M_f}(S_1) \subseteq r_{M_f}(S_2) \subseteq r_{M_f}(S_3) \subseteq \ldots$, where $S_i \subseteq M, i = 1,2,\ldots\ldots$ we get a corresponding ascending chain such as $Ann_R(S_1) \subseteq Ann_R(S_2) \subseteq \ldots\ldots$
And since M satisfies the acc. on annihilators of subsets of M in R, we have a $t \in Z^+$ such that $Ann_R(S_t) = Ann_R(S_{t+1}) = \ldots\ldots$ Now we claim $r_{M_f}(S_t) = r_{M_f}(S_{t+1}) = r_{M_f}(S_{t+2}) = \ldots$

Suppose $\alpha \in r_{M_f}(S_{t+1})$ and $\alpha \notin r_{M_f}(S_t)$. Now $S_{t+1} \bullet_f \alpha = 0$ and $S_t \bullet_f \alpha \neq 0$ and so $S_{t+1} f(\alpha) = 0$ and $S_t f(\alpha) \neq 0$ giving thereby $f(\alpha) \in Ann_R(S_{t+1})$ and $f(\alpha) \notin Ann_R(S_t)$, a Contradiction .Thus $r_{M_f}(S_t) = r_{M_f}(S_{t+1})$.
Hence our claim.//

Theorem: 4.2.2 Let M be a module over R. If M is finite dimensional then the corresponding *h-ring* M_f is also finite dimensional.

Proof: Suppose M_f is not finite dimensional. If A is a right ideal of the ring M_f, then $A \bullet_f M = Af(M)$ is a sub-module of M_R. [Lemma3.3]. Also for right ideal A_1, A_2 of M_f with $A_1 \cap A_2 = 0$

we have, $A_1 f(M) \cap A_2 f(M) \subseteq A_1 \cap A_2 = 0$ .Thus, $A_1 \oplus A_2 \oplus \ldots\ldots$, an infinite direct sum of right ideals of M_f gives rise to $A_1 f(M) \oplus A_2 f(M) \oplus \ldots\ldots$,an infinite direct sum of sub-modules of M_R, a contradiction.

4.3. f-color modules and Goldie rings

In this section we explore few ideas to convert an f -*color-module* into a Goldie ring. Here the notion of a prime module plays an important role in the way .In fact the Goldie character of an algebraic structure declares the finiteness of the chromatic number of the corresponding graph of the algebraic structure and vice-versa. The notion of the chromatic number of an arbitrary graph is accountable for what we have attempted to introduce that of a *directed-clique*, which would appear as a useful tool for study the color property of a non commutative algebraic structure.

A finite subset $C = \{a, b, \ldots, d\}$ is a *directed-clique* if for a particular labeled description of C viz. $C = \{a, b, \ldots, d\} = \{x_1, x_2, \ldots, x_n\}$ say, we have $x_i \cdot x_j = 0$ for all $i < j$ and it is to be noted that C will be a clique as in above sense, if C is a *directed-clique* for all possible labeled description of it . Now a countably infinite subset C is a *directed-clique*, if for each $n \in N$, a labeled subset viz. $X = \{a, b, \ldots, d\} = \{x_1, x_2, \ldots, x_n\} (\subseteq C)$ is a *directed-clique* and such a countably infinite set is a clique if for each $n \in N$ and for each above type of labeled description of X is a *directed- clique*.

Note:
(1) Such an infinite *directed-clique* may coincide with our ordinary clique if the same is a finite *directed-clique* for all its finite subsets with all possible permutations. If S is a finite subset of T -an infinite *directed-clique* and each of all possible permutations of S is a *directed-clique* then T is a clique.
(2) The problem of coloring of elements of such a *directed-* f *- clique* may be related with an infinite f *-clique* which has been described in above and one may expect to get more interesting results

Theorem: 4.3.1 M be a finite dimensional module over a *right sub- commutative ring* R and M_f is without infinite *directed* f *- Clique.* Then M_f is right Goldie ring if M_f is reduced.

Proof: Since M_R is a finite dimensional module, so is M_f [Theorem: 4.2.2.].Now we consider one strictly ascending chain of right annihilators of subsets of M, say $r_{M_f}(S_1) \subset r_{M_f}(S_2) \subset r_{M_f}(S_3) \subset \ldots..$ which is not stationary .Let $x_i \in r_{M_f}(S_i) \setminus r_{M_f}(S_{i-1})$, $i = 2,3,\ldots. \Rightarrow x_i \in r_{M_f}(S_i)$ and $x_i \notin r_{M_f}(S_{i-1})$ .Then $S_i \bullet_f x_i = 0$ and $S_{i-1} \bullet_f x_i \neq 0$ Consider the elements $y_n = s_{n-1} \bullet_f x_n = s_{n-1} f(x_n)$,n≥2. Then $y_i = s_{i-1} \bullet_f x_i$, $y_j = s_{j-1} \bullet_f x_j$ .Now for $i > j$, $r(S_j) \subset r(S_i)$.Therefore $x_j \in r(s_{i-1})$ .i.e. $S_{i-1} \bullet_f x_j = 0$. $y_i y_j = 0$. Now $z_n = y_{i-n}$ for all n. And if $y_i = y_j$, then $y_i^{\,2} = y_j^{\,2} = 0$ and is possible only when $y_i = 0 = y_j$ as M_f is with zero *nil-radical* .Now since $r(S_1) \subset r(S_2) \subset \ldots..$is not stationary therefore $\{z_n\}_{n=1}^{\alpha}$ is an infinite *directed- f-clique* of M_f, a contradiction .Thus M_f is right Goldie.//

Here we get a very fascinating corollary, which may be called as a sufficient condition for an *h-ring* of being partially Goldie, which reads as:

Corollary: 4.3.2 (a) M is a module over a *right sub-commutative* ring R. If M_f is a reduced ring and is without infinite *directed* f *-Clique*, then M_f has the acc. on right annihilators of subsets M in ring M_f.

Corollary: 4.3.2(b) Let M be a finite dimensional module over a ring R such that $f(M)$ is commutative and let M_f be without infinite f *- Clique*. Then M_f is right Goldie ring if *nil- radical* of M_f is zero**.**

Corollary: 4.3.2(c) Let R be a finite dimensional *right sub-commutative* ring which is without infinite *directed -clique* .Then R is right Goldie ring if *nil- radical* of R is zero.

Theorem: 4.3.3 Let M be a f *-color module* over a *right sub- commutative ring* R where M_f is reduced then $Z(M) = \bigcup_{P \in Ass(M)} P$.

Proof: Let $r \in Z(M)$. Then $mr = 0, m \neq 0 \in M$ gives $r \in Ann(m)$.Thus $r \in Ann(S_i)$ for some $S_i \neq 0$, $Ann(S_i)$ is a maximal ideal [Corollary 4.4.2 (a)]. It can be shown that $Ann(S_i)$ is a prime ideal.

Hence $r \in \bigcup_{P \in Ass(M)} P$ .Thus $Z(M) \subseteq \bigcup_{P \in Ass(M)} P$ .Conversely let $x \in \bigcup_{P \in Ass(M)} P \Rightarrow x \in P$, for some $P \in Ass(M)$ gives $x \in Ann(S_i)$ .Here $P = Ann(S_i), S_i \neq 0$ .Thus $S_i x_i = 0$ .This gives $s_i x_i = 0$ [for some $s_i \neq 0$].Thus $x \in Z(M)$ giving thereby $Z(M) = \bigcup_{P \in Ass(M)} P$ //

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