

**SOME RESULTS ON INTERVAL VALUED  
INTUITIONISTIC FUZZY n- NORMED LINEAR SPACE**

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**ABSTRACT**

*In this paper we first define concept of interval-valued-intuitionistic –fuzzy n-normed space and then we obtain some results in this newly defined space.*

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## 1. INTRODUCTION

The theory of Fuzzy sets was introduced by L.Zadeh [13] in 1965. Subsequently a progressive development was made in the theory of interval-valued fuzzy subsets in [14]. J.H.Park[9] studied the notion of intuitionistic fuzzy metric spaces. Many authors introduced the definition of fuzzy inner product space and fuzzy normed linear space in [4, 5, 6]. The origin and development of intuitionistic fuzzy set theory can be found in [7, 10, 11, 12]. The motive of this paper is to introduce the notion of interval- valued – intuitionistic –fuzzy n-normed space as generalization of intuitionistic-fuzzy n-normed space [20] and we obtain some results in this space. Further we introduce the generalized Cartesian product of the interval-valued-intuitionistic –fuzzy n-normed space and prove some of its properties.

## 2. PRELIMINARIES

In this section we provide the essential definitions and results necessary for the development of our theory.

**Definition 2.1 [14]:** An interval number on  $[0, 1]$ , say  $\bar{a}$ , is a closed sub interval of  $[0, 1]$  of the form  $\bar{a} = [a^-, a^+]$ , where  $0 \leq a^- \leq a^+ \leq 1$ . Let  $D[0, 1]$  denote the family of all closed sub-intervals of  $[0, 1]$ , that is,

$$D[0, 1] = \{ \bar{a} = [a^-, a^+]: a^- \leq a^+ \text{ and } a^-, a^+ \in [0, 1] \}$$

**Definition 2.2 [14]:** Let  $\bar{a}_i = [a_i^-, a_i^+] \in D[0, 1]$  for all  $i \in \Omega$ ,  $\Omega$  an index set.

Define

$$(a) \inf^i \{ \bar{a}_i : i \in \Omega \} = [ \inf_{i \in \Omega} a_i^-, \inf_{i \in \Omega} a_i^+ ]$$

$$(b) \sup^i \{ \bar{a}_i : i \in \Omega \} = [ \sup_{i \in \Omega} a_i^-, \sup_{i \in \Omega} a_i^+ ]$$

In particular, whenever  $\bar{a} = [a^-, a^+]$ ,  $\bar{b} = [b^-, b^+]$ , in  $D[0, 1]$ , we define

- (i)  $\bar{a} \leq \bar{b}$  if and only if  $a^- \leq b^-$  and  $a^+ \leq b^+$
- (ii)  $\bar{a} = \bar{b}$  if and only if  $a^- = b^-$  and  $a^+ = b^+$
- (iii)  $\bar{a} < \bar{b}$  if and only if  $a^- < b^-$  and  $a^+ < b^+$ .
- (iv)  $\min^i \{ \bar{a}, \bar{b} \} = [\min \{ a^-, b^- \}, \min \{ a^+, b^+ \}]$
- (v)  $\max^i \{ \bar{a}, \bar{b} \} = [\max \{ a^-, b^- \}, \max \{ a^+, b^+ \}]$ .

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**Definition 2.3 [14]:** Let  $X$  be a set. A mapping  $\bar{A} : X \rightarrow D[0, 1]$  is called an Interval-valued fuzzy subset (briefly, an i-v fuzzy subset) of  $X$ , where  $\bar{A}(x) = [A^-(x), A^+(x)]$ , and  $A^-$  and  $A^+$  are fuzzy subsets in  $X$  such that  $A^-(x) \leq A^+(x)$  for all  $x \in X$ .

**Definition 2.4 [14]:** Let  $\bar{A}$  be an interval-valued fuzzy subset of  $X$  and  $[t_1, t_2] \in D[0, 1]$ . Then the set

$$\bar{U}(\bar{A}; [t_1, t_2]) = \{x \in X : \bar{A}(x) \geq [t_1, t_2]\}$$

Note that

$$\bar{U}(\bar{A}; [t_1, t_2]) = U(A^-, t_1) \cap U(A^+, t_2) \text{ where}$$

$$U(A^-; t_1) = \{x \in X : A^-(x) \geq t_1\} \text{ and}$$

$$U(A^+; t_2) = \{x \in X : A^+(x) \geq t_2\}.$$

**Definition 2.5 [2]:** Let  $V$  denote a vector space of dimension  $n$  over a field  $F$ . A fuzzy subspace is a fuzzy subset  $\mu$  of  $V$  such that

$$\mu(\alpha x + \beta y) \geq \mu(x) \wedge \mu(y), x, y \in V, \alpha, \beta \in F(\text{field}), \text{ where } \wedge \text{ stands for intersection.}$$

**Definition 2.6:** Let  $V$  denote a vector space of dimension  $n$  over a field  $F$ . An intuitionistic fuzzy subspace is a fuzzy subset  $\mu, \eta$  of  $V$  such that

$$\mu(\alpha x + \beta y) \geq \mu(x) \wedge \mu(y)$$

$$\text{and } \eta(\alpha x + \beta y) \leq \eta(x) \vee \eta(y), x, y \in V, \alpha, \beta \in F(\text{field}),$$

where  $\wedge$  and  $\vee$  stands for intersection and union respectively.

**Remark 2.7:** A subset  $\{x_1, x_2, \dots, x_m\}$  of  $V$  is intuitionistic-fuzzy linearly independent if it is linearly independent

$$\text{and } \mu\left(\sum_{i=1}^m \alpha_i x_i\right) = \bigwedge_{i=1}^m \mu(x_i),$$

$$\eta\left(\sum_{i=1}^m \alpha_i x_i\right) = \bigvee_{i=1}^m \eta(x_i) \text{ for all } \alpha_i \in F, \alpha_i \neq 0, i = 1, \dots, m, \wedge, \vee \text{ stands for intersection and union respectively.}$$

**Definition 2.8 [20]:** An intuitionistic fuzzy n-normed linear space (or) in short i-f-n-NLS is an object of the form

$$A = \{(X, N(x_1, x_2, \dots, x_n, t), M(x_1, x_2, \dots, x_n, t) : (x_1, x_2, \dots, x_n) \in X^n\}$$

where  $X$  is a linear space over a field  $F$  and  $N, M$  are fuzzy sets on  $X^n \times (0, 1)$ ,  $N$  denotes the degree of membership and  $M$  denotes the degree of non-membership of  $(x_1, x_2, \dots, x_n) \in X^n \times (0, 1)$  satisfying the following conditions:

- (i)  $N(x_1, x_2, \dots, x_n, t) + M(x_1, x_2, \dots, x_n, t) \leq 1$ ;
- (ii)  $N(x_1, x_2, \dots, x_n, t) > 0$ ;
- (iii)  $N(x_1, x_2, \dots, x_n, t) = 1$  if and only if  $x_1, x_2, \dots, x_n$  are linearly dependent;
- (iv)  $N(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ .
- (v)  $N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ , if  $c \neq 0, c \in F(\text{field})$ .
- (vi)  $N(x_1, x_2, \dots, x_n + x_n^{'}, s+t) \geq \min \{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x_n^{'}, t)\}$
- (vii)  $N(x_1, x_2, \dots, x_n, t) : (0;1) \rightarrow [0;1]$  is continuous in  $t$ ;
- (viii)  $M(x_1, x_2, \dots, x_n, t) > 0$ ;
- (ix)  $M(x_1, x_2, \dots, x_n, t) = 0$  if and only if  $x_1, x_2, \dots, x_n$  are linearly dependent;
- (x)  $M(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ .
- (xi)  $M(x_1, x_2, \dots, cx_n, t) = M(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ , if  $c \neq 0, c \in F(\text{field})$ ;
- (xii)  $M(x_1, x_2, \dots, x_n + x_n^{'}, s+t) \leq \max \{M(x_1, x_2, \dots, x_n, s), M(x_1, x_2, \dots, x_n^{'}, t)\}$
- (xiii)  $M(x_1, x_2, \dots, x_n, t) : (0;1) \rightarrow [0;1]$  is continuous in  $t$ .

**Remark 2.9:**

- (i) Let  $V$  be a vector space over a field  $F$  and let  $U_1, U_2, \dots, U_n$  be the subspaces of  $V$ .  $V$  is said to be direct sum of  $U_1, U_2, \dots, U_n$  if every element  $v \in V$  can be written in one and only one way  
 $v = u_1 + u_2 + \dots + u_n$  where  $u_i \in U_i$  and denoted as  $V = U_1 \oplus U_2 \oplus \dots \oplus U_n$ .
- (ii) Let  $V$  and  $W$  be vector spaces over a field  $F$ . Then the tensor product of two vectors is denoted by  $V \otimes W$  and given  $t \in V \otimes W$ ,  $t$  can be uniquely written as,  $t = \sum_{i,j} t_{ij} x_i \otimes y_j$  where the sum is taken over all  $i, j$  for which  $t_{ij} \neq 0$ .

### 3. INTERVAL-VALUED-INTUITIONSTIC- FUZZY-n-NORMED LINEAR SUBSPACES

As a generalization of definition 2.8 we have the following notion of interval- valued-intuitionstic-fuzzy n-normed linear subspaces.

**Definition 3.1:** Let  $X$  be a linear space over a field  $F$ . An interval-valued fuzzy subsets  $\overline{N}, \overline{M}$  of  $X^n \times R$  is called an interval-valued- intuitionstic fuzzy n-norm if and only if

1.  $\overline{N}(x_1, x_2, \dots, x_n, t) + \overline{M}(x_1, x_2, \dots, x_n, t) \leq \bar{1}$
2. For all  $t \in R$  with  $t \leq 0$ ,  $\overline{N}(x_1, x_2, \dots, x_n, t) = \bar{0}$ .
3. For all  $t \in R$  with  $t > 0$ ,  $\overline{N}(x_1, x_2, \dots, x_n, t) = \bar{1}$  iff  $x_1, x_2, \dots, x_n$  are linearly dependent.
4.  $\overline{N}(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ .
5. For all  $t \in R$  with  $t > 0$   $\overline{N}(x_1, x_2, \dots, cx_n, t) = \overline{N}(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ , if  $c \neq 0, c \in F$ (field).
6. For all  $s, t \in R$   $\overline{N}(x_1, x_2, \dots, x_n + x_n^+, s + t) \geq \min^i \{\overline{N}(x_1, x_2, \dots, x_n, s), \overline{N}(x_1, x_2, \dots, x_n^+, t)\}$
7.  $\overline{N}(x_1, x_2, \dots, x_n, t)$  is a non decreasing function of  $t \in R$  and  $\lim_{t \rightarrow \infty} \overline{N}(x_1, x_2, \dots, x_n, t) = \bar{1}$ .
8. For all  $t \in R$  with  $t \leq 0$ ,  $\overline{M}(x_1, x_2, \dots, x_n, t) > \bar{0}$ .
9. For all  $t \in R$  with  $t > 0$ ,  $\overline{M}(x_1, x_2, \dots, x_n, t) = \bar{0}$  iff  $x_1, x_2, \dots, x_n$  are linearly dependent.
10.  $\overline{M}(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ .
11. For all  $t \in R$  with  $t > 0$   $\overline{M}(x_1, x_2, \dots, cx_n, t) = \overline{M}(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ , if  $c \neq 0, c \in F$ (field).
12. For all  $s, t \in R$   $\overline{M}(x_1, x_2, \dots, x_n + x_n^+, s + t) \leq \max^i \{\overline{M}(x_1, x_2, \dots, x_n, s), \overline{M}(x_1, x_2, \dots, x_n^+, t)\}$
13.  $\overline{M}(x_1, x_2, \dots, x_n, t)$  is a non increasing function of  $t \in R$  and  $\lim_{t \rightarrow \infty} \overline{M}(x_1, x_2, \dots, x_n, t) = \bar{0}$

Then  $(X, \overline{N}, \overline{M})$  is called an interval- valued Intuitionstic fuzzy n-normed linear space in short i-v-i-f-n NLS.

The following example agrees with our notion of i-v-i-f-n-NLS

**Example 3.2:** Let  $(X, \|\bullet, \bullet, \dots, \bullet\|)$  be an n-normed space . Define

$$\overline{N}(x_1, x_2, \dots, x_n, t) = \begin{cases} \bar{0} & \text{when } t \leq \|x_1, x_2, \dots, x_n\| \\ \bar{1} & \text{when } t > \|x_1, x_2, \dots, x_n\| \end{cases}$$

and  $\overline{M}(x_1, x_2, \dots, x_n, t) = \begin{cases} \bar{1} & \text{when } t \leq \|x_1, x_2, \dots, x_n\| \\ \bar{0} & \text{when } t > \|x_1, x_2, \dots, x_n\| \end{cases}$

Then  $(X, \overline{N}, \overline{M})$  is an i-v-i-f-n-NLS where  $\bar{0} = [0,0]$  and  $\bar{1} = [1,1]$ .

Now, we give a result that intersection of two i-v-i-f-n-NLS, and hence result holds for finite intersection of i-v-i-f-n-NLS.

**Theorem 3.3:** Let  $(X, \overline{N}_1, \overline{M}_1)$  and  $(X, \overline{N}_2, \overline{M}_2)$  be two i-v-i-f-n-NLS. Define

$$\begin{aligned}\overline{N}(x_1, x_2, \dots, x_n, t) &= (\overline{N}_1 \cap \overline{N}_2)(x_1, x_2, \dots, x_n, t) \\ &= \min^i \left\{ \overline{N}_1(x_1, x_2, \dots, x_n, t), \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\}\end{aligned}$$

and

$$\begin{aligned}\overline{M}(x_1, x_2, \dots, x_n, t) &= (\overline{M}_1 \cap \overline{M}_2)(x_1, x_2, \dots, x_n, t) \\ &= \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t) \right\}\end{aligned}$$

for all  $(x_1, x_2, \dots, x_n, t) \in X^n \times R$ . Then  $(X, \overline{N}_1 \cap \overline{N}_2, \overline{M}_1 \cap \overline{M}_2)$  or  $(X, \overline{N}, \overline{M})$  is an i-v-i-f-n-NLS.

**Proof:** (1) Since  $\overline{N}_1(x_1, x_2, \dots, x_n, t) + \overline{M}_1(x_1, x_2, \dots, x_n, t) \leq \bar{1}$ .

$$\text{and } \overline{N}_2(x_1, x_2, \dots, x_n, t) + \overline{M}_2(x_1, x_2, \dots, x_n, t) \leq \bar{1}.$$

It follows that

$$\overline{M}_1(x_1, x_2, \dots, x_n, t) \leq \bar{1} - \overline{N}_1(x_1, x_2, \dots, x_n, t)$$

$$\text{and } \overline{M}_2(x_1, x_2, \dots, x_n, t) \leq \bar{1} - \overline{N}_2(x_1, x_2, \dots, x_n, t)$$

By definition

$$\begin{aligned}\max^i \left\{ \bar{1} - \overline{N}_1(x_1, x_2, \dots, x_n, t), \bar{1} - \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\} \\ \geq \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t) \right\}\end{aligned}$$

$$\begin{aligned}\Rightarrow \min^i \left\{ \overline{N}_1(x_1, x_2, \dots, x_n, t), \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\} \\ = \bar{1} - \left( \max^i \left\{ \bar{1} - \overline{N}_1(x_1, x_2, \dots, x_n, t), \bar{1} - \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\} \right) \\ \leq \bar{1} - \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t) \right\}\end{aligned}$$

$$\begin{aligned}\Rightarrow \min^i \left\{ \overline{N}_1(x_1, x_2, \dots, x_n, t), \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\} \\ + \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t) \right\} \leq \bar{1}\end{aligned}$$

$$\Rightarrow \overline{N}(x_1, x_2, \dots, x_n, t) + \overline{M}(x_1, x_2, \dots, x_n, t) = \bar{1}$$

(2) For all  $t \in R$  with  $t \leq 0$ , since  $\overline{N}_1(x_1, x_2, \dots, x_n, t) = \bar{0}$  and  $\overline{N}_2(x_1, x_2, \dots, x_n, t) = \bar{0}$

$$\Rightarrow \min^i \left\{ \overline{N}_1(x_1, x_2, \dots, x_n, t), \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\} = \bar{0}$$

$$\Rightarrow \overline{N}(x_1, x_2, \dots, x_n, t) = \bar{0}.$$

(3) Since for all  $t \in R$  with  $t > 0$ ,  $\overline{N}_1(x_1, x_2, \dots, x_n, t) = \bar{1}$  and  $\overline{N}_2(x_1, x_2, \dots, x_n, t) = \bar{1}$  iff  $x_1, x_2, \dots, x_n$  are linearly dependent.

$$\Rightarrow \min^i \left\{ \overline{N}_1(x_1, x_2, \dots, x_n, t), \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\} = \bar{1} \text{ iff } x_1, x_2, \dots, x_n \text{ are linearly dependent.}$$

$$\Rightarrow \overline{N}(x_1, x_2, \dots, x_n, t) = \bar{1} \text{ iff } x_1, x_2, \dots, x_n \text{ are linearly dependent.}$$

(4) Since  $\overline{N}_1(x_1, x_2, \dots, x_n, t)$  and  $\overline{N}_2(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ .

$$\Rightarrow \min^i \left\{ \overline{N}_1(x_1, x_2, \dots, x_n, t), \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\} \text{ is also invariant under any permutation of } x_1, x_2, \dots, x_n.$$

$\Rightarrow \overline{N}(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ .

(5) Since for all  $t \in R$  with  $t > 0$ ,  $\overline{N}_1(x_1, x_2, \dots, cx_n, t) = \overline{N}_1(x_1, x_2, \dots, x_n, \frac{t}{|c|})$  and  $\overline{N}_2(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ , if  $c \neq 0, c \in F(\text{field})$ .

$$\begin{aligned} \Rightarrow \min^i & \left\{ \overline{N_1}(x_1, x_2, \dots, cx_n, t), \overline{N_2}(x_1, x_2, \dots, cx_n, t) \right\} \\ &= \min^i \left\{ \overline{N_1}(x_1, x_2, \dots, x_n, \frac{t}{|c|}), \overline{N_2}(x_1, x_2, \dots, x_n, \frac{t}{|c|}) \right\} \\ \Rightarrow \overline{N}(x_1, x_2, \dots, cx_n, t) &= \overline{N}(x_1, x_2, \dots, x_n, \frac{t}{|c|}). \end{aligned}$$

(6) For all  $s, t \in R$   $\overline{N}(x_1, x_2, \dots, x_n + x_n^+, s + t) \geq \min^i \left\{ \overline{N}(x_1, x_2, \dots, x_n, s), \overline{N}(x_1, x_2, \dots, x_n^+, t) \right\}$ , since  
 $\overline{N}(x_1, x_2, \dots, x_n + x_n^+, s + t) = \min^i \left\{ \overline{N_1}(x_1, x_2, \dots, x_n + x_n^+, s + t), \overline{N_2}(x_1, x_2, \dots, x_n + x_n^+, t + s) \right\}$  and  
 $\min^i \left\{ \overline{N_1}(x_1, x_2, \dots, x_n + x_n^+, s + t), \overline{N_2}(x_1, x_2, \dots, x_n + x_n^+, t + s) \right\}$   
 $\geq \min^i \left\{ \min^i \left\{ \overline{N_1}(x_1, x_2, \dots, x_n, s), \overline{N_1}(x_1, x_2, \dots, x_n^+, t) \right\}, \min^i \left\{ \overline{N_2}(x_1, x_2, \dots, x_n, s), \overline{N_2}(x_1, x_2, \dots, x_n^+, t) \right\} \right\}$   
 $\geq \min^i \left\{ \min^i \left\{ \overline{N_1}(x_1, x_2, \dots, x_n, s), \overline{N_2}(x_1, x_2, \dots, x_n, s) \right\}, \min^i \left\{ \overline{N_1}(x_1, x_2, \dots, x_n^+, t), \overline{N_2}(x_1, x_2, \dots, x_n^+, t) \right\} \right\}$   
 $\geq \min^i \left\{ \overline{N}(x_1, x_2, \dots, x_n, s), \overline{N}(x_1, x_2, \dots, x_n^+, t) \right\}$

(7)  $\overline{N_1}(x_1, x_2, \dots, x_n, t), \overline{N_2}(x_1, x_2, \dots, x_n, t)$  is a non decreasing function of  $t \in R$  and

$$\lim_{t \rightarrow \infty} \overline{N_1}(x_1, x_2, \dots, x_n, t) = \bar{1}, \quad \lim_{t \rightarrow \infty} \overline{N_2}(x_1, x_2, \dots, x_n, t) = \bar{1}$$

$$\Rightarrow \overline{N}(x_1, x_2, \dots, x_n, t) \text{ is a non decreasing function of } t \in R \text{ and } \lim_{t \rightarrow \infty} \overline{N}(x_1, x_2, \dots, x_n, t) = \bar{1}$$

(8) For all  $t \in R$  with  $t \leq 0$ ,  $\overline{M}_1(x_1, x_2, \dots, x_n, t) > \bar{0}$ ,  $\overline{M}_2(x_1, x_2, \dots, x_n, t) > \bar{0}$  and

$$\begin{aligned} \overline{M}(x_1, x_2, \dots, x_n, t) &= (\overline{M}_1 \cap \overline{M}_2)(x_1, x_2, \dots, x_n, t) \\ &= \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t) \right\} \\ \Rightarrow \overline{M}(x_1, x_2, \dots, x_n, t) &> \bar{0}. \end{aligned}$$

(9) For all  $t \in R$  with  $t > 0$ ,  $\overline{M}_1(x_1, x_2, \dots, x_n, t) = \bar{0}$ ,  $\overline{M}_2(x_1, x_2, \dots, x_n, t) = \bar{0}$  iff  $x_1, x_2, \dots, x_n$  are linearly dependent. So,

$$\begin{aligned} \overline{M}(x_1, x_2, \dots, x_n, t) &= (\overline{M}_1 \cap \overline{M}_2)(x_1, x_2, \dots, x_n, t) \\ &= \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t) \right\} = 0 \end{aligned}$$

iff  $x_1, x_2, \dots, x_n$  are linearly dependent.

(10) By definition of  $\overline{M}(x_1, x_2, \dots, x_n, t)$ , and using the fact that  $\overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ . It is clear that  $\overline{M}(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, x_2, \dots, x_n$ .

(11) For all  $t \in R$  with  $t > 0$   $\overline{M}_1(x_1, x_2, \dots, cx_n, t) = \overline{M}_1(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ , and  $\overline{M}_2(x_1, x_2, \dots, cx_n, t) = \overline{M}_2(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ , if  $c \neq 0, c \in F(\text{field})$ .

$$\begin{aligned} \Rightarrow \overline{M}(x_1, x_2, \dots, cx_n, t) &= (\overline{M}_1 \cap \overline{M}_2)(x_1, x_2, \dots, cx_n, t) \\ &= \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, cx_n, t), \overline{M}_2(x_1, x_2, \dots, cx_n, t) \right\} \\ &= \max^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, \frac{t}{|c|}), \overline{M}_2(x_1, x_2, \dots, x_n, \frac{t}{|c|}) \right\} \\ &= \overline{M}(x_1, x_2, \dots, x_n, \frac{t}{|c|}) \end{aligned}$$

$$\begin{aligned}
 (12) \text{ For all } s, t \in \mathbb{R} \quad & \overline{M_1}(x_1, x_2, \dots, x_n + x_n^+, s+t) \leq \max^i \left\{ \overline{M_1}(x_1, x_2, \dots, x_n, s), \overline{M_1}(x_1, x_2, \dots, x_n^+, t) \right\} \\
 \text{and} \quad & \overline{M_2}(x_1, x_2, \dots, x_n + x_n^+, s+t) \leq \max^i \left\{ \overline{M_2}(x_1, x_2, \dots, x_n, s), \overline{M_2}(x_1, x_2, \dots, x_n^+, t) \right\} \\
 \Rightarrow \quad & \overline{M}(x_1, x_2, \dots, x_n + x_n^+, s+t) = (\overline{M_1} \cap \overline{M_2})(x_1, x_2, \dots, x_n + x_n^+, s+t) \\
 & = \max^i \left\{ \overline{M_1}(x_1, x_2, \dots, x_n + x_n^+, s+t), \overline{M_2}(x_1, x_2, \dots, x_n + x_n^+, s+t) \right\}. \\
 \leq \max^i \left\{ \max^i \{ \overline{M_1}(x_1, x_2, \dots, x_n, s), \overline{M_1}(x_1, x_2, \dots, x_n^+, t) \}, \max^i \{ \overline{M_2}(x_1, x_2, \dots, x_n, s), \overline{M_2}(x_1, x_2, \dots, x_n^+, t) \} \right\} \\
 \leq \max^i \left\{ \max^i \{ \overline{M_1}(x_1, x_2, \dots, x_n, s), \overline{M_2}(x_1, x_2, \dots, x_n, s) \}, \max^i \{ \overline{M_1}(x_1, x_2, \dots, x_n^+, t), \overline{M_2}(x_1, x_2, \dots, x_n^+, t) \} \right\} \\
 \leq \max^i \left\{ \overline{M}(x_1, x_2, \dots, x_n, s), \overline{M}(x_1, x_2, \dots, x_n^+, t) \right\}.
 \end{aligned}$$

Hence  $(X, \overline{N}_1 \cap \overline{N}_2, \overline{M}_1 \cap \overline{M}_2)$  or  $(X, \overline{N}, \overline{M})$  is an i-v-i-f-n-NLS.

**Remark 3.4:** Let  $(X, \overline{N}_1, \overline{M}_1)$  and  $(X, \overline{N}_2, \overline{M}_2)$  be two i-v-i-f-n-NLS. Define

$$\begin{aligned}
 \overline{N}(x_1, x_2, \dots, x_n, t) &= (\overline{N}_1 \cup \overline{N}_2)(x_1, x_2, \dots, x_n, t) \\
 &= \max^i \left\{ \overline{N}_1(x_1, x_2, \dots, x_n, t), \overline{N}_2(x_1, x_2, \dots, x_n, t) \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 \overline{M}(x_1, x_2, \dots, x_n, t) &= (\overline{M}_1 \cup \overline{M}_2)(x_1, x_2, \dots, x_n, t) \\
 &= \min^i \left\{ \overline{M}_1(x_1, x_2, \dots, x_n, t), \overline{M}_2(x_1, x_2, \dots, x_n, t) \right\}
 \end{aligned}$$

for all  $(x_1, x_2, \dots, x_n, t) \in X^n \times R$ . Then  $(X, \overline{N}_1 \cup \overline{N}_2, \overline{M}_1 \cup \overline{M}_2)$  or  $(X, \overline{N}, \overline{M})$  is an not i-v-i-f-n-NLS.

#### 4. GENERALIZED CARTESIAN PRODUCT OF THE INTERVAL-VALUED-INTUITIONISTIC FUZZY-n-NORMED LINEAR SPACES

We now proceed to our new notion of generalized cartesian product of the Interval-valued-intuitionistic fuzzy  $n$ -normed linear spaces in the following theorem.

**Theorem 4.1:** Let

$$S = \{(X, (\overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t); \overline{\mathbf{B}}_1(x_1, x_2, \dots, x_n, t)); (x_1, x_2, \dots, x_n) \in X^n)\}$$

and

$$T = \{(Y, (\overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t); \overline{\mathbf{B}}_2(y_1, y_2, \dots, y_n, t)); (y_1, y_2, \dots, y_n) \in Y^n)\}$$

be two interval-valued-intuitionistic fuzzy  $n$ -normed linear spaces. Then

$S \times_{*, \diamond} T = \{(X \times Y, (\overline{\mathbf{A}}(z_1, z_2, \dots, z_n, t); \overline{\mathbf{B}}(z_1, z_2, \dots, z_n, t)); (z_1, z_2, \dots, z_n) \in (X \times Y)^n)\}$  is an i-v-i-f-n-NLS with

$$\overline{\mathbf{A}}(z_1, z_2, \dots, z_n, t) = \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t)$$

and

$$\overline{\mathbf{B}}(z_1, z_2, \dots, z_n, t) = \overline{\mathbf{B}}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{\mathbf{B}}_2(y_1, y_2, \dots, y_n, t);$$

where  $z_i = (x_i, y_i)$ ,  $i = 1, 2, \dots, n$ .

**Proof:** As  $\overline{A}_1(x_1, x_2, \dots, x_n, t) + \overline{B}_1(x_1, x_2, \dots, x_n, t) \leq \bar{1}$ . (1)

and it follows that

$$\overline{B}_1(x_1, x_2, \dots, x_n, t) \leq \bar{1} - \overline{A}_1(x_1, x_2, \dots, x_n, t)$$

$$\text{and } \overline{B}_2(x_1, x_2, \dots, x_n, t) \leq \bar{1} - \overline{A}_2(x_1, x_2, \dots, x_n, t)$$

So,

$$(1 - \overline{A}_1(x_1, x_2, \dots, x_n, t)) \diamond (1 - \overline{A}_2(x_1, x_2, \dots, x_n, t)) \geq \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(x_1, x_2, \dots, x_n, t)$$

Then,  $\overline{A}_1(x_1, x_2, \dots, x_n, t) \overset{\wedge}{\diamond} \overline{A}_2(x_1, x_2, \dots, x_n, t)$

$$\begin{aligned}
 &= 1 - (\overline{A}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(x_1, x_2, \dots, x_n, t)) \\
 &\leq 1 - (\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(x_1, x_2, \dots, x_n, t))
 \end{aligned} \tag{2}$$

Thus,

$$\overline{A}_1(x_1, x_2, \dots, x_n, t) \hat{\diamond} \overline{A}_2(x_1, x_2, \dots, x_n, t) \leq 1 - (\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(x_1, x_2, \dots, x_n, t))$$

where  $a \hat{\diamond} b = 1 - ((1 - a) \diamond (1 - b))$  is defined as the dual  $t$ -norm with respect to  $\diamond$ . So, if

$$\overline{A}_1(x_1, x_2, \dots, x_n, t) * \overline{A}_2(y_1, y_2, \dots, y_n, t) \leq \overline{A}_1(x_1, x_2, \dots, x_n, t) \hat{\diamond} \overline{A}_2(y_1, y_2, \dots, y_n, t)$$

then by (2), we have,

$$\begin{aligned}
 &\overline{A}_1(x_1, x_2, \dots, x_n, t) * \overline{A}_2(y_1, y_2, \dots, y_n, t) \leq 1 - (\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t)) \\
 &(\overline{A}_1(x_1, x_2, \dots, x_n, t) * \overline{A}_2(y_1, y_2, \dots, y_n, t)) + (\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t)) \leq 1 \\
 &\overline{A}(z_1, z_2, \dots, z_n, t) + \overline{B}(z_1, z_2, \dots, z_n, t) \leq 1.
 \end{aligned}$$

Similarly we can verify the other conditions. Thus,  $S \times_{*,\diamond} T$  is an i-f-n-NLS.

**Theorem 4.2:** The generalized Cartesian product of the interval-valued-intuitionistic-fuzzy n-normed linear spaces is commutative. In other words if  $S$  and  $T$  are two interval-valued-intuitionistic-fuzzy n-normed linear spaces. Then  $S = T \Rightarrow S \times_{*,\diamond} T = T \times_{*,\diamond} S$ .

**Proof:** Assume  $S = T$ ,  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ . Then,

$$\overline{A}_1(x_1, x_2, \dots, x_n, t) * \overline{A}_2(y_1, y_2, \dots, y_n, t) = \overline{A}_2(x_1, x_2, \dots, x_n, t) * \overline{A}_1(y_1, y_2, \dots, y_n, t)$$

$$\text{and } \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t) = \overline{B}_2(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_1(y_1, y_2, \dots, y_n, t)$$

Thus,  $S \times_{*,\diamond} T = T \times_{*,\diamond} S$ . However the converse is not true. For example, Let

$$S = \{(X, (\overline{A}_1(x_1, x_2, \dots, x_n, t), \overline{B}_1(x_1, x_2, \dots, x_n, t)); \overline{A}_1(x_1, x_2, \dots, x_n, t) = \bar{\alpha},$$

$$\overline{B}_1(x_1, x_2, \dots, x_n, t) = \bar{\beta}, (x_1, x_2, \dots, x_n) \in X^n\}$$

$$T = \{(Y, (\overline{A}_2(x_1, x_2, \dots, x_n, t), \overline{B}_2(x_1, x_2, \dots, x_n, t)); \overline{A}_2(x_1, x_2, \dots, x_n) = \bar{\gamma},$$

$$\overline{B}_2(x_1, x_2, \dots, x_n) = \bar{\delta}, (x_1, x_2, \dots, x_n) \in X^n\}$$

Then

$$\overline{A}_1(x_1, x_2, \dots, x_n, t) * \overline{A}_2(x_1, x_2, \dots, x_n, t) = \bar{\alpha} * \bar{\gamma} = \bar{\gamma} * \bar{\alpha} = \overline{A}_2(x_1, x_2, \dots, x_n, t) * \overline{A}_1(y_1, y_2, \dots, y_n, t).$$

and

$$\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(x_1, x_2, \dots, x_n, t) = \bar{\beta} \diamond \bar{\delta} = \bar{\delta} \diamond \bar{\beta} = \overline{B}_2(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_1(y_1, y_2, \dots, y_n, t).$$

So, we obtain  $S \times_{*,\diamond} T = T \times_{*,\diamond} S$ , but  $S \neq T$ , if  $\bar{\alpha} \neq \bar{\gamma}$  or  $\bar{\beta} \neq \bar{\delta}$ .

**Theorem 4.3:** The generalized Cartesian product of the interval-valued-intuitionistic-fuzzy n-normed linear spaces is distributive with respect to union and intersections. In other words if

$$S = \{(X, (\overline{A}_1(x_1, x_2, \dots, x_n, t), \overline{B}_1(x_1, x_2, \dots, x_n, t)); (x_1, x_2, \dots, x_n) \in X^n\}$$

$$\text{and } T = \{(Y, (\overline{A}_2(y_1, y_2, \dots, y_n, t), \overline{B}_2(y_1, y_2, \dots, y_n, t)); (y_1, y_2, \dots, y_n) \in Y^n\}$$

$$\text{and } U = \{(Z, (\overline{A}_3(z_1, z_2, \dots, z_n, t), \overline{B}_3(z_1, z_2, \dots, z_n, t)); (z_1, z_2, \dots, z_n) \in Z^n\}$$

are the interval-valued-intuitionistic-fuzzy n-normed linear spaces, then

$$S \times_{*,\diamond} (T \cap U) = (S \times_{*,\diamond} T) \cap (S \times_{*,\diamond} U)$$

$$\text{and } S \times_{*,\diamond} (T \cup U) = (S \times_{*,\diamond} T) \cup (S \times_{*,\diamond} U)$$

**Proof:** We have,

$$S \times_{*,\diamond} (T \cap U) =$$

$$\{(X \times Y, (\overline{A}_1(x_1, x_2, \dots, x_n, t) * \min^i\{\overline{A}_2(y_1, y_2, \dots, y_n, t), \overline{A}_3(z_1, z_2, \dots, z_n, t)\})$$

$$\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \max^i\{\overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_3(z_1, z_2, \dots, z_n, t)\}), (z_1, z_2, \dots, z_n) \in (X \times Y)^n\}$$

and

$$\begin{aligned}
 & (S \times_{*,\diamond} T) \cap (S \times_{*,\diamond} U) = \{(X \times Y, \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \\
 & \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \\
 & \overline{B}_2(y_1, y_2, \dots, y_n, t), (z_1, z_2, \dots, z_n) \in (X \times Y)^n\} \cap \{(X \times Y, \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \\
 & \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t), \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t), (z_1, z_2, \dots, z_n) \in (X \times Y)^n\} \\
 & = \{(X \times Y, \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \\
 & \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t)\}, \max^i \{ \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \\
 & \overline{B}_3(y_1, y_2, \dots, y_n, t)\}, (z_1, z_2, \dots, z_n) \in (X \times Y)^n\}
 \end{aligned}$$

So it is enough to prove that,

$$\begin{aligned}
 & \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t)\} \\
 & = \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_3 \\
 & (y_1, y_2, \dots, y_n, t) \}
 \end{aligned} \tag{1}$$

and

$$\begin{aligned}
 & \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \max^i \{ \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_3(y_1, y_2, \dots, y_n, t)\} \\
 & = \max^i \{ \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t)\}
 \end{aligned} \tag{2}$$

Let

$$\overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t) \leq \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t) \tag{3}$$

Then

$$\begin{aligned}
 & \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t) \\
 & \leq \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t)
 \end{aligned} \tag{4}$$

Therefore by (3) and (4),

LHS of (1)

$$\begin{aligned}
 & \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t)\} \\
 & = \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t) \\
 & = \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_3 \\
 & (y_1, y_2, \dots, y_n, t)\} \\
 & = \text{RHS of (1)}.
 \end{aligned}$$

Let

$$\overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t) > \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t) \tag{5}$$

$$\begin{aligned}
 \text{So, } & \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t) \\
 & > \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t)
 \end{aligned} \tag{6}$$

Therefore by (5) and (6),

LHS of (1)

$$\begin{aligned}
 & \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t)\} \\
 & = \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_3(y_1, y_2, \dots, y_n, t) \\
 & = \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_3 \\
 & (y_1, y_2, \dots, y_n, t)\} \\
 & = \text{RHS of (1)}.
 \end{aligned}$$

Thus equality holds in (1).

Let

$$\overline{B}_2(y_1, y_2, \dots, y_n, t) \leq \overline{B}_3(y_1, y_2, \dots, y_n, t) \quad (7)$$

Then

$$\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t) \leq \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t) \quad (8)$$

Therefore by (7) and (8),

LHS of (2)

$$\begin{aligned} &= \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \max^i \{ \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_3(y_1, y_2, \dots, y_n, t) \} \\ &= \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t), \\ &= \max^i \{ \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t) \} \\ &= \text{RHS of (2)} \end{aligned}$$

Let

$$\overline{B}_2(y_1, y_2, \dots, y_n, t) > \overline{B}_3(y_1, y_2, \dots, y_n, t) \quad (9)$$

So,

$$\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t) > \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t) \quad (10)$$

Therefore by (9) and (10),

LHS of (2)

$$\begin{aligned} &= \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \max^i \{ \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_3(y_1, y_2, \dots, y_n, t) \} \\ &= \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t) \\ &= \max^i \{ \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t) \} \\ &= \text{RHS of (2)}. \end{aligned}$$

Thus equality holds in (2). Finally from (1) and (2), we have

$$S \times_{*,\diamond} (T \cap U) = (S \times_{*,\diamond} T) \cap (S \times_{*,\diamond} U). \text{ Similarly, we can prove that}$$

$$S \times_{*,\diamond} (T \cup U) = (S \times_{*,\diamond} T) \cup (S \times_{*,\diamond} U)$$

**Theorem 4.4:** The generalized cartesian product of the interval-valued- intuitionistic fuzzy n-normed linear spaces is distributive with respect to difference. In other words If

$$S = \{(X, (\overline{A}_1(x_1, x_2, \dots, x_n, t); \overline{B}_1(y_1, y_2, \dots, y_n, t)); (x_1, x_2, \dots, x_n) \in X^n)\}$$

$$\text{and } T = \{(Y, (\overline{A}_2(y_1, y_2, \dots, y_n, t); \overline{B}_2(y_1, y_2, \dots, y_n, t)); (y_1, y_2, \dots, y_n) \in Y^n)\}$$

$$\text{and } U = \{(Y, (\overline{A}_3(y_1, y_2, \dots, y_n, t); \overline{B}_3(y_1, y_2, \dots, y_n, t)); (y_1, y_2, \dots, y_n) \in Y^n)\}$$

are the interval-valued-intuitionistic fuzzy n-normed linear spaces, then

$$S \times_{*,\diamond} (T \setminus U) \subseteq (S \times_{*,\diamond} T) \setminus (S \times_{*,\diamond} U)$$

$$T = \{(Y, (\overline{A}_2(y_1, y_2, \dots, y_n, t) = \overline{1}, \overline{B}_2(y_1, y_2, \dots, y_n, t) = \overline{0}); (y_1, y_2, \dots, y_n) \in Y^n)\}$$

$$U \subseteq S, * = \min, \diamond = \max, \text{ then equality holds.}$$

**Proof:** We need to prove,  $S \times_{*,\diamond} (T \setminus U) \subseteq (S \times_{*,\diamond} T) \setminus (S \times_{*,\diamond} U)$ . It is enough to prove,

$$\begin{aligned} &\overline{A}_1(x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{A}_2(y_1, y_2, \dots, y_n, t), \overline{B}_3(y_1, y_2, \dots, y_n, t) \} \\ &\leq \min^i \{ \overline{A}_1(x_1, x_2, \dots, x_n, t) * \overline{A}_2(y_1, y_2, \dots, y_n, t), \\ &\quad \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_3(y_1, y_2, \dots, y_n, t) \} \end{aligned} \quad (1)$$

and

$$\begin{aligned} &\overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \max^i \{ \overline{B}_2(y_1, y_2, \dots, y_n, t), \overline{A}_3(y_1, y_2, \dots, y_n, t) \} \\ &\geq \max^i \{ \overline{B}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{B}_2(y_1, y_2, \dots, y_n, t), \\ &\quad \overline{A}_1(x_1, x_2, \dots, x_n, t) * \overline{A}_3(y_1, y_2, \dots, y_n, t) \} \end{aligned} \quad (2)$$

**Case-(i):** Let

$$\overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) < \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \quad (3)$$

and using the fact

$$a * b \leq \min\{a, b\} \leq a \leq \max\{a, c\} \leq a \diamond c. \quad (4)$$

Then (4),

(17)

$$\begin{aligned} & \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) \\ & < \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \leq \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \\ & < \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) \diamond \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \\ \Rightarrow & \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) \\ & < \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) \diamond \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \end{aligned} \quad (5)$$

Therefore by (3) and (5),

LHS of (1)

$$\begin{aligned} & = \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \} \\ & = \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) \\ & \quad \min^i \{ \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) \diamond \\ & \quad \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \} \\ & = \text{RHS of (1)}. \end{aligned}$$

Thus equality holds in (1).

**Case-(ii):** Let

$$\overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) \geq \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \quad (6)$$

Thus,

$$\begin{aligned} & \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) \\ & \geq \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \end{aligned} \quad (7)$$

Therefore by (6) and (7),

$$\begin{aligned} & \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \} \\ & = \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \\ & \leq \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) \\ \Rightarrow & \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{B}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \} \\ & \leq \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t) \end{aligned} \quad (8)$$

By (4) and (6),

$$\begin{aligned} & \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \} \\ & = \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \\ & \leq \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \leq \overline{\mathbf{B}}_1 (x_1, x_2, \dots, x_n, t) \diamond \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \end{aligned} \quad (9)$$

From (8) and (9) we have,

$$\begin{aligned} & \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{B}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \} \\ & \leq \min^i \{ \overline{\mathbf{A}}_1 (x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_1 (x_1, x_2, \dots, x_n, t) \diamond \overline{\mathbf{B}}_3 (y_1, y_2, \dots, y_n, t) \\ & \quad (y_1, y_2, \dots, y_n, t) \} \end{aligned}$$

Thus we have proved (1) and (2) can be proved similarly. So,

$$S \times_{*,\diamond} (T \setminus U) \subseteq (S \times_{*,\diamond} T) \setminus (S \times_{*,\diamond} U).$$

Let

$$\begin{aligned} T &= \{(Y, \overline{\mathbf{A}}_2 (y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_2 (y_1, y_2, \dots, y_n, t)); (y_1, y_2, \dots, y_n) \in Y^n\} \\ U &\subseteq S, * = \min, \diamond = \max, \end{aligned}$$

LHS of (1)

$$\begin{aligned}
 &= \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \min^i \{ \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_3(y_1, y_2, \dots, y_n, t) \} \\
 &= \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \min^i \{ 1, \overline{\mathbf{B}}_3(y_1, y_2, \dots, y_n, t) \} \\
 &= \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{B}}_3(y_1, y_2, \dots, y_n, t) \\
 &= \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t), \overline{\mathbf{B}}_3(y_1, y_2, \dots, y_n, t) \}
 \end{aligned}$$

RHS of (1)

$$\begin{aligned}
 &= \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t) * \overline{\mathbf{A}}_2(y_1, y_2, \dots, y_n, t), \overline{\mathbf{B}}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{\mathbf{B}}_3 \\
 &\quad (y_1, y_2, \dots, y_n, t) \} \\
 &= \min^i \{ \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t), 1 \}, \max^i \{ \overline{\mathbf{B}}_1(x_1, x_2, \dots, x_n, t) \diamond \overline{\mathbf{B}}_3(y_1, y_2, \dots, y_n, t) \} \} \\
 &= \min^i \{ \overline{\mathbf{A}}_1(x_1, x_2, \dots, x_n, t), \overline{\mathbf{B}}_3(y_1, y_2, \dots, y_n, t) \}
 \end{aligned}$$

Thus equality holds in (1). Similarly we can prove the equality in (2).

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