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ANTI-INVARIANT SUBMANIFOLDS OF (ϵ) –SASAKIAN MANIFOLD

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ABSTRACT

*The object of the present paper is to study anti-invariant submanifolds M of (ϵ) - Sasakian manifold $\overline{M}$. It is shown that if M is totally umbilical then M is totally geodesic. Also results have been obtained connecting totally geodesicty and anti-invariance of M. Also we find the necessary and sufficient condition for anti-invariant submanifolds of (ϵ)-Sasakian manifold to be **T**-invariant and anti-invariant and condition for integrability of the distribution **D**.*

AMS Subject Classification: *53C15, 53C20, 53C50.*

Key Words: *Anti-invariant submanifold, (ϵ)-Sasakian manifold, **T** -invariant, Totally geodesic, Totally umbilical, Integrable condition.*

INTRODUCTION

The index of a metric is very important in differential geometry as it gives rise to vector fields such as space-like, time like, and light-like fields. K.L.Duggal and A.Bejancu [9], introduced and studied (ϵ)-Sasakian manifolds with the help of these vector fields and further such manifolds were investigated by Xufeng and Xiaoli [2], and others ([1], [3], [4]) and study of light like submanifolds is carried out by ([6], [17], [25]) because Sasakian manifolds with indefinite metrics play crucial role in Physics. The research work on the geometry of invariant submanifols of contact and complex manifolds is carried out by M.Kon [29], in 1973, C.S.Bagewadi [27], in 1982, K.Yano and M.Kon [28], in 1984 and other authors ([7],[8][10][20]). Also the study of geometry of anti-invariant submanifolds is carried out by ([11], [13], [14], [15], [16], [21], [22], [23], [24], [26]) invarious contact manifolds. Motivated by the studies of the above authors, we study antiinvariant submanifolds of (ϵ)-Sasakian manifold. The paper is organised as follows: the section 1 consists of preliminaries of (ϵ)-Sasakian manifold, and section 2 contains the results as stated in abstract.

1. PRELIMINARIES

A (2n+1)-dimensional differentiable manifold $\overline{M}$ end owed with an almost contact structure (φ, ξ, η), where φ is a tensor field of type (1, 1), η is a 1-form and ξ is a vector field on $\overline{M}$ Satisfying

$$\phi^2 X = -X + \eta(X)\xi,\ \ \eta(\xi) = 1, \tag{1.1}$$
$$\eta(\varphi X) = 0,\ \ \varphi(\xi) = 0$$

is called an almost contact manifold. If there exists a semi-Riemannian metric g satisfying,

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y) \tag{1.2}$$

then (φ, ξ, η, g) is called an (ϵ)-almost contact metric structure and M is known as (ϵ)- almost contact manifold for all X,Y ∈ T(M) where ϵ= ±1 , For an (ϵ)-almost contact manifold.

We also have

$$\eta(X) = \epsilon g(X, \xi)$$

for all X∈TM, ϵ=g(ξ, ξ). Hence ξ is never a light like vector field on M and we have two classes of (ϵ)-Sasakian manifolds. when ϵ=-1 and the index of g is odd then M is time like Sasakian manifold and M is a space like Sasakian manifold when ϵ=-1 and the index of g is even. For ϵ= 1 and index of g is zero we obtain usual Sasakian manifold and for ϵ = 1 and index of g is one then M is a Lorentz - Sasakian manifold. If dη(X, Y) = g(φX, Y) then M is said to have (ϵ)-contact metric structure (φ, η, ξ, g). If moreover this structure is normal then the (ϵ)-contact metric structure is called (ϵ)-Sasakian structure and the manifold endowed with this structure is called an (ϵ)- Sasakian manifold. Also the (ϵ)-contact metric structure is an (ϵ) -Sasakian structure if and only if

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$$(\overline{\nabla}_X \varphi)Y = g(X,Y)\xi - \epsilon\eta(Y)X \tag{1.3}$$
$$\overline{\nabla}_X\xi = -\epsilon\varphi X \tag{1.4}$$

Let M be a submanifold of $\overline{M}$. Let $T_x(M)$ and $T^{\perp}{}_x(M)$ denote the tangent and normal space of M at x∈M respectively. The Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{1.5}$$
$$\overline{\nabla}_X N = -\mathrm{A}_N\ X + \nabla^{\perp}{}_X N \tag{1.6}$$

for any vector fields X, Y tangent to M and any vector field N normal to M, where of $\overline{\nabla}$ and ∇ are the operator of covariant differentiation on and of $\overline{M}$ *and* M, is $\nabla^{\perp}$ the linear connection induced in the normal space $T^{\perp}{}_x(M)$ Both A_N and σ are called the Shape operator and the second fundamental form and they satisfy

$$g(\sigma(X,Y),N) = g(\mathrm{A}_N X, Y) \tag{1.7}$$

If the second fundamental form σ of M is of the form σ(X, Y) = g(X, Y)μ, then M is called totally umbilical. where μ is the mean curvature. If the second fundamental form vanishes identically then M is said to be totally geodesic. If μ = 0, then M is said to be minimal.

A submanifold M of a (ϵ)-Sasakian manifold $\overline{M}$ is said to be invariant if the structure vector field ξ of $\overline{M}$ is tangent to M and φ($T_x(M) \subset T_x(M)$, where $T_x(M)$ is the tangent space for all x ∈ M and If φ($T_x(M) \subset T^{\perp}{}_x(M)$ where $T^{\perp}{}_x(M)$ is the normal space at x ∈ M then M is said to be anti-invariant in $\overline{M}$.

Now we define (ϵ) -Sasakian manifold with constant φ-holomorphic sectional curvature; A plane section π on $\overline{M}$ is called an invariant φ-section if it is determined by the plane formed by an orthonormal pair X and φX spanning the section. The sectional curvature of the plane section φ is called the φ-sectional curvature. If $\overline{M}$ is an (ϵ)-Sasakian manifold of constant φ-sectional curvature k, then its curvature tensor has the form

$$\overline{R}\,(X\ Y, Z) = (K+3\epsilon)/4\ \{g(Y,Z)X-g(X,Z)Y+(K-\epsilon)/4\{\eta(X)\eta(Z)\ Y-\eta(Y)\eta(Z)X +g(X, Z)\eta(Y)\xi\}-g(Y, Z)\eta(X)+g(\phi Y, Z)\phi X-g(\phi X, Z)\phi Y-2g(\phi X, Y)\phi Z)\} \tag{1.8}$$

For any vector fields X, Y, Z on $\overline{M}$

Define a Tensor field T on $\overline{M}$ by [12] setting

$$T(X,Y,Z)= \overline{R}\,(X\ Y,Z) + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z \tag{1.9}$$

for any vector fields X, Y, Z on $\overline{M}$

A sub manifold M is said to $\overline{R}$ –invariant and **T**-invariant if and only if

$\overline{R}(X,Y)T_x(M) \subset T_x(M)$ and (T(X, Y)) $T_x(M) \subset T_x(M)$ respectively.

2. SOME THEOREMS

Theorem 2.1: Let M be a submanifold tangent to the structure vector field ξ of an (ϵ)-Sasakian manifold $\overline{M}$ is totally umbilical then M is totally geodesic.

Proof: Since ξ is tangent to M, we have from Gauss formu *la*

$$\overline{\nabla}_X\xi = \nabla_X\xi + \sigma(X,\xi)$$

Using (1.4) we have

$$-\epsilon\varphi X = \nabla_X\xi + \sigma(X,\xi)$$

Equating tangential and normal components

$$(\epsilon(\varphi X)^T = -\nabla_X\xi, \quad (\epsilon(\varphi X)^{\perp} = \sigma(X,\xi)$$

Putting X = ξ in second equation then by (1.1) we have $\sigma(\xi,\xi) = 0$. Let us assume that M is totally umbilical then $\sigma(X,\ Y) = g(X,Y\,)\mu$ for all X, Y ∈ TM where μ is the mean curvature vector, Putting in $X = Y = \xi$ we get

$$\sigma(\xi,\xi) = g(\xi,\xi)\mu$$

This shows that $\mu = 0$, Hence $\sigma(X,\ Y) = g(X,\ Y)\mu$ $i$mplies $\sigma(X,\ Y) = 0$.

The second fundamental form σ = 0 thus M is totally geodesic.

Remark 2.1: If M is totally geodesic then $(\epsilon(\varphi X)^{\perp} = \sigma(X, \xi) = 0$, i.e $\epsilon^2(\varphi X)^{\perp} = 0$

i.e $(\varphi X)^{\perp} = 0$, φX is tangent to M and hence M is invariant submanifold of (Є)-Sasakian manifold. Therefore M will also be (Є)-Sasakian manifold.

Theorem 2.2: Let M be a submanifold of a (Є)-Sasakian manifold $\overline{M}$ tangent to the structure vector field ξ of $\overline{M}$ then ξ is parallel with respect to the induced connection on M if and only if M is anti-invariant submanifold in $\overline{M}$.

Proof: Suppose the structure vector field ξ is tangent to M. By Gauss formula

$$-\epsilon(\varphi X) = \overline{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi) \tag{2.1}$$

Next suppose ξ is parallel w.r.t induced connection on M, then we have $\nabla_X \xi = 0$ from equation (2.1) we have

$$-\epsilon\varphi X = \sigma(X, \xi) \quad \text{i.e} \quad \varphi X = -\epsilon\,\sigma(X, \xi)$$

Hence ЄφX is normal to M, $\varphi X \in T^{\perp}{}_x(M)$ Thus M is anti-invariant.

Conversely: suppose M is anti-invariant, then by dentition of anti-invariant if $X \in T_x(M)$ Then $\varphi X \in T^{\perp}{}_x(M)$ so $\varphi X = \sigma(X, \xi)$ for conviennce we choose $-\epsilon(\varphi X) = \sigma(X, \xi)$

Hence from (2.1), We have $\nabla_X \xi = 0$
This shows that ξ is parallel w.r.to the induced connection on M.
Hence the theorem.

Theorem 2.3: Let M be a sub manifold of (Є)-Sasakian manifold $\overline{M}$ If ξ is normal to M then M is totally geodesic if and only if M is anti-invariant submanifold.

Proof: Suppose ξ is normal to M then Weingarten formula implies

$$\overline{\nabla}_X \xi = -A_N\,\xi + \nabla^{\perp}{}_X \xi$$

Using (1.4) and (2.2) we have

$$g(-\epsilon\varphi X, Y) = g(\overline{\nabla}_X \xi, Y) = g(-A_\xi X, Y) + g(\nabla^{\perp}{}_X \xi, Y) = -g(A_\xi X, Y)$$

for any X and Y tangent on M, that is,

$$g(\varphi X, Y) = g(A_\xi X, Y) \tag{2.3}$$

Interchange X and Y in the above and adding and by virtue of (1.2) we have

$$g(A_\xi X, Y) + g(A_\xi Y, X) = 0$$

and $\quad g(\sigma(X, Y), \xi) = g(A_\xi X, Y)$

and $\quad A_\xi X$ *i*s symmetric we must have

$$g(A_\xi X, Y) = 0$$

If M is totally geodesic, then $\sigma(X, Y) = 0$, *i.e* $A_\xi X = 0$, then by (1.4) $-\epsilon\,\varphi X = \nabla^{\perp}{}_X \xi$
Hence M is anti-invariant

Conversely: suppose M is anti-invariant then $\epsilon\,\varphi X \in T^{\perp}{}_x(M)$ then from (2.2) we get

$$-g\big(A_\xi X, Y, Y\big) = 0 \quad \text{i.e by (1.7)}, g(\sigma(X, Y), \xi) = 0$$

i.e $\sigma(X, Y) = 0$

Hence M is totally geodesic.

We have the following known result;

Proposition 2.1: [11] Let M be a submanifold tangent to the structure vectorfield ξ of a normal almost para contact metric manifold with constant c($c \neq 3$) Then M is T-invariant if and only if M is invariant or anti-invariant.

On the basis of the above we can prove the following Theorem.

Theorem 2.4: Let M be a submanifold tangent to ξ the structure vector field of (Є)-Sasakian manifold $\overline{M}$ with constant k ($k \neq 3$) then M is T-invariant if and only if M is invariant or anti-invariant.

Proof: Easily follows from the Proposition 2.1

Theorem 2.5: Let M be an anti-invariant submanifold tangent to ξ the structure vectorfield of (Є)-Sasakian manifold $\overline{M}$ with constant k. If $A_N X = 0$ for any N ∈ $T^{\perp}{}_x(M)$, then φ($T_x(M)$) is parallel w .r. t the normal connection.

Proof: To show that φ($T_x(M)$) is parallel w.r.t to the normal connection, $\nabla^{\perp}$ we have to show that for every local section φY ∈ φ($T_x(M)$ is also a local section in φ($T_x(M)$.

Using Gauss and Weingarten formula

$$\nabla^{\perp}{}_X \phi Y = \overline{\nabla}_X \phi Y + A_{\phi Y}\ X$$

$$\nabla^{\perp}{}_X \phi Y = \overline{\nabla}_X \phi Y + \phi(\overline{\nabla}_X Y) + A_{\phi Y}\ X$$

$$= \overline{\nabla}_X \phi Y + g(X,Y)\xi - \epsilon\square(Y)X + A_{\phi Y}\ X$$

By virtue (1.3) and (1.5).

Since $A_N X = 0$ for any N ∈ $T^{\perp}{}_x(M)$ we have

$$g(\nabla^{\perp}{}_X \phi Y, N) = g(X,Y)g(\xi,N) - \epsilon\eta(Y)g(X,N) + g(\varphi\,\nabla_X Y, N) + g(\sigma(X,Y),N) + g(A_{\phi Y}\ X, N)$$

$$= -g(\nabla_X Y, \phi N) - g(\sigma(X,Y), \Phi N) + g(A_{\phi Y}\ X, N)$$

$$= -g(\nabla_X Y, \phi N) - g(A_{\phi N}\ X, Y) + g(A_{\phi Y}\ X, N)$$

Since, φN is also in $T^{\perp}{}_x(M)$, R.H.S of the above equation is zero

Hence $g(\nabla^{\perp}{}_X \phi Y, N) = 0$

Hence the result.

If D denotes the orthogonal subspace of T $\overline{M}$ to ξ then we can *write* $T\,\overline{M} = D \oplus \{\xi\}$.

We Prove the following Theorem.

Theorem 2.6: Let M be a submanifold of an (Є)-Sasakian manifold $\overline{M}$ then M is anti-invariant if and only if D is integrable.

Proof: Let X, Y ∈ D then X,Y ∈ $\boldsymbol{T\,\overline{M}}$

$$g([X,Y],\xi) = g(\overline{\nabla}_X Y - \overline{\nabla}_Y X, \xi)$$

$$= g(\overline{\nabla}_X Y, \xi) - g(\overline{\nabla}_Y X, \xi) = Xg(Y,\xi) - g(Y, \overline{\nabla}_X \xi) - Y\,g(X,\xi) + g(X, \overline{\nabla}_Y \xi)$$

Using (1.4) we have

$$g([X,Y],\xi) = -g(Y, -\epsilon\varphi X) + g(X, -\epsilon\varphi X)$$

$$= \epsilon[g(Y,\varphi X) - g(X,\varphi Y)]$$

i.*e* $g([X,Y],\xi) = 2g\,\epsilon\,(\varphi X, Y)$ *Thus* $[X,Y] \in D$ if and only if Φx is normal to Y

i.*e* $[X,\ Y] \in D$ if and only *if* $\varphi X \in T^{\perp}{}_x(M)$ *i.e* $[X,Y] \in D$ if and only if M is anti-invariant

i.e D is integrable if and only if M is anti-invariant.

Hence the theorem.

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