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# Common Fixed Point Theorems in Fuzzy Metric Spaces with E. A. Property

**V. H. Badshah**

*School of Studies in Mathematics, Vikram University, Ujjain-456010, (M.P.), India*

**Rekha Jain***

*Medi-Caps Institute of Technology and Management, Indore, (M.P.), India*

*E-mail: rjain5129@yahoo.com*

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*ABSTRACT*

*In this paper we prove some common fixed point theorems for a pair of self mappings which possess the property E.A and satisfy certain sufficient conditions in the setting of a fuzzy metric space.*

***Mathematics Subject Classification:*** *47H10, 54A40, 54E99*

**Keywords**: Fuzzy metric space, compatible maps, weakly compatible maps, E.A property.

## 1. INTRODUCTION AND PRELIMINARIES:

The concept of Fuzzy sets was initially investigated by Zadeh [12] as a new way to represent the vagueness in daily life. In mathematical programming, problems are expressed as optimizing some goal function given certain constraints, and there are real life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution, that is quite useful, is the one using fuzzy sets it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. Many authors have ex-pensively developed the theory of fuzzy sets and its applications. Kramosil and Michalek [7], Kaleva and Seikkala [6] introduced the concept of fuzzy metric spaces in different ways. George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7]. They also obtained a Hausdorff topology for this kind of fuzzy metric space which has very important applications in quantum particle physics, particularly in connection with both string and $\varepsilon^{\infty}$ theory and later Grabiec [4] obtained the fuzzy version of Banach contraction principle. Many authors proved fixed point theorems for contractive maps in fuzzy metric spaces. In 1986 Jungck [5] generalized the concept of commutativity by introducing compatibility. Mishra et al [9] proved common fixed point theorems for compatible maps on fuzzy metric spaces. Sharma [10] obtained common fixed point theorems for six self mappings satisfying compatibility of type α conditions. Aamir and Moutawakil [1] further generalized the concept of non compatibility by introducing the notion of E.A. property in the classical settings of a metric space. In this paper we extend the concept of E.A. property to fuzzy metric space and obtain common fixed point theorems for a pair of self mappings under sufficient contractive type conditions which generalize and improve the result of Vijayraju and Sajath [11] Now we begin with some known definitions and preliminary concepts.

**Definition: 1.1** A binary operation *: $[0,1]\times[0,1]\rightarrow[0,1]$ is called a continuous $t$ norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b\leq c*d$ whenever $a\leq c$ and $c\leq d$ for all $a,b,c,d\in[0,1]$.

**Example: 1.1** for t- norms are $a*b=ab$ and $a*b=\min\{a,b\}$.

**Definition: 1.2** The tuple $t*t\geq t$ $(X,M,*)$ is called a fuzzy metric space, if $X$ is an arbitary set, * is a continuous t-norm and M is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions.

For all $x,y,z$ in $X$ and $s,t>0$

***Corresponding author: Rekha Jain*,*E-mail:** rjain5129@yahoo.com

(i) $M(x,y,0)=0$

(ii) $M(x,y,t)=1$, for all $t>0$ and only if $x=y$

(iii) $M(x,y,t)=M(y,x,t)$

(iv) $M(x,y,t)*M(y,z,s)\le(x,z,t+s)$

(v) $M(x,y,.):[0,\infty)\to[0,1]$ is left continuous,

(vi) $\lim_{t\to\infty}M(x,y,t)=1$

**Definition: 1.3** Let $(X,M,*)$ be a fuzzy metric space.

A sequence $\{x_n\}$ in $X$ is called Cauchy sequence if and only if $\lim_{n\to\infty}M(x_{n+p},x_n,t)=1$ for each $p>0,t>0$.

A sequence $\{x_n\}$ in $X$ is said to converge to $x$ in $X$ if and only if $\lim_{n\to\infty}M(x_n,x,t)=1$ for each $t>0$.

A fuzzy metric space $(X,M,*)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent in $X$.

**Lemma: 1.1 [2]** $M(x,y,.)$ is nondecreasing for all $x,y$ in $X$.

**Definition: 1.4** Two self mappings $S$ and $T$ be two self mappings of a fuzzy metric space $(X,M,*)$ are said to be weakly commuting if $M(STx,TSx,t)\ge M(Sx,Tx,t)$ for all $t>0$ and for all $x\in X$.

Clearly two commuting mappings are weakly commuting.

**Definition: 1.5** Let $T$ and $S$ be two self mappings of a fuzzy metric space $(X,M,*)$. $S$ and $T$ are said to be compatible if $\lim_{n\to\infty}M(STx_n,TSx_n,t)=1$ whenever $(x_n)$ is a sequence in $X$ such that $\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Tx_n=x_o$

Obviously two weakly commuting mappings are compatible.

**Definition: 1.6** Two self mappings $T$ and $S$ of a fuzzy metric space $(X,M,*)$ are said to be weakly compatible if they commute at their coincidence points; (i.e) if $Tu=Su$ for same $u\in X$, then $TSu=STu$.

**Definition: 1.7** [2] Let $S$ and $T$ be self maps on a fuzzy metric space $(X,M,*)$ A pair $\{S,T\}$ is said to be $S$ compatible of type (I) if for all $t>0$,

$$\lim_{n\to\infty}M(STx_n,x,t)\le M(Tx,x,t)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Tx_n=x$ for some $x\in X$

$T$ compatible of type (II) if the pair $\{T,S\}$ is compatible of type (I).

It is easy to see that two compatible maps are weakly compatible.

**Definition: 1.8** Let $T$ and $S$ be two self mappings of a fuzzy metric space $(X,M,*)$. We say that $T$ and $S$ satisfy E.A property, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}Sx_n=x_o$, for some $x_o\in X$; i.e. $\lim_{n\to\infty}M(Tx_n,x_o,t)=\lim_{n\to\infty}M(Sx_n,x_o,t)=1$ for all $t\in[0,\infty)$

**Example: 1.2** Let $X = [0,\infty)$. Let $M(x,y,t) = \dfrac{t}{t+|x-y|}$.

Define $T,S : X \to [0,\infty)$ by $Tx = \dfrac{x}{5}$ and $Sx = \dfrac{2x}{5}$ for all $x \in X$. Then $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$ where $x_n = \dfrac{1}{n}$.

**Definition: 1.9** Mappings A, B, S and T on a fuzzy metric space $(X, M, *)$ are said to satisfy common (EA) property if there exists sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = x, \text{ For some } x \in X$$

For more on (EA) and common (EA) properties, we refer to [1] and [8].

Note that compatible, non compatible, compatible of type (I) and compatible of type (II) satisfy (EA) property but converse is not true in general.

**Example: 1.3** Let $(X, M, *)$ be a fuzzy metric space, where $X = [0,2]$ with minimum t-norm, and $M(x,y,t) = \dfrac{t}{t+d(x,y)}$ for all $t > 0$ and for all $x, y \in X$.

Define the self maps S and T as follows:

$$Sx = \begin{cases} 2, when \quad x \in [0,1] \\ \dfrac{x}{2}, when \ 1 < x \le 2 \end{cases}$$

$$Tx = \begin{cases} 0, when \quad x = 1 \\ \dfrac{x+3}{5}, otherwise \end{cases}$$

Now, suppose $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ By definition of S and T, we have $z \in \{1\}$. Thus $\{S,T\}$ satisfies (EA) property.Note that $\{S,T\}$ is not compatible. Indeed, if $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$

then it must be $x_n \in 2^-$ and so $\lim_{n\to\infty} TSx_n = \dfrac{4}{5}$ and $\lim_{n\to\infty} STx_n = 2$

Therefore

$\lim_{n\to\infty} M(STx_n, TSx_n, t) = M(2, \frac{4}{5}, t) = \dfrac{t}{t+\frac{6}{5}} < 1$ for all $t > 0$. Also note that, $\{S,T\}$ is not compatible of type (II). Since $\lim_{n\to\infty} M(TSx_n, x, t) = \lim_{n\to\infty} M\left(\frac{4}{5}, 1, t\right) = \dfrac{t}{t+\frac{1}{5}} > (Sx, x, t) = (2,1,t) = \dfrac{t}{1+t}$

For all $t > 0$.

**Example: 1.4** Let $(X, M, *)$ be a fuzzy metric space, where $X = [0,2]$ with minimum t-norm, and $M(x,y,t) = \dfrac{t}{t+d(x,y)}$ for all $t > 0$ and for all $x, y \in X$.

Define the self map T as follows:

$$Tx = \begin{cases} \frac{1}{2}, when \ \ 0 < x \le \frac{1}{2} \ \ or \ \ x = 1 \\ 1, when \ \ \frac{1}{2} < x \le 1 \end{cases}$$

Let S be the identity map. Then, as {S, T} is commuting, it is compatible and hence satisfy property (EA). However, {S, T} is not compatible of type (I). Indeed,

suppose $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$

definition of S and T, we have $z \in \{\frac{1}{2}, 1\}$

Now if $z = \frac{1}{2}$

we can consider $x_n = \frac{1}{2} - \frac{1}{n}$

Therefore, $\lim_{n\to\infty} M\ (STx_n, z, t) = M\ (\frac{1}{2}, \frac{1}{2}, t) = 1 > \frac{t}{1/(2+t)} = M\ (Tz, z, t)$ for all $t > 0$.

if $z = 1$, we can consider $x_n = 1 - \frac{1}{n}$,

Therefore

**Lemma: .2.** [12] if for all $x, y \in X, t > 0$ with positive number $q \in (0,1)$ and $M\ (x, y, qt) \ge M\ (x, y, t)$, then $x = y$.

**2. MAIN RESULT**

**Theorem: 2.1** Let $S$ and $T$ be two weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \ge t$ such that

(i) $T$ and $S$ satisfy the E.A. property

(ii) For every $x \ne y \in X$ and for $t > 0$

$$M\ (Tx, Ty, qt) \ge$$
$$\min\{(Sx, Sy, t), [M\ (Tx, Sx, t) * M\ (Ty, Sy, t)][M\ (Ty, Sx, t) * M\ (Tx, Sy, t)], (Tx, Ty, t)\}$$

where $0 < q < 1$

(iii) $T(X) \subset S(X)$

(iv) $S(X)$ or $T(X)$ is complete subspace of $X$ Then $T$ and $S$ have a unique common fixed point.

**Proof**: Since $STa = TSa = SSa = TTa$ $T$ and $S$ satisfy the E.A. property, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x_o$ for some $x_o inX$. Suppose that $S(X)$ is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$.

$\therefore \lim_{n\to\infty} Tx_n = Sa$ by (i).

Now we show that $Ta = Sa$.

Condition (ii) implies that

 

$M\left(Tx_n, Ta, qt\right) \geq$

Letting limit $n \rightarrow \infty$

$$M\left(Sa, Ta, qt\right)$$
$$\geq \min\left\{1, \left[M\left(Sa, Sa, t\right) * M\left(Ta, Sa, t\right)\right], \left[M\left(Ta, Sa, t\right) * 1\right], (Sa, Ta, t)\right\}$$
$$= \min\left\{1, M\left(Ta, Sa, t\right), M\left(Ta, Sa, t\right), M\left(Sa, Ta, t\right)\right\}$$
$$M\left(Sa, Ta, qt\right) \geq M\left(Ta, Sa, t\right)$$
$$\therefore Ta = Sa$$

Now we show that $Ta$ is the common fixed point of $T$ and $S$. Since $T$ and $S$ are weakly compatible,

$$STa = TSa = SSa = TTa.$$
$$M\left(Ta, TTa, qt\right) \geq$$
$$\min\left\{\begin{matrix} M\left(Sa, STa, t\right), \left[M\left(Ta, Sa, t\right) * M\left(TTa, STa, t\right)\right]\left[M\left(TTa, Sa, t\right) * M\left(Ta, STa, t\right)\right], \\ M\left(Ta, TTa.t\right) \end{matrix}\right\}$$
$$= \min\left\{M\left(Ta, TTa, t\right), \left[M\left(Ta, Ta, t\right) * M\left(TTa, TTa, t\right)\right]\left[M\left(TTa, Ta, t\right) * M\left(Ta, TTa, t\right)\right], M\left(Ta, TTa.t\right)\right\}$$
$$= \min\left\{M\left(Ta, TTa, t\right), 1, M\left(Ta, TTa, t\right) * M\left(Ta, TTa, t\right), M\left(Ta, TTa.t\right)\right\}$$
$$M\left(Ta, TTa, qt\right) \geq M\left(Ta, TTa, t\right)$$

Hence $Ta$ is the common fixed point of $T$ and $S$

Even, if we assume that $T\left(X\right)$ is complete and proceed as above the result will be same.

Now it is left to prove that the fixed point is unique.

Let $x_0$ and $y_0$ be two common fixed points of $T$ and $S$.

Then

$$M\left(x_o, y_o, qt\right) = M\left(Tx_o, Ty_o, qt\right)$$
$$\geq \min\left\{\begin{matrix} M\left(Sx_o, Sy_o, t\right)\left[M\left(Tx_o, Sx_o, t\right) * M\left(Ty_o, Sy_o, t\right)\right], \left[M\left(Ty_o, Sx_o, t\right) * M\left(Tx_o, Sy_o, t\right)\right], \\ M\left(Tx_o, Ty_o, t\right) \end{matrix}\right\}$$
$$= \min\left\{M\left(x_o, y_o, t\right)\left[M\left(x_o, x_o, t\right) * M\left(y_o, y_o, t\right)\right], \left[M\left(y_o, x_o, t\right) * M\left(x_o, y_o, t\right)\right], M\left(x_o, y_o, t\right)\right\}$$
$$= \min\left\{M\left(x_o, y_o, t\right), 1, \left[M\left(y_o, x_o, t\right) * M\left(x_o, y_o, t\right)\right], M\left(x_o, y_o, t\right)\right\}$$
$$M\left(x_o, y_o, qt\right) \geq M\left(x_o, y_o, t\right)$$
$$\therefore x_o = y_o$$

**Corollary: 2.1.** Let $S$ and $T$ be two noncompatible weakly compatible self mappings of a fuzzy metric space $\left(X, M, *\right)$ with $t * t \geq$ such that

(i) $M\left(Tx, Ty, qt\right) \geq$

$$\min\left\{M\left(Sx, Sy, t\right)\left[M\left(Tx, Sx, t\right) * M\left(Ty, Sy, t\right)\right]\left[M\left(Ty, Sx, t\right) * M\left(Tx, Sy, t\right)\right], M\left(Tx, Ty, t\right)\right\}$$

 

(ii) $T(X) \subset S(X)$

If $S(X)$ or $T(X)$ is complete subspace of $X$, then $T$ and $S$ have a unique common fixed point.

**Theorem: 2.2** Let $S$ and $T$ be two weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ such that

(i) $T$ and $S$ satisfy the E.A. property

(ii) For every $x \neq t \in X$ and for $t > 0$

$$M(Tx, Ty, qt) \geq$$

$$\min\left\{M(Sx, Sy, t), \frac{[M(Tx, Sx, t) + M(Ty, Sy, t)]}{2}, \frac{[M(Tx, Sy, t) + M(Ty, Sx, t)]}{2}, M(Tx, Ty, t)\right\}$$

(iii) $T(X) \subset S(X)$

(iv) $S(X)$ or $T(X)$ is complete subspace of $X$

Then $T$ and $S$ have a unique common fixed point.

**Proof:** Since $T$ and $S$ satisfy the E.A. property, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x_0$ for some $x_0$ in $X$. Assuming $S(X)$ to be complete, we get

$\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$

$\therefore \lim_{n\to\infty} Tx_n = sa$ by (i)

We claim that $Ta = Sa$.

If $Ta \neq Sa$, then $M(Ta, Sa, t) < 1$ for all $t$.

Condition (ii) implies that

$$M(Tx_n, Ta, qt) \geq$$

$$\min\left\{\begin{array}{l} M(Sx_n, Sa, t)\dfrac{[M(Tx_n, Sx_n, t) + M(Ta, Sa, t)]}{2}, \dfrac{[M(Tx_n, Sa, t) + M(Ta, Sx_n, t)]}{2}, \\ M(Tx_n, Ta, t) \end{array}\right\}$$

Letting limit $n \to \infty$

$$M(Sa, Ta, qt) \geq \min\left\{\frac{[1 + M(Ta, Sa, t)]}{2}, \frac{[1 + M(Ta, Sa, t)]}{2}, M(Ta, Sa, t)\right\}$$

$$M(Sa, Ta, qt) \geq \frac{[1 + M(Ta, Sa, t)]}{2}$$

$$\geq M(Ta, Sa, t)$$

for all $t$

Because, if $\frac{[1 + M(Ta, Sa, t)]}{2} \leq M(Ta, Sa, t)$, then $M(Ta, Sa, t) > 1$, which is a contradiction.

Hence $Ta = Sa$.

Let us prove now that $Ta$ is the common fixed point of $T$ and $S$. Suppose that $Ta \neq TTa$.

Since $T$ and $S$ are weakly compatible $STa = TSa = SSa = TTa$.

$$M(Ta, TTa, qt) \geq$$
$$= \min\left\{\begin{array}{l} M(Sa, STa, t)\dfrac{[M(Ta, Sa, t) + M(TTa, STa, t)]}{2}, \dfrac{[M(Ta, STa, t) + M(TTa, Sa, t)]}{2}, \\ M(Ta, TTa, t) \end{array}\right\}$$
$$= \min\left\{M(Ta, TTa, t), \frac{[1 + M(TTa, TTa, t)]}{2}, M(Ta, TTa, t), M(Ta, TTa, t)\right\}$$
$$= M(Ta, TTa, t)$$

Because, $= M(Ta, TTa, t) \leq \dfrac{[1 + M(TTa, TTa, t)]}{2}$

Thus $M(Ta, TTa, qt) \geq M(Ta, TTa, t)$ for all $t$

This implies $TTa = Ta$

Hence $Ta$ is the common fixed point of $T$ and $S$

As in the above theorem 2.4, the proof will be similar if we assume that $T(X)$ is complete subspace of $X$ finally we show that the fixed point is unique.

If possible, let $x_0$ and $y_0$ be two common fixed points of $T$ and $S$. Then

$$M(x_0, y_0, qt) = M(Tx_0, Ty_0, qt)$$
$$\geq \min\left\{\begin{array}{l} M(Sx_0, Sy_0, t), \dfrac{[M(Tx_0, Sx_0, t) + M(Ty_0, Sy_0, t)]}{2}, \dfrac{[M(Tx_0, Sy_0, t) + M(Ty_0, Sx_0, t)]}{2}, \\ M(Tx_0, Ty_0, t) \end{array}\right\}$$
$$= \min\left\{M(x_0, y_0, t), \frac{[M(x_0, x_0, t) + M(y_0, y_0, t)]}{2}, \frac{[M(x_0, y_0, t) + M(y_0, x_0, t)]}{2}, M(x_0, y_0, t)\right\}$$
$$= \min\{M(x_0, y_0, t), 1, M(x_0, y_0, t), M(x_0, y_0, t)\}$$
$$\Rightarrow M(x_0, y_0, qt) \geq M(x_0, y_0, t)$$
$$\therefore x_0 = y_0$$

Hence the theorem.

**Corollary: 2.2.** Let $S$ and $T$ be two noncompatible weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ such that

(i) $M(Tx, Ty, qt) \geq$
$$\min\left\{\begin{array}{l} M(Sx, Sy, t), \dfrac{[M(Tx, Sx, t) + M(Ty, Sy, t)]}{2}, \dfrac{[M(Tx, Sy, t) + M(Ty, Sx, t)]}{2}, \\ M(Tx, Ty, t) \end{array}\right\}$$

(ii) $T(X) \supset S(X)$

If $S(X)$ or $T(X)$ is complete subspace of $X$, then $T$ and $S$ have a unique common fixed point.

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