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SOME RESULTS ON INTERVAL VALUED INTUITIONISTIC FUZZY n- NORMED LINEAR SPACE

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ABSTRACT

In this paper we first define concept of interval-valued-intuitionistic –fuzzy n-normed space and then we obtain some results in this newly defined space.

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1. INTRODUCTION

The theory of Fuzzy sets was introduced by L.Zadeh [13] in 1965. Subsequently a progressive development was made in the theory of interval-valued fuzzy subsets in [14]. J.H.Park[9] studied the notion of intuitionistic fuzzy metric spaces. Many authors introduced the definition of fuzzy inner product space and fuzzy normed linear space in [4, 5, 6]. The origin and development of intuitionstic fuzzy set theory can be found in [7, 10, 11, 12]. The motive of this paper is to introduce the notion of interval- valued – intuitionistic –fuzzy n-normed space as generalization of intuitionstic-fuzzy n-normed space [20] and we obtain some results in this space. Further we introduce the generalized Cartesian product of the interval-valued-intuitionistic –fuzzy n-normed space and prove some of its properties.

2. PRELIMINARIES

In this section we provide the essential definitions and results necessary for the development of our theory.

Definition 2.1 [14]: An interval number on [0, 1], say $\bar{a}$, is a closed sub interval of [0, 1] of the form $\bar{a} = [a-, a+]$, where $0 \le a- \le a+ \le 1$. Let D [0, 1] denote the family of all closed sub-intervals of [0, 1], that is,
D [0, 1] = { $\bar{a} = [a-, a+]$: $a- \le a+$ and $a-, a+ \in [0, 1]$ }

Definition 2.2 [14]: Let $\bar{a}_i = [a_i^-, a_i^+] \in$ D[0, 1] for all i$\in$ Ω, Ω an index set.
Define
(a) $\inf^i \{ \bar{a}_i : i \in \Omega \} = [\inf_{i\in\Omega} a_i^-, \inf_{i\in\Omega} a_i^+]$
(b) $\sup^i \{ \bar{a}_i : i \in \Omega \} = [\sup_{i\in\Omega} a_i^-, \sup_{i\in\Omega} a_i^+]$

In particular, whenever $\bar{a} = [a-, a+]$, $\bar{b} = [b-, b+]$, in D [0, 1], we define
(i) $\bar{a} \le \bar{b}$ if and only if $a- \le b-$ and $a+ \le b+$
(ii) $\bar{a} = \bar{b}$ if and only if $a- = b-$ and $a+ = b+$
(iii) $\bar{a} < \bar{b}$ if and only if $a- < b-$ and $a+ < b+$.
(iv) $\min^i \{ \bar{a}, \bar{b} \}$= [min {$a-, b-$}, min {$a+, b+$}]
(v) $\max^i \{ \bar{a}, \bar{b} \}$ = [max{$a-, b-$}, max{$a+, b+$}].

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Definition 2.3 [14]: Let *X* be a set. A mapping $\overline{A}$: *X* → D[0, 1] is called an Interval-valued fuzzy subset (briefly, an i-v fuzzy subset) of *X*, where $\overline{A}$ (*x*) = [*A*−(*x*), *A*+(*x*)], and *A*− and *A*+ are fuzzy subsets in *X* such that *A*−(*x*) ≤ *A*+(*x*) for all *x*∈ *X*.

Definition 2.4 [14]: Let $\overline{A}$ be an interval-valued fuzzy subset of *X* and [t_1, t_2] ∈ D [0, 1]. Then the set $\overline{\cup}$ ($\overline{A}$; [t_1, t_2]) = {*x* ∈ *X*: $\overline{A}$ (*x*) ≥ [t_1, t_2]} is called an upper level subset of *A*.
Note that
$\overline{\cup}$ ($\overline{A}$; [t_1, t_2]) = *U* (*A*−, t_1) ∩ *U* (*A*+; t_2) where
U (*A*−; t_1) = {*x*∈ *X* : *A*−(*x*) ≥ t_1 } and
U (*A*+; t_2) = {x∈ X : A+(x) ≥ t_2 }.

Definition 2.5 [2]: Let *V* denote a vector space of dimension *n* over a field *F*. A fuzzy subspace is a fuzzy subset *μ* of *V* such that
μ(*αx* + *βy*) ≥ *μ*(*x*) Λ *μ*(*y*), *x, y*∈ *V* , *α, β* ∈ *F*(field), where Λ stands for intersection.

Definition 2.6: Let *V* denote a vector space of dimension *n* over a field *F*. A intuitionstic fuzzy subspace is a fuzzy subset μ, η of *V* such that

$$\mu(\alpha x + \beta y) \geq \mu(x) \wedge \mu(y)$$

and $$\eta(\alpha x + \beta y) \leq \eta(x) \vee \eta(y),\ x, y \in V,\ \alpha, \beta \in F(\text{field}),$$

where Λ and ∨ stands for intersection and union respectively.

Remark 2.7: A subset { $x_1, x_2, \ldots\ldots, x_m$ } of *V* is intuitionstic-fuzzy linearly independent if it is linearly independent and $\mu(\sum_{i=1}^{m} \alpha_i x_i) = \bigwedge_{i=1}^{m} \mu(x_i)$,

$\eta(\sum_{i=1}^{m} \alpha_i x_i) = \bigvee_{i=1}^{m} \mu(x_i)$ for all $\alpha_i \in F, \alpha_i \neq 0,$ i = 1,....., m, Λ, ∨ stands for intersection and union respectively.

Definition 2.8 [20]: An intuitionistic fuzzy n-normed linear space (or) in short i-f-n-NLS is an object of the form
A = {(X, N($x_1, x_2, \ldots\ldots, x_n$, t), M($x_1, x_2, \ldots\ldots, x_n$, t) : ($x_1, x_2, \ldots\ldots, x_n$)∈ X^n }
where X is a linear space over a field F and N, M are fuzzy sets on $X^n \times (0,1)$, N denotes the degree of membership and M denotes the degree of non-membership of ($x_1, x_2, \ldots\ldots, x_n$)∈ $X^n \times (0,1)$ satisfying the following conditions:

(i) N($x_1, x_2, \ldots\ldots, x_n$, t) +M($x_1, x_2, \ldots\ldots, x_n$, t) ≤ 1;

(ii) N($x_1, x_2, \ldots\ldots, x_n$, t) > 0;

(iii) N($x_1, x_2, \ldots\ldots, x_n$,t) = 1 if and only if $x_1, x_2, \ldots\ldots, x_n$ are linearly dependent;

(iv) N($x_1, x_2, \ldots\ldots, x_n$,t) is invariant under any permutation of $x_1, x_2, \ldots\ldots, x_n$.

(v) N($x_1, x_2, \ldots\ldots, cx_n$, t) = N($x_1, x_2, \ldots\ldots, x_n$, $\frac{t}{|c|}$), if c ≠ 0, c ∈ F(field).

(vi) $N(x_1, x_2, \ldots.., x_n + x_n', s+t) \geq \min\left\{N(x_1, x_2, \ldots\ldots x_n, s), N(x_1, x_2, \ldots., x_n', t)\right\}$

(vii) N($x_1, x_2, \ldots\ldots, x_n$, t) : (0;1) → [0; 1] is continuous in t;

(viii) M($x_1, x_2, \ldots\ldots, x_n$, t) > 0;

(ix) M($x_1, x_2, \ldots\ldots, x_n$, t) = 0 if and only if $x_1, x_2, \ldots\ldots, x_n$ are linearly dependent;

(x) M($x_1, x_2, \ldots\ldots, x_n$,t) is invariant under any permutation of $x_1, x_2, \ldots\ldots, x_n$.

(xi) M($x_1, x_2, \ldots\ldots, cx_n$, t) = M($x_1, x_2, \ldots\ldots, x_n$, $\frac{t}{|c|}$), if c ≠ 0, c ∈ F(field);

(xii) $M(x_1, x_2, \ldots.., x_n + x_n', s+t) \leq \max\left\{M(x_1, x_2, \ldots\ldots x_n, s), M(x_1, x_2, \ldots., x_n', t)\right\}$

(xiii) M($x_1, x_2, \ldots\ldots, x_n$,t) : (0;1) → [0; 1] is continuous in t.

Remark 2.9:

(i) Let V be a vector space over a field F and let $U_1, U_2, \ldots\ldots, U_n$ be the subspaces of V. V is said to be direct sum of $U_1, U_2, \ldots\ldots, U_n$ if every element $v \in V$ can be written in one and only one way $v = u_1 + u_2 + \ldots\ldots + u_n$ where $u_i \in U_i$ and denoted as $V = U_1 \oplus U_2 \oplus \ldots\ldots \oplus U_n$.

(ii) Let V and W be vector spaces over a field F. Then the tensor product of two vectors is denoted by $V \otimes$ W and given $t \in V \otimes W$, t can be uniquely written as , $t = \sum_{i,j} t_{ij} x_i \otimes y_j$ where the sum is taken over all i, j for which $t_{ij} \neq 0$.

3. INTERVAL-VALUED-INTUITIONSTIC- FUZZY-n-NORMED LINEAR SUBSPACES

As a generalization of definition 2.8 we have the following notion of interval- valued-intuitionstic-fuzzy n-normed linear subspaces.

Definition 3.1: Let X be a linear space over a field F. An interval-valued fuzzy subsets $\overline{N}, \overline{M}$ of $X^n \times R$ is called an interval-valued- intituitionstic fuzzy n-norm if and only if

1. $\overline{N}(x_1, x_2, \ldots.., x_n, t) + \overline{M}(x_1, x_2, \ldots\ldots, x_n, t) \leq \bar{1}$
2. For all t$\in$ R with t $\leq$ 0, $\overline{N}(x_1, x_2, \ldots.., x_n, t) = \bar{0}$.
3. For all t $\in$ R with t > 0, $\overline{N}(x_1, x_2, \ldots.., x_n, t) = \bar{1}$ iff $x_1, x_2, \ldots\ldots.., x_n$ are linearly dependent.
4. $\overline{N}(x_1, x_2, \ldots.., x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots\ldots.., x_n$.
5. For all t$\in$R with t > 0 $\overline{N}(x_1, x_2, \ldots.., cx_n, t) = \overline{N}(x_1, x_2, \ldots.., x_n, \frac{t}{|c|})$, if c$\neq$ 0,c$\in$F(field).
6. For all s, t $\in$ R $\overline{N}(x_1, x_2, \ldots.., x_n + x_n^{'}, s+t) \geq \min^i \left\{\overline{N}(x_1, x_2, \ldots\ldots x_n, s), \overline{N}(x_1, x_2, \ldots., x_n^{'}, t)\right\}$
7. $\overline{N}(x_1, x_2, \ldots.., x_n, t)$ is a non decreasing function of t$\in$R and $\lim_{t\to\infty} \overline{N}(x_1, x_2, \ldots.., x_n, t) = \bar{1}$.
8. For all t$\in$ R with t $\leq$ 0, $\overline{M}(x_1, x_2, \ldots.., x_n, t) > \bar{0}$.
9. For all t $\in$ R with t > 0, $\overline{M}(x_1, x_2, \ldots.., x_n, t) = \bar{0}$ iff $x_1, x_2, \ldots\ldots.., x_n$ are linearly dependent.
10. $\overline{M}(x_1, x_2, \ldots.., x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots\ldots.., x_n$.
11. For all t$\in$R with t > 0 $\overline{M}(x_1, x_2, \ldots.., cx_n, t) = \overline{M}(x_1, x_2, \ldots.., x_n, \frac{t}{|c|})$, if c$\neq$ 0,c$\in$F(field).
12. For all s, t $\in$ R $\overline{M}(x_1, x_2, \ldots.., x_n + x_n^{'}, s+t) \leq \max^i \left\{\overline{M}(x_1, x_2, \ldots\ldots x_n, s), \overline{M}(x_1, x_2, \ldots., x_n^{'}, t)\right\}$
13. $\overline{M}(x_1, x_2, \ldots.., x_n, t)$ is a non increasing function of t$\in$R and $\lim_{t\to\infty} \overline{M}(x_1, x_2, \ldots.., x_n, t) = \bar{0}$

Then (X, $\overline{N}, \overline{M}$) is called an interval- valued Intuitionstic fuzzy n-normed linear space in short i-v-i-f-n NLS.

The following example agrees with our notion of i-v-i-f-n-NLS

Example 3.2: Let (X, $\|\bullet,\bullet,\ldots\ldots..,\bullet\|$) bean n-normed space . Define

$$\overline{N}(x_1, x_2, \ldots.., x_n, t) = \begin{cases} \bar{0} & when\, t \leq \|x_1, x_2, \ldots\ldots. x_n\| \\ \bar{1} & when\, t > \|x_1, x_2, \ldots.., x_n\| \end{cases}$$

and

$$\overline{M}(x_1, x_2, \ldots.., x_n, t) = \begin{cases} \bar{1} & when\, t \leq \|x_1, x_2, \ldots\ldots. x_n\| \\ \bar{0} & when\, t > \|x_1, x_2, \ldots.., x_n\| \end{cases}$$

Then (X, $\overline{N}, \overline{M}$) is an i-v-i-f-n-NLS where $\bar{0}$ =[0,0] and $\bar{1}$=[1,1].

Now, we give a result that intersection of two i-v-i-f-n-NLS, and hence result holds for finite intersection of i-v-i-f-n-NLS.

Theorem 3.3: Let (X, $\overline{N_1}$, $\overline{M_1}$) and (X, $\overline{N_2}$, $\overline{M_2}$) be two i-v-i-f-n-NLS. Define

$$\overline{N}(x_1, x_2, \ldots, x_n, t) = (\overline{N_1} \cap \overline{N_2})(x_1, x_2, \ldots, x_n, t) = \min^i \{\overline{N_1}(x_1, x_2, \ldots, x_n, t), \overline{N_2}(x_1, x_2, \ldots, x_n, t)\}$$

and

$$\overline{M}(x_1, x_2, \ldots, x_n, t) = (\overline{M_1} \cap \overline{M_2})(x_1, x_2, \ldots, x_n, t) = \max^i \{\overline{M_1}(x_1, x_2, \ldots, x_n, t), \overline{M_2}(x_1, x_2, \ldots, x_n, t)\}$$

for all $(x_1, x_2, \ldots, x_n, t) \in X^n \times R$. Then (X, $\overline{N_1} \cap \overline{N_2}$, $\overline{M_1} \cap \overline{M_2}$) or (X, $\overline{N}$, $\overline{M}$) is an i-v-i-f-n-NLS.

Proof: (1) Since $\overline{N_1}(x_1, x_2, \ldots, x_n, t) + \overline{M_1}(x_1, x_2, \ldots, x_n, t) \le \bar{1}$.

and $\overline{N_2}(x_1, x_2, \ldots, x_n, t) + \overline{M_2}(x_1, x_2, \ldots, x_n, t) \le \bar{1}$.

It follows that

$$\overline{M_1}(x_1, x_2, \ldots, x_n, t) \le \bar{1} - \overline{N_1}(x_1, x_2, \ldots, x_n, t)$$

and

$$\overline{M_2}(x_1, x_2, \ldots, x_n, t) \le \bar{1} - \overline{N_2}(x_1, x_2, \ldots, x_n, t)$$

By definition

$$\max^i \{\bar{1} - \overline{N_1}(x_1, x_2, \ldots, x_n, t), \bar{1} - \overline{N_2}(x_1, x_2, \ldots, x_n, t)\} \ge \max^i \{\overline{M_1}(x_1, x_2, \ldots, x_n, t), \overline{M_2}(x_1, x_2, \ldots, x_n, t)\}$$

$$\Rightarrow \min^i \{\overline{N_1}(x_1, x_2, \ldots, x_n, t), \overline{N_2}(x_1, x_2, \ldots, x_n, t)\} = \bar{1} - (\max^i \{\bar{1} - \overline{N_1}(x_1, x_2, \ldots, x_n, t), \bar{1} - \overline{N_2}(x_1, x_2, \ldots, x_n, t)\}) \le \bar{1} - \max^i \{\overline{M_1}(x_1, x_2, \ldots, x_n, t), \overline{M_2}(x_1, x_2, \ldots, x_n, t)\}$$

$$\Rightarrow \min^i \{\overline{N_1}(x_1, x_2, \ldots, x_n, t), \overline{N_2}(x_1, x_2, \ldots, x_n, t)\} + \max^i \{\overline{M_1}(x_1, x_2, \ldots, x_n, t), \overline{M_2}(x_1, x_2, \ldots, x_n, t)\} \le \bar{1}$$

$$\Rightarrow \overline{N}(x_1, x_2, \ldots, x_n, t) + \overline{M}(x_1, x_2, \ldots, x_n, t) = \bar{1}$$

(2) For all $t \in R$ with $t \le 0$, since $\overline{N_1}(x_1, x_2, \ldots, x_n, t) = \bar{0}$ and $\overline{N_2}(x_1, x_2, \ldots, x_n, t) = \bar{0}$

$$\Rightarrow \min^i \{\overline{N_1}(x_1, x_2, \ldots, x_n, t), \overline{N_2}(x_1, x_2, \ldots, x_n, t)\} = \bar{0}$$

$$\Rightarrow \overline{N}(x_1, x_2, \ldots, x_n, t) = \bar{0}.$$

(3) Since for all $t \in R$ with $t > 0$, $\overline{N_1}(x_1, x_2, \ldots, x_n, t) = \bar{1}$ and $\overline{N_2}(x_1, x_2, \ldots, x_n, t) = \bar{1}$ iff $x_1, x_2, \ldots, x_n$ are linearly dependent.

$\Rightarrow \min^i \{\overline{N_1}(x_1, x_2, \ldots, x_n, t), \overline{N_2}(x_1, x_2, \ldots, x_n, t)\} = \bar{1}$ iff $x_1, x_2, \ldots, x_n$ are linearly dependent.

$\Rightarrow \overline{N}(x_1, x_2, \ldots, x_n, t) = \bar{1}$ iff $x_1, x_2, \ldots, x_n$ are linearly dependent.

(4) Since $\overline{N_1}(x_1, x_2, \ldots, x_n, t)$ and $\overline{N_2}(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

$\Rightarrow \min^i \{\overline{N_1}(x_1, x_2, \ldots, x_n, t), \overline{N_2}(x_1, x_2, \ldots, x_n, t)\}$ is also invariant under any permutation of $x_1, x_2, \ldots, x_n$.

$\Rightarrow \overline{N}(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

(5) Since for all $t \in R$ with $t > 0$, $\overline{N}_1(x_1, x_2, \ldots, cx_n, t) = \overline{N_1}(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ and $\overline{N_2}(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$(field).

$$\Rightarrow \min^i\{\overline{N_1}(x_1,x_2,.....,cx_n,t),\overline{N_2}(x_1,x_2,......,cx_n,t)\}$$

$$= \min^i\left\{\overline{N_1}(x_1,x_2,.....,x_n,\frac{t}{|c|}),\overline{N_2}(x_1,x_2,......,x_n,\frac{t}{|c|})\right\}$$

$$\Rightarrow \overline{N}(x_1,x_2,.....,cx_n,t) = \overline{N}(x_1,x_2,.....,x_n,\frac{t}{|c|}).$$

(6) For all s, t $\in$ R $\overline{N}(x_1,x_2,.....,x_n+x_n',s+t) \geq \min^i\{\overline{N}(x_1,x_2,......x_n,s),\ \overline{N}(x_1,x_2,....,x_n',t)\}$, since $\overline{N}(x_1,x_2,.....,x_n+x_n',s+t) = \min^i\{\overline{N_1}(x_1,x_2,......x_n+x_n',s+t),\ \overline{N_2}(x_1,x_2,....,x_n+x_n',t+s)\}$ and

$$\min^i\{\overline{N_1}(x_1,x_2,......x_n+x_n',s+t),\ \overline{N_2}(x_1,x_2,....,x_n+x_n',t+s)\}$$

$$\geq \min^i\{\min^i\{\overline{N_1}(x_1,x_2,......x_n,s),\ \overline{N_1}(x_1,x_2,....,x_n',t)\},\min^i\{\overline{N_2}(x_1,x_2......,x_n,s),\overline{N_2}(x_1,x_2,......,x_n',t)\}\}$$

$$\geq \min^i\{\min^i\{\overline{N_1}(x_1,x_2,......x_n,s),\overline{N_2}(x_1,x_2,....,x\ ,s)\},\ \min^i\{\overline{N_1}(x_1,x_2,....,x_n',t),\overline{N_2}(x_1,x_2,.....,x_n',t)\}\}$$

$$\geq \min^i\{\overline{N}(x_1,x_2,......x_n,s),\ \overline{N}(x_1,x_2,....,x_n',t)\}$$

(7) $\overline{N_1}(x_1,x_2,.....,x_n,t)$, $\overline{N_2}(x_1,x_2,.....,x_n,t)$ is a non decreasing function of t$\in$R and

$$\lim_{t\to\infty}\overline{N_1}(x_1,x_2,.....,x_n,t)=\bar{1},\ \lim_{t\to\infty}\overline{N_2}(x_1,x_2,.....,x_n,t)=\bar{1}$$

$\Rightarrow \overline{N}(x_1,x_2,.....,x_n,t)$ is a non decreasing function of t$\in$R and $\lim_{t\to\infty}\overline{N}(x_1,x_2,.....,x_n,t)=\bar{1}$

(8) For all t$\in$ R with t $\leq$ 0, $\overline{M}_1(x_1,x_2,.....,x_n,t)>\bar{0}$, $\overline{M_2}(x_1,x_2,.....,x_n,t)>\bar{0}$ and

$$\overline{M}(x_1,x_2,........,x_n,t) = (\overline{M_1}\cap\overline{M_2})(x_1,x_2,........,x_n,t)$$

$$= \max^i\{\overline{M_1}(x_1,x_2,.....,x_n,t),\overline{M_2}(x_1,x_2,......,x_n,t)\}.$$

$$\Rightarrow \overline{M}(x_1,x_2,.....,x_n,t)>\bar{0}.$$

(9) For all t $\in$ R with t > 0, $\overline{M_1}(x_1,x_2,.....,x_n,t)=\bar{0}$, $\overline{M_2}(x_1,x_2,.....,x_n,t)=\bar{0}$ iff $x_1,x_2,........,x_n$ are linearly dependent. So,

$$\overline{M}(x_1,x_2,........,x_n,t) = (\overline{M_1}\cap\overline{M_2})(x_1,x_2,........,x_n,t)$$

$$= \max^i\{\overline{M_1}(x_1,x_2,.....,x_n,t),\overline{M_2}(x_1,x_2,......,x_n,t)\}. =0$$

iff $x_1,x_2,........,x_n$ are linearly dependent.

(10) By definition of $\overline{M}(x_1,x_2,.....,x_n,t)$, and using the fact that $\overline{M_1}(x_1,x_2,.....,x_n,t)$, $\overline{M_2}(x_1,x_2,.....,x_n,t)$ is invariant under any permutation of $x_1,x_2,........,x_n$.It is clear that $\overline{M}(x_1,x_2,.....,x_n,t)$ is invariant under any permutation of $x_1,x_2,........,x_n$.

(11) For all t$\in$R with t > 0 $\overline{M_1}(x_1,x_2,.....,cx_n,t) = \overline{M_1}(x_1,x_2,.....,x_n,\frac{t}{|c|})$, and $\overline{M_2}(x_1,x_2,.....,cx_n,t) = \overline{M_2}(x_1,x_2,.....,x_n,\frac{t}{|c|})$, if c$\neq$0,c$\in$F(field).

$$\Rightarrow \overline{M}(x_1,x_2,........,cx_n,t) = (\overline{M_1}\cap\overline{M_2})(x_1,x_2,........,cx_n,t)$$

$$= \max^i\{\overline{M_1}(x_1,x_2,.....,cx_n,t),\overline{M_2}(x_1,x_2,......,cx_n,t)\}.$$

$$= \max^i\left\{\overline{M_1}(x_1,x_2,.....,x_n,\frac{t}{|c|}t),\overline{M_2}(x_1,x_2,......,x_n,\frac{t}{|c|})\right\}.$$

$$= \overline{M}(x_1,x_2,........,x_n,\frac{t}{|c|})$$

(12) For all s, t $\in$ R $\overline{M_1}(x_1, x_2, \ldots, x_n + x_n', s+t) \le \max^i \{\overline{M_1}(x_1, x_2, \ldots x_n, s), \overline{M_1}(x_1, x_2, \ldots, x_n', t)\}$
and
$\overline{M_2}(x_1, x_2, \ldots, x_n + x_n', s+t) \le \max^i \{\overline{M_2}(x_1, x_2, \ldots x_n, s), \overline{M_2}(x_1, x_2, \ldots, x_n', t)\}$

$$\Rightarrow \overline{M}(x_1, x_2, \ldots, x_n + x_n', s+t) = (\overline{M_1} \cap \overline{M_2})(x_1, x_2, \ldots, x_n + x_n', s+t)$$
$$= \max^i \{\overline{M_1}(x_1, x_2, \ldots, x_n + x_n', s+t), \overline{M_2}(x_1, x_2, \ldots, x_n + x_n', s+t)\}.$$
$$\le \max^i \{\max^i \{\overline{M_1}(x_1, x_2, \ldots x_n, s), \overline{M_1}(x_1, x_2, \ldots, x_n', t)\}, \max^i \{\overline{M_2}(x_1, x_2, \ldots, x_n, s), \overline{M_2}(x_1, x_2, \ldots, x_n', t)\}\}$$
$$\le \max^i \{\max^i \{\overline{M_1}(x_1, x_2, \ldots x_n, s), \overline{M_2}(x_1, x_2, \ldots, x\ \ , s)\}, \max^i \{\overline{M_1}(x_1, x_2, \ldots, x_n', t), \overline{M_2}(x_1, x_2, \ldots, x_n', t)\}\}$$
$$\le \max^i \{\overline{M}(x_1, x_2, \ldots x_n, s), \overline{M}(x_1, x_2, \ldots, x_n', t)\}.$$

Hence (X, $\overline{N_1} \cap \overline{N_2}$, $\overline{M_1} \cap \overline{M_2}$) or (X, $\overline{N}$, $\overline{M}$) is an i-v-i-f-n-NLS.

Remark 3.4: Let (X, $\overline{N_1}$, $\overline{M_1}$) and (X, $\overline{N_2}$, $\overline{M_2}$) be two i-v-i-f-n-NLS. Define

$$\overline{N}(x_1, x_2, \ldots, x_n, t) = (\overline{N_1} \cup \overline{N_2})(x_1, x_2, \ldots, x_n, t)$$
$$= \max^i \{\overline{N_1}(x_1, x_2, \ldots, x_n, t), \overline{N_2}(x_1, x_2, \ldots, x_n, t)\}.$$

and

$$\overline{M}(x_1, x_2, \ldots, x_n, t) = (\overline{M_1} \cup \overline{M_2})(x_1, x_2, \ldots, x_n, t)$$
$$= \min^i \{\overline{M_1}(x_1, x_2, \ldots, x_n, t), \overline{M_2}(x_1, x_2, \ldots, x_n, t)\}.$$

for all ($x_1, x_2, \ldots, x_n$,t) $\in X^n \times R$. Then (X, $\overline{N_1} \cup \overline{N_2}$, $\overline{M_1} \cup \overline{M_2}$) or (X, $\overline{N}$, $\overline{M}$) is an not i-v-i-f-n-NLS.

4. GENERALIZED CARTESIAN PRODUCT OF THE INTERVAL-VALUED-INTUITIONISTIC FUZZY-*n*-NORMED LINEAR SPACES

We now proceed to our new notion of generalized cartesian product of the Interval-valued-intuitionistic fuzzy *n*-normed linear spaces in the following theorem.

Theorem 4.1: Let

$$S = \{(X, (\overline{\mathcal{A}_1}(x_1, x_2, \ldots, x_n, t); \overline{\mathcal{B}_1}(x_1, x_2, \ldots, x_n, t)); (x_1, x_2, \ldots, x_n) \in X^n\}$$

and

$$T = \{(Y, \overline{\mathcal{A}_2}(y_1, y_2, \ldots, y_n, t); \overline{\mathcal{B}_2}(y_1, y_2, \ldots, y_n, t)); (y_1, y_2, \ldots, y_n) \in Y^n\}$$

be two interval-valued-intuitionistic fuzzy n-normed linear spaces. Then

$S \times_{*,\lozenge} T = \{(X \times Y, \overline{\mathcal{A}}(z_1, z_2, \ldots, z_n, t), \overline{\mathcal{B}}(z_1, z_2, \ldots, z_n, t)); (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\}$ is an i-v-i-f-n-NLS with

$$\overline{\mathcal{A}}(z_1, z_2, \ldots, z_n, t) = \overline{\mathcal{A}_1}(x_1, x_2, \ldots, x_n, t) * \overline{\mathcal{A}_2}(y_1, y_2, \ldots, y_n, t)$$

and

$$\overline{\mathcal{B}}(z_1, z_2, \ldots, z_n, t) = \overline{\mathcal{B}_1}(x_1, x_2, \ldots, x_n, t) \lozenge \overline{\mathcal{B}_2}(y_1, y_2, \ldots, y_n, t);$$

where $z_i = (x_i, y_i)$, i = 1, 2,……, n.

Proof: As $\overline{A_1}(x_1, x_2, \ldots, x_n, t) + \overline{B_1}(x_1, x_2, \ldots, x_n, t) \le \bar{1}$. (1)

and it follows that

$$\overline{B_1}(x_1, x_2, \ldots, x_n, t) \le \bar{1} - \overline{A_1}(x_1, x_2, \ldots, x_n, t)$$

and $\overline{B_2}(x_1, x_2, \ldots, x_n, t) \le \bar{1} - \overline{A_2}(x_1, x_2, \ldots, x_n, t)$

So,

$$(1 - \overline{A_1}(x_1, x_2, \ldots, x_n, t)) \lozenge (\bar{1} - \overline{A_2}(x_1, x_2, \ldots, x_n, t)) \ge \overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge \overline{B_2}(x_1, x_2, \ldots, x_n, t)$$

Then, $\overline{A_1}(x_1, x_2, \ldots, x_n, t) \ \hat{\lozenge}\ \overline{A_2}(x_1, x_2, \ldots, x_n, t)$

$$= 1\text{-}\ (1-\overline{A_1}(x_1,x_2,.....,x_n,t)\lozenge \bar{1}-\overline{A_2}(x_1,x_2,.....,x_n,t))$$

$$\leq 1\text{-}(\overline{B_1}(x_1,x_2,.....,x_n,t)\lozenge \overline{B_2}(x_1,x_2,.....,x_n,t)) \quad (2)$$

Thus,

$$\overline{A_1}(x_1,x_2,.....,x_n,t)\ \hat{\lozenge}\ \overline{A_2}(x_1,x_2,.....,x_n,t) \leq 1\text{-}(\overline{B_1}(x_1,x_2,.....,x_n,t)\lozenge \overline{B_2}(x_1,x_2,.....,x_n,t))$$

where $a\ \hat{\lozenge}\ b = 1 - ((1-a) \lozenge (1-b))$ is defined as the dual *t*-norm with respect to $\lozenge$ So, if

$$\overline{A_1}(x_1,x_2,.....,x_n,t) * \overline{A_2}(y_1,y_2,......,y_n,t) \leq \overline{A_1}(x_1,x_2,.....,x_n,t)\ \hat{\lozenge}\ \overline{A_2}(y_1,y_2,......,y_n,t)$$

then by (2), we have,

$$\overline{A_1}(x_1,x_2,.....,x_n,t) * \overline{A_2}(y_1,y_2,......,y_n,t) \leq 1\text{-}(\overline{B_1}(x_1,x_2,.....,x_n,t)\lozenge \overline{B_2}(y_1,y_2,.....,y_n,t))$$

$$(\overline{A_1}(x_1,x_2,.....,x_n,t) * \overline{A_2}(y_1,y_2,......,y_n,t)) + (\overline{B_1}(x_1,x_2,.....,x_n,t)\lozenge \overline{B_2}(y_1,y_2,.....,y_n,t)) \leq 1$$

$$\overline{A}(z_1,z_2,.......,z_n,t) + \overline{B}(z_1,z_2,.......,z_n,t) \leq 1.$$

Similarly we can verify the other conditions. Thus, S $\times_{*,\lozenge}$ T is an i-f-n-NLS.

Theorem 4.2: The generalized Cartesian product of the interval-valued-intuitionistic-fuzzy n-normed linear spaces is commutative. In other words if S and T are two interval-valued-intuitionistic-fuzzy n-normed linear spaces. Then S = T $\Rightarrow$ S $\times_{*,\lozenge}$ T = T $\times_{*,\lozenge}$ S.

Proof: Assume S=T, $(x_1,x_2,.......,x_n)$, $(y_1,y_2,.......,y_n) \in X^n$. Then,

$$\overline{A_1}(x_1,x_2,.....,x_n,t) * \overline{A_2}(y_1,y_2,......,y_n,t) = \overline{A_2}(x_1,x_2,.....,x_n,t) * \overline{A_1}(y_1,y_2,......,y_n,t)$$

and $\overline{B_1}(x_1,x_2,.....,x_n,t)\lozenge\overline{B_2}(y_1,y_2,.....,y_n,t) = \overline{B_2}(x_1,x_2,.....,x_n,t)\lozenge\overline{B_1}(x_1,x_2,.....,x_n,t)$

Thus, S $\times_{*,\lozenge}$ T = T $\times_{*,\lozenge}$ S. However the converse is not true. For example, Let

S = {(X, ($\overline{\boldsymbol{A}_1}$ $(x_1,x_2,.......,x_n$, t), $\overline{\boldsymbol{B}_1}$ $(x_1,x_2,.......,x_n$, t)); $\overline{A_1}$ $(x_1,x_2,.......,x_n$,t)= $\bar{\alpha}$,

$\overline{B_1}$ $(x_1,x_2,.......,x_n$,t) = $\bar{\beta}$, $(x_1,x_2,.......,x_n) \in X^n$}

T = {(Y, $\overline{\boldsymbol{A}_2}$ $(x_1,x_2,.......,x_n$, t), $\overline{\boldsymbol{B}_2}$ $(x_1,x_2,.......,x_n$, t)); $\overline{A}_2(x_1,x_2,.......,x_n)$= $\bar{\gamma}$,

$\overline{B_2}$ $(x_1,x_2,.......,x_n)$ = $\bar{\delta}$, $(x_1,x_2,.......,x_n) \in X^n$}

Then

$$\overline{A_1}(x_1,x_2,.....,x_n,t) * \overline{A_2}(x_1,x_2,......,x_n,t) = \bar{\alpha} * \bar{\gamma} = \bar{\gamma} * \bar{\alpha} = \overline{A_2}(x_1,x_2,.....,x_n,t) * \overline{A_1}(y_1,y_2,......,y_n,t) .$$

and

$$\overline{B_1}(x_1,x_2,.....,x_n,t)\lozenge\overline{B_2}(x_1,x_2,.....,x_n,t) = \bar{\beta} \lozenge \bar{\delta} = \bar{\delta} \lozenge \bar{\beta} = \overline{B_2}(x_1,x_2,.....,x_n,t)\lozenge\overline{B_1}(x_1,x_2,.....,x_n,t) .$$

So, we obtain S $\times_{*,\lozenge}$ T = T $\times_{*,\lozenge}$ S, but S $\neq$ T, if $\bar{\alpha} \neq \bar{\gamma}$ or $\bar{\beta} \neq \bar{\delta}$.

Theorem 4.3: The generalized Cartesian product of the interval-valued-intuitionistic-fuzzy -n-normed linear spaces is distributive with respect to union and intersections. In other words if

S = {(X, ($\overline{\boldsymbol{A}_1}$ $(x_1,x_2,.......,x_n$, t); $\overline{\boldsymbol{B}_1}$ $(x_1,x_2,.......,x_n$, t)); $(x_1,x_2,.......,x_n) \in X^n$}

and T = {(Y, $\overline{\boldsymbol{A}_2}$ $(y_1,y_2,.......,y_n$, t); $\overline{\boldsymbol{B}_2}$ $(y_1,y_2,.......,y_n$, t)); $(y_1,y_2,.......,y_n) \in Y^n$}

and U= {(Y, $\overline{\boldsymbol{A}_3}$ $(y_1,y_2,.......,y_n$, t); $\overline{\boldsymbol{B}_3}$ $(y_1,y_2,.......,y_n$, t)); $(y_1,y_2,.......,y_n) \in Y^n$}

are the interval-valued-intuitionistic-fuzzy n-normed linear spaces, then

S $\times_{*,\lozenge}$ (T $\cap$ U) = (S $\times_{*,\lozenge}$ T) $\cap$ (S $\times_{*,\lozenge}$ U)

and S $\times_{*,\lozenge}$ (T $\cup$ U) =(S $\times_{*,\lozenge}$ T) $\cup$ (S $\times_{*,\lozenge}$ U)

Proof: We have,

S $\times_{*,\lozenge}$ (T $\cap$ U) =

{(X$\times$Y, $\overline{\boldsymbol{A}_1}$ $(x_1,x_2,.......,x_n$, t) $*$ min[i]{ $\overline{\boldsymbol{A}_2}$ $(y_1,y_2,.......,y_n$, t), $\overline{A}_3(y_1,y_2,.......,y_n$,t)}

$\overline{B_1}(x_1,x_2,.....,x_n,t)\lozenge$ max[i]{ $\overline{B_2}$ $(y_1,y_2,.......,y_n$,t), $\overline{B}_3(z_1,z_2,.......,z_n)$},$(z_1,z_2,.......,z_n) \in (X\times Y)^n$}

and

$(S \times_{*,\lozenge} T) \cap (S \times_{*,\lozenge} U) = \{(X \times Y, \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t),$
$\overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge$
$\overline{B_2}(y_1, y_2, \ldots, y_n, t), (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\} \cap \{(X \times Y, \overline{A_1}(x_1, x_2, \ldots, x_n, t) *$
$\overline{A_3}(y_1, y_2, \ldots, y_n, t), \overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge \overline{B_3}(y_1, y_2, \ldots, y_n, t), (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\}$
$= \{(X \times Y, \min^i\{\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t), \overline{A_1}(x_1, x_2, \ldots, x_n, t) *$
$\overline{A_3}(y_1, y_2, \ldots, y_n, t)\}, \max^i\{\overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge \overline{B_2}(y_1, y_2, \ldots, y_n, t), \overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge$
$\overline{B_3}(y_1, y_2, \ldots, y_n, t), (z_1, z_2, \ldots, z_n) \in (X \times Y)^n\}$

So it is enough to prove that,

$\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \min^i\{\overline{A_2}(y_1, y_2, \ldots, y_n, t), \overline{A_3}(y_1, y_2, \ldots, y_n, t)\}$
$= \min^i\{\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t), \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_3}$
$(y_1, y_2, \ldots, y_n, t)\}$ (1)

and

$\overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge \max^i\{\overline{B_2}(y_1, y_2, \ldots, y_n, t), \overline{B_3}(y_1, y_2, \ldots, y_n, t)\}$
$= \max^i\{\overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge \overline{B_2}(y_1, y_2, \ldots, y_n, t), \overline{B_1}(x_1, x_2, \ldots, x_n, t) \lozenge \overline{B_3}(y_1, y_2, \ldots, y_n, t)\}$ (2)

Let

$$\overline{A_2}(y_1, y_2, \ldots, y_n, t) \leq \overline{A_3}(y_1, y_2, \ldots, y_n, t) \quad (3)$$

Then

$\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t)$
$\leq \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_3}(y_1, y_2, \ldots, y_n, t)$ (4)

Therefore by (3) and (4),

LHS of (1)

$\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \min^i\{\overline{A_2}(y_1, y_2, \ldots, y_n, t), \overline{A_3}(y_1, y_2, \ldots, y_n, t)\}$
$= \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t)$
$= \min^i\{\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t), \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_3}$
$(y_1, y_2, \ldots, y_n, t)\}$
= RHS of (1).

Let

$$\overline{A_2}(y_1, y_2, \ldots, y_n, t) > \overline{A_3}(y_1, y_2, \ldots, y_n, t) \quad (5)$$

So, $\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t)$
$> \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_3}(y_1, y_2, \ldots, y_n, t)$ (6)

Therefore by (5) and (6),

LHS of (1)

$\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \min^i\{\overline{A_2}(y_1, y_2, \ldots, y_n, t), \overline{A_3}(y_1, y_2, \ldots, y_n, t)\}$
$= \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_3}(y_1, y_2, \ldots, y_n, t)$
$= \min^i\{\overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_2}(y_1, y_2, \ldots, y_n, t), \overline{A_1}(x_1, x_2, \ldots, x_n, t) * \overline{A_3}$
$(y_1, y_2, \ldots, y_n, t)\}$
= RHS of (1).

Thus equality holds in (1).
Let

$$\overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t) \le \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t) \qquad (7)$$

Then

$$\overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t) \le \overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t) \qquad (8)$$

Therefore by (7) and (8),

LHS of (2)

$= \overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \max^i\{\overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t), \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\}$

$\overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t),$

$= \max^i\{\overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t), \overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\}$

= RHS of (2)

Let

$$\overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t) > \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t) \qquad (9)$$

So,

$$\overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t) > \overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t) \qquad (10)$$

Therefore by (9) and (10),

LHS of (2)

$= \overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \max^i\{\overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t), \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\}$

$= \overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t)$

$= \max^i\{\overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t), \overline{B_1}(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\}$

= RHS of (2).

Thus equality holds in (2). Finally from (1) and (2), we have

$S \times_{*,\Diamond} (T \cap U) = (S \times_{*,\Diamond} T) \cap (S \times_{*,\Diamond} U)$. Similarly, we can prove that

$S \times_{*,\Diamond} (T \cup U) = (S \times_{*,\Diamond} T) \cup (S \times_{*,\Diamond} U)$

Theorem 4.4: The generalized cartesian product of the interval-valued- intuitionistic fuzzy n-normed linear spaces is distributive with respect to diference. In other words If

$S = \{(X, (\overline{A_1}\,(x_1, x_2, \ldots\ldots, x_n, t); \overline{B_1}\,(x_1, x_2, \ldots\ldots, x_n, t)); (x_1, x_2, \ldots\ldots, x_n) \in X^n\}$

and $T = \{(Y, \overline{A_2}\,(y_1, y_2, \ldots\ldots, y_n, t); \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t)); (y_1, y_2, \ldots\ldots, y_n) \in Y^n\}$

and $U = \{(Y, \overline{A_3}\,(y_1, y_2, \ldots\ldots, y_n, t); \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t)); (y_1, y_2, \ldots\ldots, y_n) \in Y^n\}$

are the interval-valued-intuitionistic fuzzy n-normed linear spaces, then

$S \times_{*,\Diamond} (T \setminus U) \subseteq (S \times_{*,\Diamond} T) \setminus (S \times_{*,\Diamond} U)$

$T = \{(Y, \overline{A_2}\,(y_1, y_2, \ldots\ldots, y_n, t) = \overline{1}, \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t) = \overline{0}); (y_1, y_2, \ldots\ldots, y_n) \in Y^n\}$

$U \subseteq S$, $* = \min$, $\Diamond = \max$, then equality holds.

Proof: We need to prove, $S \times_{*,\Diamond} (T \setminus U) \subseteq (S \times_{*,\Diamond} T) \setminus (S \times_{*,\Diamond} U)$. It is enough to prove,

$$\overline{A_1}\,(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{\overline{A_2}\,(y_1, y_2, \ldots\ldots, y_n, t), \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\}$$
$$\le \min^i\{\overline{A_1}\,(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A_2}\,(y_1, y_2, \ldots\ldots, y_n, t),$$
$$\overline{B_1}\,(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\} \qquad (1)$$

and

$$\overline{B_1}\,(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \max^i\{\overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t), \overline{A_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\}$$
$$\ge \max^i\{\overline{B_1}\,(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B_2}\,(y_1, y_2, \ldots\ldots, y_n, t),$$
$$\overline{A_1}\,(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A_3}\,(y_1, y_2, \ldots\ldots, y_n, t)\} \qquad (2)$$

Case-(i): Let

$$\overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t) < \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t) \quad (3)$$

and using the fact

$$a * b \le \min\{a,b\} \le a \le \max\{a,c\} \le a \lozenge c. \quad (4)$$

Then (4),

(17)

$$\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t)$$
$$< \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t) \le \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)$$
$$< \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) \lozenge \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t) \quad (5)$$
$$\Rightarrow \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t)$$
$$< \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) \lozenge \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)$$

Therefore by (3) and (5),

LHS of (1)

$$= \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{\overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$$
$$= \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t)$$
$$\min^i\{\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) \lozenge$$
$$\overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$$

= RHS of (1).

Thus equality holds in (1).

Case-(ii): Let

$$\overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t) \ge \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t) \quad (6)$$

Thus,

$$\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t)$$
$$\ge \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t) \quad (7)$$

Therefore by (6) and (7),

$$\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{\overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$$
$$= \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)$$
$$\le \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t)$$
$$\Rightarrow \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{\overline{B}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$$
$$\le \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t) \quad (8)$$

By (4) and (6),

$$\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{\overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$$
$$= \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)$$
$$\le \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t) \le \overline{B}_1(x_1, x_2, \ldots\ldots, x_n, t) \lozenge \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t) \quad (9)$$

From (8) and (9) we have,

$$\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{\overline{B}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$$
$$\le \min^i\{\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_1(x_1, x_2, \ldots\ldots, x_n, t) \lozenge \overline{B}_2$$
$$(y_1, y_2, \ldots\ldots, y_n, t)\}$$

Thus we have proved (1) and (2) can be proved similarly. So,

$$S \times_{*,\lozenge} (T \setminus U) \subseteq (S \times_{*,\lozenge} T) \setminus (S \times_{*,\lozenge} U).$$

Let

$$T = \{(Y, \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t); \overline{B}_2(y_1, y_2, \ldots\ldots, y_n, t)); (y_1, y_2, \ldots\ldots, y_n) \in Y^n\}$$

$U \subseteq S$, $* = \min$, $\lozenge = \max$, LHS of (1)

$= \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{\overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$

$= \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \min^i\{1, \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$

$= \overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)$

$= \min^i\{\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$

RHS of (1)

$= \min^i\{\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t) * \overline{A}_2(y_1, y_2, \ldots\ldots, y_n, t), \overline{B}_1(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B}_3$
$(y_1, y_2, \ldots\ldots, y_n, t)\}$

$= \min^i\{\min^i\{\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t), 1\}, \max^i\{\overline{B}_1(x_1, x_2, \ldots\ldots, x_n, t) \Diamond \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}\}$

$= \min^i\{\overline{A}_1(x_1, x_2, \ldots\ldots, x_n, t), \overline{B}_3(y_1, y_2, \ldots\ldots, y_n, t)\}$

Thus equality holds in (1).Similarly we can prove the equality in (2).

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