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NUMERICAL SOLUTION OF FRACTIONAL PREDATOR-PREY MODEL BY TRAPEZOIDAL BASED HOMOTOPY PERTURBATION METHOD

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ABSTRACT

In this paper, we study the fractional predator-prey with Holling Type-II functional response by investigating the dynamic behavior of the system. Solution of the system is obtained using Trapezoidal rule base Homotopy perturbation method. Numerical results agree with the dynamic behavior and also establish the convenience of handling the Fractional differential equation with singularity.

Keywords: *Trapezoidal rule, Homotopy perturbation method, Caputo derivative, Holling Tpye-II functional response.*

1. INTRODUCTION

One of the classical applications of mathematics to biology is the differential equation modeling the interactions between species, first introduces by Lotka-Volterra [1, 2] popularly known as predator-prey model. They had used simple functional response to express the interactions between predator and prey. One of the most widely studied predator-prey models is the model involving logistic growth [3]. Later Feller [4] did comprehensive investigation on this. Based on experiments, Holling [5] suggested somekinds of functional responses to model the phenomena of predation, which made the standard Lotka-Volterra system more realistic. There are many other types of functional responses such as Michaelis-Menten type [6], Beddington-DeAngelis type [7] etc. The asymptotic behavior of a stochastic predator-prey with Holling Type-II functional response is studied in [8]. The existence and asymptotical stability of equilibrium and limit cycles for predator-prey systems with Holling Type-II functional response is studied in [9]

Recently, fractional calculus has penetrated in all branches of sciences and engineering where scientists are able to give a generalized flavor to all popular scientific models of fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, optics and signal processing, rheology etc. [10-14]. In [15] Das S *et al.* have investigated fractional Lotka equation. In [16] Deng W. *et.al.* have studied stability analysis of fractional order predator-prey and rabies models.

In this paper, we are concerned about fractional predator-prey model with Holling type-II functional response. We have studied the stability of the equilibrium points. Numerical solution of the model is given using Trapezoidal rule based on Homotopy Perturbation method (TRHP).

2. PRILIMINARIES

In this section we give the basic definition of fractional differentiation and integration [17-18].

Definition 1: The Riemann-Liouville fractional integral operator (J^{α}) of order $\alpha \geq 0$ of a function$f \in C_{\mu}, \mu \geq -1$

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad (\alpha > 0) \tag{1}$$

$$J^0 f(t) = f(t)$$

where Γ is the well-known Gamma function.

1. $J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$,
2. $J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t)$,
3. $J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$

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Definition 2: The Riemann –Liouville fractional derivative of order $\alpha > 0$ for a function $f \epsilon C^{\mu}_{-e}$ is defined as

$$D^{\alpha}_{a} f(x) = D^{m} J^{m-\alpha} f(x)$$

$$D^{\alpha}_{a} f(x) = \frac{d^m}{dx^m}\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f(\tau)d\tau\alpha > 0,\ \tau > 0 \tag{2}$$

Definition 3: The fractional derivative $D^{\alpha}$of a function $f: \mathbb{R}^{+} \to \mathbb{R}$ in the caputo's sense is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau \tag{3}$$

where $m-1 < \alpha \leq m,\ m \epsilon \mathbb{N},\ t > 0,\ f(t)$ has absolutely continuous derivative up to order $(m-1)$. Also, we need here two of its basic properties:

$m-1 < \alpha \leq m, m \epsilon N,\ t > 0,\ f \epsilon C^{m}_{\mu},\ \mu \geq -1$, then

$$(D^{\alpha} J^{\alpha}) f(x) = f(x)$$

$$(J^{\alpha} D^{\alpha}) f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\frac{x^k}{k!},\quad x > 0$$

3. FRACTIONAL PREDATOR-PREY MODEL WITH HOLLING TYPE-II FUNCTIONAL RESPONSE

We consider the following fractional predator-prey model with Holling type-II functional response

$$d^{\alpha}x(t) = x(t)\left[a - bx(t) - \frac{\gamma y(t)}{1+\beta x(t)}\right]dt^{\alpha}$$

$$d^{\alpha}y(t) = y(t)\left[-e + \frac{k\gamma x(t)}{1+\beta x(t)}\right]dt^{\alpha} \tag{4}$$

where $x(t)$ and $y(t)$ represent the population densities of prey and predator at time t respectively. The parameters a, b, γ, β, e, and k are the positive constants that stand for prey intrinsic growth rate, carrying capacity, the maximum ingestion rate, half-saturation constant, predator death rate and is the conversion factor, respectively.

4. STABILITY OF EQUILIBRIUM POINTS

Consider the fractional-order systems of the form

$$D^{\alpha}x(t) = f_1(x,y),\ D^{\alpha}y(t) = f_2(x,y),\ \alpha \epsilon (0,1],\ x(0) = x_0,\ y(0) = y_0 \tag{5}$$

With an equilibrium point (x_e, y_e).

Lemma 1: The equilibrium point (x_e, y_e) of the fractional system (5) is locally asymptotically stable if and only if all eigenvalues $\lambda_i, i = 1,2$ of the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

evaluated at the equilibrium point (x_e, y_e) satisfy the condition that $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$ [19].

We expect that "The fractional-order differential equations are, at least, as stable as their integer-order counterpart". The equilibria of (4) are the points of intersections at which $D^{\alpha}x(t) = 0$ and $D^{\alpha}y(t) = 0$.

To find point of equilibrium we consider $\frac{d^{\alpha}x(t)}{dt^{\alpha}} = 0$ and $\frac{d^{\alpha}y(t)}{dt^{\alpha}} = 0$ in equation (4) which yields

$$x(t)[a - bx(t)] - \frac{\gamma\, x(t)y(t)}{1+\beta x(t)} = 0 \tag{6}$$

$$-ey(t) + \frac{k\,\gamma\, x(t)y(t)}{1+\beta x(t)} = 0 \tag{7}$$

Solving equation (6) and (7), we get the equilibrium points $E_0 = (0,0)$, $E_1 = \left(\frac{a}{b}, 0\right)$ and

$$E_2 = \left(-\frac{e}{e\beta - k\gamma}\ ,\frac{-bek - aek\beta + ak^2\gamma}{(e\beta - k\gamma)^2}\right)$$

Jacobi matrix of the system (4) is $J(x,y) = \begin{pmatrix} (a-2bx) - \frac{\gamma\, y}{(1+\beta x)^2} & -\frac{\gamma x}{1+\beta x} \\ \frac{k\gamma y}{(1+\beta x)^2} & -e + \frac{k\gamma x}{1+\beta x} \end{pmatrix}$

Case-(i): Equilibrium point E_0

At (0,0) the eigenvalues of $J(x,y)$are $\lambda_1 = a > 0$ and $\lambda_2 = -e < 0$

Hence, the system is unstable.

Case-(ii): The equilibrium pointE_1

The eigenvalues of $J(x,y)$at $\left(\frac{a}{b},0\right)$are $\lambda_1 = -a < 0$ and $\lambda_2 = -e + \frac{k\gamma a}{b+\beta a} = -\left[\frac{eb+e\beta a-k\gamma a}{b+\beta a}\right]$

Case-(iii): Equilibrium point E_2

Real part of the Eigenvalues is $\frac{-ae^2\beta^2+aek\beta\gamma-be(e\beta+k\gamma)}{2k\gamma(-e\beta+k\gamma)}$

Theorem 1: The semi-trivial equilibrium point E_1 of the system (4) is asymptotically stable, if $r_0 > a$.

Theorem 2: The fixed point E_3 is stable spiral, if $r_0 < a < r_1$.
Where $r_0 = \frac{eb}{k\gamma - e\beta}$ and $r_1 = \frac{b(e\beta+k\gamma)}{\beta(k\gamma-e\beta)}$.

5. HOMOTOPY PERTURBATION METHOD (HPM)

To illustrate the basic idea of this method [20], we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{8}$$

With boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma, \tag{9}$$

where A is a general differential operator, B is a boundary operator, r is a known analytic function, and Γ is the boundary of the domain Ω. In general, the operator A can be divided in to two parts L and N,where$L$ is linear, while N is nonlinear. Equation (8) therefor can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{10}$$

By the homotopy technique [21-22], we construct a homotopy V(r,p) : Ω × [0,1] → R which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0$$
$$p \in [0,1], r \in \Omega \tag{11}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{12}$$

where p ∈ [0,1] is an embedding parameter and u_0 is an initial approximation of the solution which satisfies the boundary conditions.

From (12), we have

$$H(v,0) = L(v) - L(u_0) = 0$$
$$H(v,1) = A(v) - f(r) = 0 \tag{13}$$

The change in the process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and$L(v) - L(u_0)$, and $A(v) - f(r)$ are called Homotopic.

Now, assume that the solution of (11) and (12) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{14}$$

The approximate solution of (8) can be obtained by setting p=1:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{15}$$

6. TRAPEZOIDAL RULE BASED HOMOTOPY PERTURBATION METHOD

To derive a Trapezoidal rule based Homotopy Perturbation (TRHP) method we use the Caputo derivative because of its applicability to real world models and also it allows the traditional initial and boundary conditions to be included in the formulation of the problems. The definition of Riemann-Liouville fractional integral of a function y read as

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\, y(t)d\tau, t > 0, n-1 < \alpha < n \,\epsilon\, z^+,$$

And the Caputo fractional derivative of y is defined as

$$D^\alpha y(t) = J^{n-\alpha} y^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\, y^{(n)}(t)d\tau,$$
$$t > 0, n-1 < \alpha < n \,\epsilon\, z^+$$

The nonlinear fractional differential equation with Caputo derivative is the following form

$$D^\alpha x(t) = f\big(t, x(t)\big), x^k(0) = x_0^{(k)},$$
$$n-1 < \alpha < n \,\epsilon\, z^+,\ k = 0,1, \ldots \lceil\alpha\rceil - 1 \tag{16}$$

where $\lceil\alpha\rceil = n$. It is well known that the initial value problem (16) is equivalent to a Volterra integral equation [23],

$$x(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} x_0{}^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(t, x(t)\big) d\tau \tag{17}$$

In the sense that a continuous function solves (17) if and only if it solves (16).

In [24], Diethelm, *et al.*, successfully constructed a Predictor-Corrector method for a fractional differential equation with Caputo derivative, called “Fractional Adams method” for brevity. And the error analysis for this method was given in [25].

Firstly the product trapezoidal quadrature formula is applied to replacing the integrals of (17). Set $h = \frac{T}{N}$, $t_n = nh$, $n = 0, 1, \dots N \in z^+$. Then (17) can be discretized as follows,

$$x(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} x_0{}^{(k)} \frac{t^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n+1} a_{j,n+1}\, f\left(t_j, x(t_j)\right), \tag{18}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$

Eq. (18) may be rewritten as

$$x(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} x_0{}^{(k)} \frac{t^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f\big(t_{n+1}, x(t_{n+1})\big) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1}\, f\left(t_j, x(t_j)\right) \tag{19}$$

The right hand side of system (19) contains term $x(t_{n+1})$, so it is an implicit scheme. We shall evaluate the term $x(t_{n+1})$ in right hand side by HPM and the new scheme is named as TRHP method.

7. FRACTIONAL PREDATOR-PREY MODEL WITH HOLLING TYPE-II FUNCTIONAL RESPONSE BY TRHP METHOD

Applying TRHP method to model (4) yields

$$\begin{aligned} u(t_{n+1}) = {} & \frac{h^\alpha}{\Gamma(\alpha+2)} \left[u(t_{n+1})\big(a - bu(t_{n+1})\big) - \frac{\gamma u(t_{n+1}) v(t_{n+1})}{1 + \beta u(t_{n+1})}\right] + \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{u_0^{(k)} t_{n+1}^k}{k!} \\ & + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \left[a_{1,j,n+1} \left\{u(t_j)\left(a - bu(t_j)\right) - \frac{\gamma u(t_j) v(t_j)}{b + \beta u(t_j)}\right\}\right] \end{aligned} \tag{20}$$

$$\begin{aligned} v(t_{n+1}) = {} & \frac{h^\alpha}{\Gamma(\alpha+2)} \left[\frac{k\gamma u(t_{n+1}) v(t_{n+1})}{1 + \beta u(t_{n+1})} - ev(t_{n+1})\right] + \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{v_0^{(k)} t_{n+1}^k}{k!} \\ & + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \left[a_{2,j,n+1} \left\{\frac{k\gamma u(t_j) v(t_j)}{1 + \beta u(t_j)} - ev(t_j)\right\}\right] \end{aligned} \tag{21}$$

where $a_{i,j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$

where $i = 1, 2, \dots..$

We represent the terms $u(t_{n+1})$ and $v(t_{n+1})$ in the right hand side by $u_p(t_{n+1})$ and $v_p(t_{n+1})$ and find them by applying the HPM. We may constructed the homotopy as

$$u_p(t_{n+1}) = u_{10}(t_{n+1}) + p \frac{h^\alpha}{\Gamma(\alpha+2)} \left[u_p(t_{n+1})\left(a - bu_p(t_{n+1})\right) - \frac{\gamma u_p(t_{n+1}) v_p(t_{n+1})}{1 + \beta u_p(t_{n+1})}\right]$$

$$v_p(t_{n+1}) = v_{10}(t_{n+1}) + p \frac{h^\alpha}{\Gamma(\alpha+2)} \left[\frac{k\gamma u_p(t_{n+1}) v_p(t_{n+1})}{1 + \beta u_p(t_{n+1})} - ev_p(t_{n+1})\right]$$

where

$$u_{10}(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{u_0^{(k)} t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \left[a_{1,j,n+1} \left\{u(t_j)\left(a - bu(t_j)\right) - \frac{r\gamma u(t_j) v(t_j)}{1 + \beta u(t_j)}\right\}\right]$$

$$v_{10}(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{u_0^{(k)} t_{n+1}^k}{k!} + + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} \left[a_{2,j,n+1} \left\{\frac{k\gamma u(t_j) v(t_j)}{1 + \beta u(t_j)} - ev(t_j)\right\}\right]$$

According to HPM substitute in the above equations

$$u_p(t_{n+1}) = \sum_{r=0}^{\infty} p^r u_r(t_{n+1})$$

$$v_p(t_{n+1}) = \sum_{r=0}^{\infty} p^r v_r(t_{n+1})$$

We obtain the following equations:

$$u_1(t_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)}\left[u_0(t_{n+1})\big(a - bu_0(t_{n+1})\big) - \frac{\gamma u_0(t_{n+1})v_0(t_{n+1})}{1+\beta u_0(t_{n+1})}\right]$$

$$v_1(t_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)}\left[\frac{k\gamma\lambda u_0(t_{n+1})v_0(t_{n+1})}{1+\beta u_0(t_{n+1})} - ev_0(t_{n+1})\right]$$

$$u_2(t_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)}\left[u_1(t_{n+1})\big(a - bu_1(t_{n+1})\big) - \frac{\gamma u_1(t_{n+1})v_1(t_{n+1})}{1+\beta u_1(t_{n+1})}\right]$$

$$v_2(t_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)}\left[\frac{k\gamma u_1(t_{n+1})v_1(t_{n+1})}{1+\beta u_1(t_{n+1})} - ev_1(t_{n+1})\right]$$

$$u_3(t_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)}\left[u_2(t_{n+1})\big(a - bu_2(t_{n+1})\big) - \frac{\gamma u_2(t_{n+1})v_2(t_{n+1})}{1+\beta u_2(t_{n+1})}\right]$$

$$v_3(t_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)}\left[\frac{k\gamma u_2(t_{n+1})v_2(t_{n+1})}{1+\beta u_2(t_{n+1})} - ev_2(t_{n+1})\right]$$

and so on

8. NUMERICAL SIMULATION

In Fig. 1-7 we displayed the solution of the model (1) for step size 0.02 and different $0 < \alpha \leq 1$. We take

Case-(i): let us consider the parameters of the system (4) as $h = 0.02,\ \ a = 2,\ b = 0.2,\ e = 0.3\ \beta = 0.2,\ k = 0.5$ *and* $\gamma = 0.2$ in appropriate units for different values of fractional derivative such as $\alpha = 1,\ \alpha = 0.98$ and $\alpha = 0.85$.

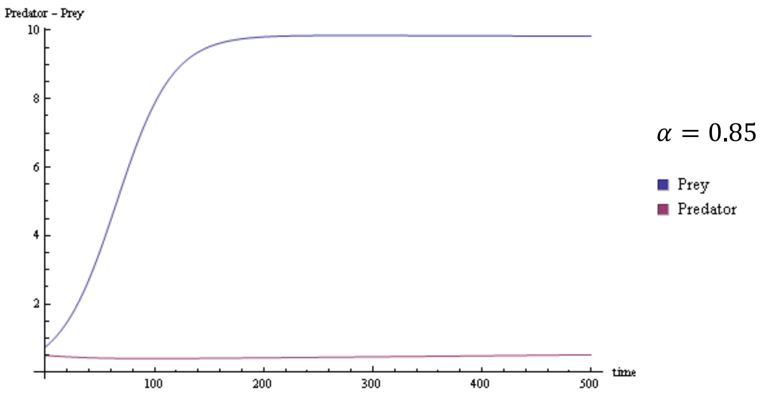


Figure-1

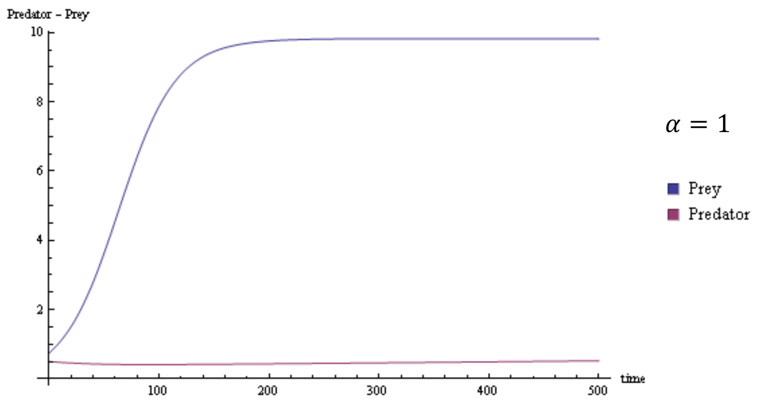


Figure-2

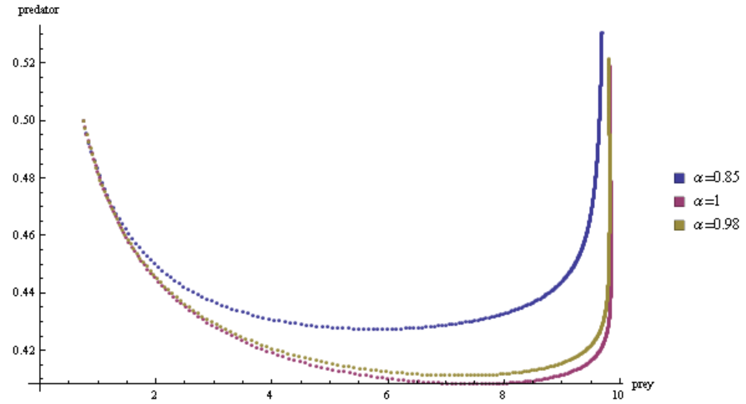


Figure-3

Case-(ii): let us consider the parameters of the system (4) as $h = 0.02$, $a = 0.8$, $b = 0.16$, $e = 0.3$ $\beta = 0.01$, $k = 0.5$, $\gamma = 0.5$ in appropriate units for different values of fractional derivative such as $\alpha = 1$, $\alpha = 0.98$ and $\alpha = 0.85$.

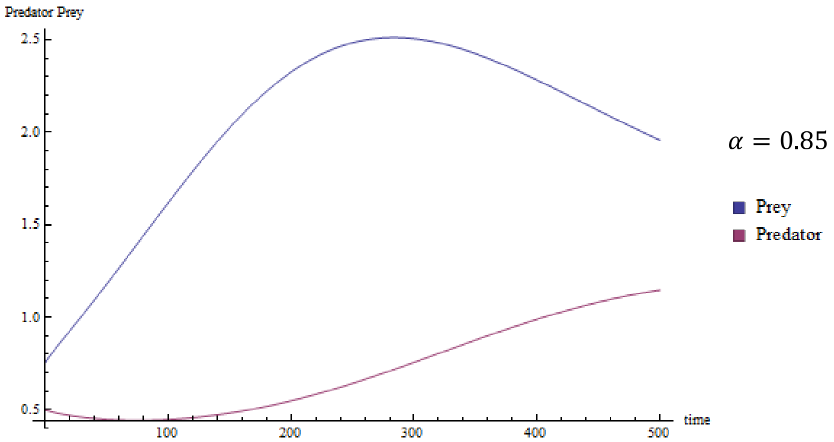


Figure-4

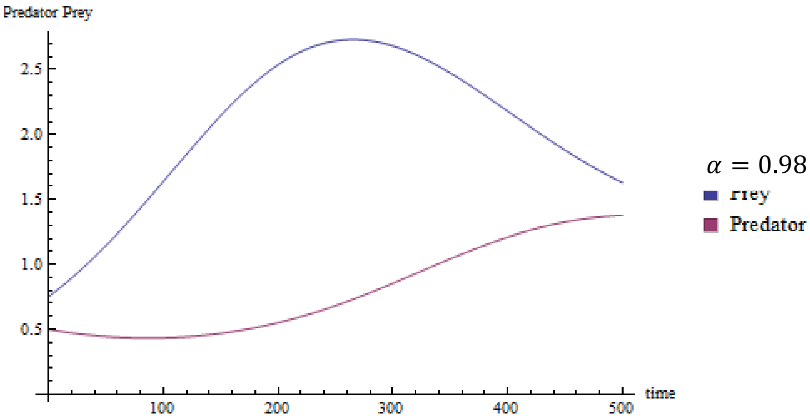


Figure-5

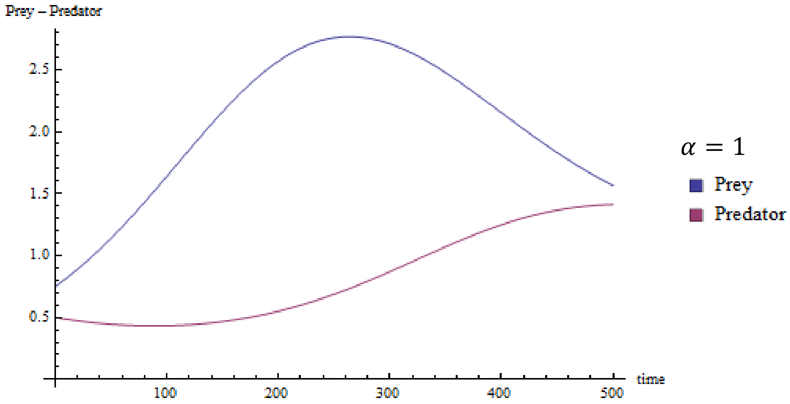
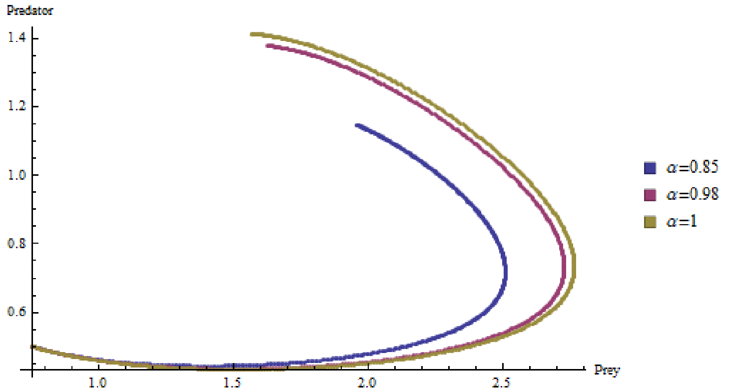


Figure-6



Figure—7

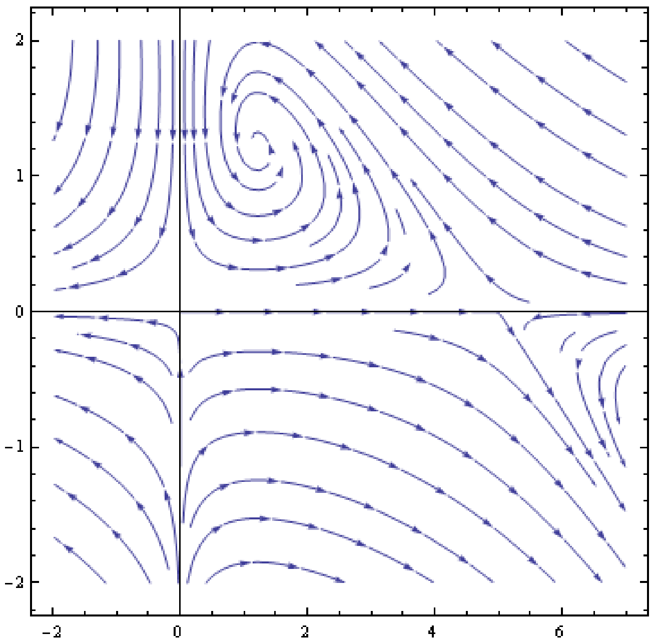


Figure-8

For the above values of the parameters, fixed point are $E_0 = (0,0)$, $E_1 = (5,0)$ and E_2 =(1.215, 1.226). The phase portrait in Fig-8 represents the same behavior that is shown by the solutions graphs in Fig-7 for various values of α.

From the graphical analysis it is clear that the numerical solution obtained by the proposed method agrees with the dynamical behavior of the solution.

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