

**STABILITY OF THE EQUILIBRIUM POSITION
 OF THE CENTRE OF MASS OF AN EXTENSIBLE CABLE CONNECTED SATELLITES SYSTEM
 IN THE ELLIPTIC ORBIT UNDER THE INFLUENCE OF PERTURBATIVE FORCES**

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ABSTRACT:

During the motion of a cable connected satellites system, at least one equilibrium point exists when perturbative forces like Air-resistance, shadow of the earth due to solar radiation radiation pressure, magnetic force and oblateness of the earth act simultaneously. We have obtained two equilibrium points out of which only one is found to be stable in case of perturbative forces like shadow of the earth due to solar radiation pressure, magnetic force and oblateness of the earth act simultaneously on the motion of two extensible cable-connected satellites. Lyapunov's theorem has been used to examine the stability of the equilibrium points.

Keywords: Stability, perturbative forces, equilibrium Point, satellites, Elliptic orbit.

1. INTRODUCTION

The present paper is devoted to examine the stability of the equilibrium points of the centre of mass of a system of two satellites connected by a light, flexible and extensible string under the influence of shadow of the earth due to solar radiation pressure, magnetic force and oblateness of the earth acting simultaneously in elliptic orbit. Beletsky; V.V[1] is the pioneer worker in this field. It is the generalisation of the works done by Beletsky[1], Singh[2], Sinha[3], Singh[4] and Das *et. al* [7].

2. EQUATION OF MOTION

The equation of motion of one of the two satellites moving along a keplerian elliptic orbit in Nechvill's coordinates can be obtained by exploiting Lagrange's equation of motion of first kind in the form:

$$\begin{aligned} x'' - 2y' - 3x\rho - \frac{4Bx}{\rho} + A\Psi_1\rho^3 \cos \epsilon \cos(\nu - \alpha) &= -\bar{\lambda}_\alpha \rho^4 \left[1 - \frac{l_0}{\rho r} \right] x - \frac{D \cos i}{\rho} \\ y'' + 2x' + \frac{By}{\rho} - A\Psi_1\rho^3 \cos \epsilon \sin(\nu - \alpha) &= -\bar{\lambda}_\alpha \rho^4 \left[1 - \frac{l_0}{\rho r} \right] y \end{aligned} \quad (1)$$

where, $\rho = \frac{1}{1 + e \cos \nu}$; $r = \sqrt{x^2 + y^2}$

ν = True anomaly of the centre of mass of the system

$A = \frac{P^3}{\mu} \left(\frac{B_1}{m_1} - \frac{B_2}{m_2} \right)$ = Solar pressure parameter

$B = \frac{3k_2}{P^2}$ = Oblateness force parameter.

$C = \left(\frac{Q_1}{m_1} - \frac{Q_2}{m_2} \right) \sqrt{\frac{\mu_E}{\mu p}}$ = Magnetic force parameter.

Ψ_1 = Shadow function parameter.

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$$\bar{\lambda}_\alpha = \frac{p^3}{\mu} \frac{\lambda}{\ell_0} \frac{(m_1 + m_2)}{m_1 m_2}; \ell_0 = \text{Natural length of the string}$$

$\lambda = \text{modulus of elasticity}$

Where μ denotes the product of gravitational constant and mass of the earth. Here B_1 and B_2 are the absolute values of forces due to the direct solar pressure exerted on the masses m_1 and m_2 and α be the angular separation of solar position vector projected on the orbital plane from the orbit perigee. Here ϵ is the inclination of the osculating plane of the orbit of the centre of mass of the system with the plane of ecliptic and p and e are focal parameter and eccentricity of the earth.

Here, dashes denote the differentiation with respect to the true anomaly v of the elliptic orbit of the centre of mass .

The condition of constraint is given by

$$x^2 + y^2 \leq \frac{\ell_0^2}{\rho^2} \tag{2}$$

To find the Jacobian integral of the problem, the averaged values of the secular terms due to periodic terms presents in the equations of motion (1) can be deduced as given follow :

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\rho} dv &= 1; \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\rho^2} dv = \left(1 + \frac{e^2}{2}\right) \mathbb{R} \\ \frac{1}{2\pi} \int_0^{2\pi} \rho dv &= \frac{1}{(1-e^2)^{\frac{1}{2}}}; \quad \frac{1}{2\pi} \int_0^{2\pi} \rho^3 dv = \frac{2+e^2}{2(1-e^2)^{\frac{5}{2}}}; \quad \frac{1}{2\pi} \int_0^{2\pi} \rho^4 dv = \frac{2+3e^2}{2(1-e^2)^{\frac{7}{2}}}; \\ \frac{1}{2\pi} \int_0^{2\pi} A\rho^3 \psi_1 \cos \epsilon \cos(v-\alpha) dv &= \frac{1}{2\pi} \left[\int_{\psi_1=0}^{\theta} A\rho^3 \psi_1 \cos \epsilon \cos(v-\alpha) dv + \int_{\psi_1=1}^{2\pi} A\rho^3 \psi_1 \cos \epsilon \cos(v-\alpha) dv \right] \\ &= \frac{-(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A \cos \epsilon \cos \alpha \sin \theta}{\pi} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} A\rho^3 \psi_1 \cos \epsilon \sin(v-\alpha) dv &= \frac{1}{2\pi} \left[\int_{\psi_1=0}^{\theta} A\rho^3 \psi_1 \cos \epsilon \sin(v-\alpha) dv + \int_{\psi_1=1}^{2\pi} A\rho^3 \psi_1 \cos \epsilon \sin(v-\alpha) dv \right] \\ &= \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A \cos \epsilon \sin \alpha \sin \theta}{\pi} \end{aligned} \tag{3}$$

where θ is taken to be constant.

Using (3) in (1), we get

$$\begin{aligned} x'' - 2y' - \frac{3x}{(1-e^2)^{\frac{1}{2}}} - 4Bx - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A \cos \epsilon \cos \alpha \sin \theta}{\pi} &= -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_0}{2r(1-e^2)^{\frac{5}{2}}} \right] x - D \cos i \\ \text{and } y'' + 2x' + By - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A \cos \epsilon \sin \alpha \sin \theta}{\pi} &= -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_0}{2r(1-e^2)^{\frac{5}{2}}} \right] y \end{aligned} \tag{4}$$

where, $r = \sqrt{x^2 + y^2}$

The condition of constraint given by (2) takes the form

$$x^2 + y^2 \leq \left(1 + \frac{e^2}{2}\right) \ell_0^2 \quad (5)$$

From equations of motion (4) it follows that the true anomaly v does not appear explicitly in the equations of motion, so there must exist Jacobian integral for the problem.

Multiplying first and second equations of (4) by $2x'$ and $2y'$ respectively and adding and integrating, we get the Jacobian integral in the form.

$$\begin{aligned} x'^2 + y'^2 - \frac{3x^2}{(1-e^2)^{\frac{1}{2}}} - 4Bx^2 + By^2 - \frac{(2+e^2)Ax \cos \epsilon \cos \alpha \sin \theta}{\pi(1-e^2)^{\frac{5}{2}}} - \frac{(2+e^2)Ay \cos \epsilon \sin \alpha \sin \theta}{\pi(1-e^2)^{\frac{5}{2}}} \\ + \frac{\bar{\lambda}_\alpha (2+3e^2)(x^2+y^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{\bar{\lambda}_\alpha \ell_o (2+e^2)(x^2+y^2)^{\frac{1}{2}}}{(1-e^2)^{\frac{5}{2}}} + Dx \cos i = h \end{aligned} \quad (6)$$

where, h is the constant of integration.

The curve of zero velocity is obtained by putting $x'^2 + y'^2 = 0$ in (6) as

$$\begin{aligned} \frac{3x^2}{(1-e^2)^{\frac{1}{2}}} + 4Bx^2 - By^2 + \frac{(2+e^2)Ax \cos \epsilon \cos \alpha \sin \theta}{\pi(1-e^2)^{\frac{5}{2}}} + \frac{(2+e^2)Ay \cos \epsilon \sin \alpha \sin \theta}{\pi(1-e^2)^{\frac{5}{2}}} \\ - \frac{\bar{\lambda}_\alpha (2+3e^2)(x^2+y^2)}{2(1-e^2)^{\frac{7}{2}}} + \frac{\bar{\lambda}_\alpha \ell_o (2+e^2)(x^2+y^2)^{\frac{1}{2}}}{(1-e^2)^{\frac{5}{2}}} - Dx \cos i + h = 0 \end{aligned} \quad (7)$$

Hence we conclude that the satellite by mass m_1 will move inside the boundaries of different curves of zero velocity represented by (7) of (6) for different values of Jacobian constant h .

3. EQUILIBRIUM POSITION OF THE SYSTEM

We have obtained the system of equations (4) of the particle of mass m_1 of the system in rotating frame of reference. It has been assumed that the system is moving with effective constants and hence the string connecting the two satellites of masses m_1 and m_2 will remain always tight

The equilibrium positions of the system are given by the constant values of the coordinates in the rotating frame of reference Now, let $x = x_0$ and $y = y_0$ give the equilibrium position where x_0 and y_0 are constants.

$$\therefore x' = 0 = x'' \quad \text{and} \quad y' = 0 = y''$$

Thus, equations given by (4) take the following form:

$$\begin{aligned} \frac{-3x_0}{(1-e^2)^{\frac{1}{2}}} - 4Bx_0 - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A \cos \epsilon \cos \alpha \sin \theta}{\pi} + D \cos i = -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}} \right] x_0 \\ By_0 - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A \cos \epsilon \sin \alpha \sin \theta}{\pi} = -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}} \right] y_0 \end{aligned} \quad (8)$$

where $r_0 = \sqrt{x_0^2 + y_0^2}$

Actually, it is impossible to find the solution of the algebraic equations (8) in its present form.

Therefore, for our further investigation, it has been assumed that $\alpha=0$

This means that the sun rays is in the line of the perigee of the elliptical orbit of the centre of mass of the system. Putting $\alpha = 0$, the equations of motion given by (8) take the form:

$$-\left[\frac{3}{(1-e^2)^{\frac{1}{2}}} + 4B \right] x_0 - \frac{(2+e^2)A_1 \sin \theta}{2(1-e^2)^{\frac{5}{2}}} = -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}} \right] x_0$$

$$\text{and } B y_0 = -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}} \right] y_0 \quad (9)$$

From (9), we get the equilibrium points as

$$[a_1, 0] = \left[\frac{(2+e^2)(\bar{\lambda}_\alpha \ell_o + A_1 \sin \theta) - 2(1-e^2)^{\frac{5}{2}} D \cos i}{\bar{\lambda}_\alpha (2+3e^2) - 2(1-e^2)^3 \left\{ 3 + 4B(1-e^2)^{\frac{1}{2}} \right\}}, 0 \right] \quad (10)$$

$$[b_1, c_1] = \left[\frac{2(1-e^2)^{\frac{5}{2}} D \cos i - (2+e^2)A_1 \sin \theta}{2(1-e^2)^2 \left\{ 3 + 5B(1-e^2)^{\frac{1}{2}} \right\}}, \sqrt{\left[\frac{\bar{\lambda}_\alpha \ell_o (2+e^2)(1-e^2)}{\bar{\lambda}_\alpha (2+3e^2) + 2B(1-e^2)^{\frac{7}{2}}} \right]^2 - \left[\frac{D \cos i - (2+e^2)A_1 \sin \theta}{2(1-e^2) \left\{ 3 + 5B(1-e^2)^{\frac{1}{2}} \right\}} \right]^2} \right] \quad (11)$$

Now, Now, it can be easily seen that the equilibrium point given by (10) only gives a meaningful value of the Hook's modulus of elasticity.

4. STABILITY OF THE SYSTEM

We shall study the stability of the equilibrium position given by (10) of the system in the sense of Liapunov. For this, let us assume that there are small variation in the coordinate at the given equilibrium point $[a_1, 0]$. let η_1 and η_2 be small variation in x and y-coordinates respectively for the given position of equilibrium:

$$\begin{aligned} \therefore x &= a_1 + \eta_1 & \text{and} & & y &= \eta_2 \\ \therefore x' &= \eta_1' & \text{and} & & y' &= \eta_2' \\ x'' &= \eta_1'' & \text{and} & & y'' &= \eta_2'' \end{aligned} \quad (12)$$

Putting the values of x, y, x', y', x'' and y'' from (12) in (4), we get on putting and, as $\frac{A \cos \epsilon}{\pi} = A_1 \alpha = 0$

$$\eta_1'' - 2\eta_1' - \left\{ \frac{3}{\sqrt{1-e^2}} + 4B \right\} (a_1 + \eta_1) - \frac{(2+e^2)A_1 \sin \theta}{2(1-e^2)^{\frac{5}{2}}} + D \cos i = -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}} \right] (a_1 + \eta_1)$$

$$\text{and } \eta_2'' - 2\eta_2' + B\eta_2 = -\bar{\lambda}_\alpha \left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}} \right] \eta_2 \quad (13)$$

where, $r = \sqrt{(a_1 + \eta_1)^2 + \eta_2^2}$

To obtain Jacobian integral of the equations of motion (13), we multiply first equation of (13) by $2(a_1 + \eta_1)'$ and second equation of (13) by η_2' and add them together, we get after integrating

$$\eta_1'^2 + \eta_2'^2 - \left\{ \frac{3}{\sqrt{1-e^2}} + 4B \right\} (a_1 + \eta_1)^2 + 2D(a_1 + \eta_1) \cos i + B\eta_2^2 - \frac{(2+e^2)(a+\eta_1)A_1 \sin \theta}{2(1-e^2)^{\frac{5}{2}}} + \frac{\bar{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} \left\{ (a_1 + \eta_1)^2 + \eta_2^2 \right\} - \frac{\bar{\lambda}_\alpha (2+e^2)l_0}{(1-e^2)^{\frac{5}{2}}} \left\{ (a_1 + \eta_1)^2 + \eta_2^2 \right\}^{\frac{1}{2}} = h_1 \quad (14)$$

where h_1 is the constant of integration.

To test the stability in the sense of Liapunov, we take Jacobian integral (14) as Liapunov's function $V(\eta_1', \eta_2', \eta_1, \eta_2)$ and is obtained by expanding the terms of (14) as:

$$V(\eta_1', \eta_2', \eta_1, \eta_2) = \eta_1'^2 + \eta_2'^2 + \eta_1^2 \left[\frac{\bar{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - 4B - \frac{3}{\sqrt{1-e^2}} \right] + \eta_2^2 \left[B + \frac{\bar{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{\bar{\lambda}_\alpha l_0 (2+e^2)}{2a_1 (1-e^2)^{\frac{5}{2}}} \right] + \eta_1 \left[\frac{-6a_1}{\sqrt{1-e^2}} - 8Ba_1 - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}} A_1 \sin \theta + 2D \cos i + \frac{\bar{\lambda}_\alpha (2+3e^2)a_1}{(1-e^2)^{\frac{7}{2}}} - \frac{\bar{\lambda}_\alpha (2+e^2)l_0}{(1-e^2)^{\frac{5}{2}}} \right] + \left[-4Ba_1^2 - \frac{3a_1^2}{\sqrt{1-e^2}} - \frac{\bar{\lambda}_\alpha (2+e^2)l_0 a_1}{(1-e^2)^{\frac{5}{2}}} + \frac{\bar{\lambda}_\alpha (2+3e^2)a_1^2}{2(1-e^2)^{\frac{7}{2}}} - \frac{a_1(2+e^2)A_1 \sin \theta}{(1-e^2)^{\frac{5}{2}}} + 2Da_1 \cos i \right] + 0(3) = h_1 \quad (15)$$

where 0 (3) stand for the third and higher order terms in the small quantities η_1 and η_2 .

Now, by Lyapunov theorem on stability it follows that the only criterion for the given equilibrium position $(a_1, 0)$ to be stable is that v defined by (15) must be positive definite and for this the following conditions must be satisfied:

$$\begin{aligned} \text{(i)} \quad & \frac{-6a_1}{(1-e^2)^{\frac{1}{2}}} - 8Ba_1 - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}} A_1 \sin \theta + \frac{\bar{\lambda}_\alpha (2+3e^2)a_1}{(1-e^2)^{\frac{7}{2}}} - \frac{\bar{\lambda}_\alpha (2+e^2)l_0}{(1-e^2)^{\frac{5}{2}}} + 2D \cos i = 0 \\ \text{(ii)} \quad & \frac{\bar{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - 4B - \frac{3}{(1-e^2)^{\frac{1}{2}}} > 0 \\ \text{(iii)} \quad & B + \frac{\bar{\lambda}_\alpha (2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{\bar{\lambda}_\alpha l_0 (2+e^2)}{2a_1 (1-e^2)^{\frac{5}{2}}} > 0 \end{aligned} \quad (16)$$

Condition (i): First of all it is to be noted that

$$a_1 = \frac{\left[(2+e^2)(\bar{\lambda}_\alpha l_0 + A_1 \sin \theta) - 2(1-e^2)^{\frac{5}{2}} D \cos i \right]}{\left[\bar{\lambda}_\alpha (2+3e^2) - 2(1-e^2)^3 \left\{ 3 + 4B(1-e^2)^{\frac{1}{2}} \right\} \right]} \text{ is positive.}$$

Putting the values a_1 on the L.H.S of (i) we get

$$\frac{\text{L.H.S of (i)} = \left[\bar{\lambda}_\alpha (2 + 3e^2) - 6(1 - e^2)^3 - 8B(1 - e^2)^{\frac{7}{2}} \right] a_1}{(1 - e^2)^{\frac{7}{2}}} - \frac{\left[\bar{\lambda}_\alpha (2 + 3e^2) - 6(1 - e^2)^3 - 8B(1 - e^2)^{\frac{7}{2}} \right] a_1}{\left((1 - e^2)^{\frac{7}{2}} \right)} = 0$$

Hence the first condition is identically satisfied.

Condition (ii):

$$\frac{\bar{\lambda}_\alpha (2 + 3e^2)}{2(1 - e^2)^{\frac{7}{2}}} - 4B - \frac{3}{(1 - e^2)^{\frac{1}{2}}} = \frac{\left[\bar{\lambda}_\alpha (2 + 3e^2) - 8B(1 - e^2)^{\frac{7}{2}} - 6(1 - e^2)^3 \right]}{2(1 - e^2)^{\frac{7}{2}}} = +ve$$

Since the denominator of a_1 is positive.

Condition (iii):

$$\begin{aligned} B + \frac{\bar{\lambda}_\alpha (2 + 3e^2)}{2(1 - e^2)^{\frac{7}{2}}} - \frac{\bar{\lambda}_\alpha \ell_0 (2 + e^2)}{2a_1 (1 - e^2)^{\frac{5}{2}}} &= B + \frac{\bar{\lambda}_\alpha \left[(2 + 3e^2)a_1 - \ell_0 (2 + e^2)(1 - e^2) \right]}{2a_1 (1 - e^2)^{\frac{7}{2}}} \\ &= B + \frac{\bar{\lambda}_\alpha \left[2(a_1 - \ell_0) + e^2(3a_1 + \ell_0) + \ell_0 e^4 \right]}{2a_1 (1 - e^2)^{\frac{7}{2}}} > 0 \text{ if } a_1 > \ell_0 \end{aligned}$$

Thus the third condition is identically satisfied if $a_1 > \ell_0$.

CONCLUSION

we conclude that the equilibrium position $[a_1, 0]$ of the system is stable in the sense of Liapunov if $a_1 > l_0$.

REFERENCES

1. Beletsky; V.V, About the Relative Motion of Two Connected Bodies in orbit. Kosmicheskiya Isseldovania, vol.7, No. 6, p.p. 827 - 840, 1969 (Russian).
2. Singh; R.B, The motion of two connected bodies in an elliptical orbit. Bulletin of the Moscow state university mathematics-mechanics. No. 3, P.P.82-86, 1973 (Russian).
3. Sinha; S.K., Effects of solar pressure on the motion and Stability of the system of two inter-connected satellites in an elliptical orbit. Astrophysics and space science. N0. 140, P.P. 49-54, 1988.
4. Singh; C.P, Motion and stability of inter-connected Satellites system in the gravitational field of oblate earth. Ph.D thesis, submitted to B.R.A. Bihar University, Muzaffarpur, 1983.
5. Das; S.K., Bhattacharya; P.K., Singh; R.B, Effects of magnetic force on the motion of system of two cable connected satellite in orbit, Proc. Nat. Acad. Sci. India (1976), 287-299.
6. Kumar. V and Kamari. N, Jacobian integral and Stability of the equilibrium position of the centre of mass of an extensible cable connected satellites system in the elliptic orbit motions. IJSER, Volume 4, Issue 9, September-2013.

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