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FIXED POINT THEOREMS IN FUZZY METRIC SPACES USINGS IMPACT OF COMMON PROPERTY (E. A.)

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ABSTRACT

In this paper, we observe that the notation of common property (E. A.) relaxes the required containment of range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. As a consequence, a multitude of recent fixed point theorems of the existing literature are sharped and enrich.

Key words: Fuzzy *Metric Space, Property(E. A.) , Common Property (E. A.),Common Fixed point, Generalized Fuzzy contraction.*

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1. INTRODUCTION AND PRELIMINARIES

The evolution of fuzzy mathematics solely rests on the notation of fuzzy sets which was introduced by Zadeh [24] in 1965 with the view to represent the vagueness in everyday life. In mathematical programming, the problems are often expressed as optimizing some goal functions equipped with specific constraints suggested by some concrete practical situations. There exists many real life problems that consider multiple objectives, and generally, it is very difficult to get a feasible solution that brings us to the optimum of all the objective functions. Thus, a feasible method of resolving such problems is the use of fuzzy sets [20]. In fact, the richness of applications has engineered the all round development of fuzzy mathematics. Then, the study of fuzzy metric spaces has been carried out in several ways [4,9]. George and Veeramani [6] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10] with a view to obtain a Housdorff topology on fuzzy metric spaces, and this has recently found very fruitful applications in quantum particle physics, particulary in connection with both string and ε^{∞} theory [5]. In recent years, many authors have proved fixed point theorems and common fixed point theorems in fuzzy metric spaces. To mention a few, we cite [2, 17, 19]. As patterned in Jungct [7], a metrical common fixed point theorem generally involves conditions on commutatively, continuity, completeness together with a suitable condition on containment of ranges of involved mappings by an appropriate contraction condition. Thus researches in this domain are aimed at weakening one or more of these conditions. In this paper, we observe that the notion of common property relatively relaxes the required containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. Consequently, we obtain some common fixed point theorems in fuzzy metric spaces which improve many known earlier results[11][18][23]. Before presenting our results, we collect relevant background material as follows,

Definition: 1 [22] Let X be any set. A fuzzy set in X is a function with domain X and values in $[0,1]$.

Definition: 2 [10] A binary operation $\star : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous $t-\text{norm}$ if , $\star$ is satisfying the following condition:

(1) $\star$ is commutative and associative,
(2) $\star$ is continuous,
(3) $a \star 1 = a$ for all $a \in [0,1]$
(4) $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$ Examples of $t-\text{norm}$ are $a \star b = \min\{a, b\}$ and $a \star b = ab$.

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Definition: 3 [6] A triplet $(X, M, \star)$ is a fuzzy metric space whenever X is an arbitrary set, $\star$ is continuous $t-$norm and M is fuzzy set on $X \times X \times (0, \infty^+)$ satisfying, for every $x, y, z \in X$ and $s, t > 0$, the following condition:

(i) $M(x, y, t) > 0$
(ii) $M(x, y, t) = 1$ iff $x = y$
(iii) $M(x, y, t) = M(y, x, t)$
(iv) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$
(v) $M(x, y, \cdot) : (0, \infty^+) \to [0,1]$ is continuous.

Note that, $M(x, y, t)$ can be realized as the measure of nearness between x and y with respect to t. It is known that $M(x, y, \cdot)$ is non decreasing for all$x, y \in X$. Let $M(x, y, \star)$ be a fuzzy metric space, for $t > 0$, the open ball $B(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}$.

Now, the collection$\{B(x, r, t): x \in X, 0 < r < 1, t > 0\}$ is a neighborhood system for a topology τ on X induced by the fuzzy metric M. This topology is Housdroff and first countable.

Definition: 4 [6] A sequence $\{x_n\}$ in X converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

Definition: 5[8] A pair of self mapping (S, T) defined on a fuzzy metric space $(X, M, \star)$ is said to be compatible if for all $t > 0$, $\lim_{n\to\infty^+} M(STx_n, TSx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty^+} Sx_n = \lim_{n\to\infty^+} Tx_n = z$, for some $z \in X$.

Definition: 6 [2] A pair of self mappings (S, T) defined on fuzzy metric space $(X, M, \star)$ is said to satisfy the property $(E. A.)$ if there exists a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty^+} Sx_n = \lim_{n\to\infty^+} Tx_n = z$, for some $z \in X$.

Definition: 7 [2] Two pair of self mappings (A, S) and (B, T) defined on a fuzzy metric space $(X, M, \star)$ are said to be share common property $(E. A.)$ if there exists sequence $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty^+} Ax_n = \lim_{n\to\infty^+} Sx_n = \lim_{n\to\infty^+} By_n = \lim_{n\to\infty^+} Ty_n = z$, for some $z \in X$.

Definition: 8 [2] Two self mapping S and T on a fuzzy metric space $(X, M, \star)$ are called weakly compatible if they commute at their point of coincidence, that is, $Sx = Tx$ implies $STx = TSx$.

Definition: 9 [23] Let $(X, M, \star)$ be a fuzzy metric space and S, $T : X \to X$ be a pair of maps. The map S is called fuzzy contraction with respect to T if there exists an upper semi continuous function $r : [0, \infty^+] \to [0, \infty^+]$ with $\mathbf{r}(\tau) < \tau$ for every $\tau > 0$ such that,

$$\frac{1}{M(Sx,Sy,t)} - 1 \leq r\left(\frac{1}{m(S,T,x,y,t)} - 1\right)$$

For every x, $y \in X$ and each $t > 0$, where

$$m(S, T, x, y, t) = \min\{M(Sx, Tx, t), M(Sy, Ty, t), M(Tx, Ty, t)\}$$

Definition: 10 [23] Let $(X, M, \star)$ be a fuzzy metric space and S, $T : X \to X$ be a pair of maps. The map S is called fuzzy $k-$contraction with respect to T if there exists $k \in (0,1)$ such that,

$$\frac{1}{M(Sx,Sy,t)} - 1 \leq k\left(\frac{1}{m(S,T,x,y,t)} - 1\right)$$

For every $x, y \in X$ and each $t > 0$, where

$$m(S, T, x, y, t) = \min\{M(Sx, Tx, t), M(Sy, Ty, t), M(Tx, Ty, t), M(Sx, Sy, t), M(Sy, Tx, t)\}$$

Definition: 11 Let $(X, M, \star)$ be a fuzzy metric space and $A, B, S, T : X \to X$ be four maps. The maps A and B are called generalized fuzzy contraction with respect to S and T if there exists an upper semi continuous function $r : [0, \infty^+] \to [0, \infty^+]$ with $\mathbf{r}(\tau) < \tau$ for every $\tau > 0$
such that,

$$\frac{1}{M(Ax,By,t)} - 1 \leq r\left(\frac{1}{m(A,B,S,T,x,y,t)} - 1\right)$$

For every $x, y \in X$ and each $t > 0$, where

$m(A,B,S,T,x,y,t) = \min\{M(Sx,Ty,t), M(Sx,By,t), M(Ty,By,t), M(Ax,Sx,t), M(Ax,Ty,t)\}$

Now, we state and prove our main theorem as follows,

2. MAIN RESULTS

Theorem: 2. 1 Let (X,M,t) be a fuzzy metric space with $t(x,y) = \min\{x,y\}$ for all $x,y \in [0,1]$. Let A, B, S, T be mapping of X into itself such that,

(i) $A(X) \subset T(X)$ and $B(X) \subset S(X)$
(ii) (A,S) or (B,T) setisfy the property $(E.A.)$,
(iii) There exists a number $k \in (0,1)$ such that,

$$\frac{1}{M(Ax,By,t)} - 1 \le r\left(\frac{1}{\min\{M(Sx,Ty,t),M(Sx,By,t),M(Ty,By,t),M(Ax,Sx,t),M(Ax,Ty,t)\}} - 1\right)$$

For every $x,y \in X$ and each $t > 0$

(i) (A,S) and (B,T) are weakly compatible,
(ii) one of $A(X)$, $B(X)$, $S(X)$, $T(X)$ is a closed subset of X.

Then A, B, S, T, have a unique common fixed point in X.

Proof: We suppose that, (B,T) satisfies the property $(E.A)$. then there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty^+} Bx_n = \lim_{n\to\infty^+} Tx_n = z$. for some $z \in X$.

Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X, such that $Bx_n = Sy_n$. hence

$\lim_{n\to\infty} Sy_n = z$. let us show that $\lim_{n\to\infty} Ay_n = z$. indeed, in view of (iii),

we have,

$$\frac{1}{M(Ay_n,Bx_n,t)} - 1 \le r\left(\frac{1}{\min\{M(Sy_n,Tx_n,t),M(Sy_n,Bx_n,t),M(Tx_n,Bx_n,t),M(Ay_n,Sy_n,t),M(Ay_n,Tx_n,t)\}} - 1\right)$$

$$\frac{1}{M(Ay_n,Bx_n,t)} - 1 \le r\left(\frac{1}{\min\{M(Bx_n,Tx_n,t),M(Tx_n,Bx_n,t),M(Bx_n,Bx_n,t),M(Ay_n,Bx_n,t),M(Ay_n,Tx_n,t)\}} - 1\right)$$

$$\frac{1}{M(Ay_n,Bx_n,t)} - 1 \le r\left(\frac{1}{M(Ay_n,Bx_n,t)} - 1\right)$$

Which contradiction, of our hypothesis, therefore we deduce that $\lim_{n\to\infty^+} Ay_n = z$.

Suppose $S(X)$ is a closed subset of X, then $z = Su$ for some $u \in X$. subsequently, we have

$$\lim_{n\to\infty^+} Ay_n = \lim_{n\to\infty^+} Sy_n = \lim_{n\to\infty^+} Bx_n = \lim_{n\to\infty^+} Tx_n = Su.$$

Also

$$\frac{1}{M(Au,Bx_n,t)} - 1 \le r\left(\frac{1}{\min\{M(Su,Tx_n,t), M(Su,Bx_n,t), M(Tx_n,Bx_n,t), M(Au,Su,t), M(Au,Tx_n,t)\}} - 1\right)$$

As $n \to \infty$, then

$$\frac{1}{M(Au,Su,t)} - 1 \le r\left(\frac{1}{M(Au,Su,t)} - 1\right)$$

Which contradiction. Therefore, we have $Au = Su$. then by weak compatibility of A and S implies that,

$ASu = SAu$ and then $AAu = ASu = SSu$.

On the other hand, since $A(X) \subset T(X)$, then

$$\frac{1}{M(Au,Bv,t)} - 1 \leq r\left(\frac{1}{\min\{M(Su,Tv,t),M(Su,Bv,t),M(Tv,Bv,t),M(Au,Su,t),M(Au,Tv,t)\}} - 1\right)$$

$$\frac{1}{M(Au,Bv,t)} - 1 \leq r\left(\frac{1}{M(Au,Bv,t)} - 1\right)$$

Which is a contradiction, therefore, we have $Av = Bv$. thus $Au = Su = Tv = Bv$.

The weak compatibility of B and T implies that, $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

let us show that Au is a common fixed point of A, B, S, T. It follows that,

$$\frac{1}{M(Au,AAv,t)} - 1 = \frac{1}{M(AAu,Av,t)} - 1$$

$$\frac{1}{M(AAu,Av,t)} - 1 \leq r\left(\frac{1}{\min\{M(SAu,Tv,t),M(SAu,Bv,t),M(Tv,Bv,t),M(AAu,Su,t),M(AAu,Tv,t)\}} - 1\right)$$

$$\frac{1}{M(AAu,Av,t)} - 1 \leq r\left(\frac{1}{M(AAu,Av,t)} - 1\right)$$

Which contradiction, therefore, we have $Au = AAu = SAu$ and Au is a common fixed point of A and S.

Similarly, we can prove that Bv is a common fixed point of B and T. since $Au = Bv$, we conclude that, Au is a common fixed point of A, B, S, T.

The proof is similar when $T(X)$ is assumed to be a closed subset of X. the cases in which $A(X)$ or $B(X)$ is closed subset of X are similar to the cases in which $T(X)$ or $S(X)$, respectively, is closed since $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

If $Au = Bu = Su = Tu = u$ and $Av = Bv = Sv = Tv = v$, then we have,

$$\frac{1}{M(u,v,t)} - 1 = \frac{1}{M(Au,Bv,t)} - 1$$

$$\frac{1}{M(Au,Bv,t)} - 1 \leq r\left(\frac{1}{\min\{M(Su,Tv,t),M(Su,Bv,t),M(Tv,Bv,t),M(Au,Su,t),M(Au,Tv,t)\}} - 1\right)$$

$$\frac{1}{M(u,v,t)} - 1 = r\left(\frac{1}{M(u,v,t)} - 1\right)$$

Which contradiction, thus we have $u = v$ and the common fixed point is unique.This complete proof of the theorem.

Corollary: 2.2 Let (X, M, t) be a fuzzy metric space with $t(x, y) = \min\{x, y\}$ for all$x, y \in [0,1]$. Let $A, B, S,$ be mapping of X into itself such that,

(i) $A(X) \subset S(X)$ and $B(X) \subset S(X)$
(ii) (A, S) or (B, T) setisfy the property $(E. A.)$,
(iii) There exists a number $k \in (0,1)$ such that,

$$\frac{1}{M(Ax,By,t)} - 1 \leq r\left(\frac{1}{\min\{M(Sx,Sy,t),M(Sx,By,t),M(Sy,By,t),M(Ax,Sx,t),M(Ax,Sy,t)\}} - 1\right)$$

For every $x, y \in X$ and each $t > 0$

(iv) (A, S) and (B, S) are weakly compatible,
(v) one of $A(X)$, $B(X)$, $S(X)$, is a closed subset of X.

Then A, B, S, have a unique common fixed point in X.

Theorem: 2.3 Let A, B, S, T be four self mappings of a fuzzy metric space $(X, M, \star)$. assume that there exists a lebesgue integrable function $\varphi : R \to R$ and a function $\phi : [0,1]^6 \to R$, such that,

$$\int_0^{\phi(u,u,u,u,1,1)} \varphi(s)ds \geq 0, \quad \int_0^{\phi(u,,u,u,,1,1,u)} \varphi(s)ds \geq 0,$$

$$\int_0^{\phi(u,1,u,1,u,u)} \varphi(s)ds \geq 0 \qquad .(2.3.1)$$

Implies $u = 1$. suppose that the pair (A,S) and (B,T) share the common property $(E.A.)$ and $S(X)$ and $T(X)$ are closed subset of X. if for $t > 0$,

$$\int_0^{\phi(M(Ax,By,t),M(Ax,Ty,t),M(Sx,Ty,t),M(Sx,Ax,t),M(By,Ty,t),M(Sx,By,t))} \varphi(s)ds \geq 0 \qquad (2.3.2)$$

for all distinct $x, y \in X$,

Then the pair (A,S) and $(B,T,)$ have a point of coincidence each. Further A,B,S,T have a unique common fixed point provided both pairs (A,S) and (B,T) are weakly compatible.

Proof: Since the pair (A,S) and (B,T) share the common property $(E.A)$, then there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that,

$$\lim_{n\to\infty^+} Ax_n = \lim_{n\to\infty^+} Sx_n = \lim_{n\to\infty^+} By_n = \lim_{n\to\infty^+} Ty_n = z$$

For some $z \in X$.

Since $S(X)$ is a closed subset of X, then $\lim_{n\to\infty} Sx_n = z \in S(X)$. Therefore there exists a point $u \in X$ such that $Su = z$. now we assert that, $Au = Su$.

If not, by$(2.3.2)$, we have

$$\int_0^{\phi(M(Au,By_n,t),M(Au,Ty_n,t),M(Su,Ty_n,t),M(Su,Au,t),M(By_n,Ty_n,t),M(Su,By_n,t))} \varphi(s)ds \geq 0$$

On taking as $n \to \infty^+$

$$\int_0^{\phi(M(Au,z,t),M(Au,z,t),M(Su,z,t),M(Su,Au,t),M(z,z,t),M(Su,z,t))} \varphi(s)ds \geq 0$$

$$\int_0^{\phi(M(Au,z,t),M(Au,z,t),M(Au,z,t),1,1,M(Au,z,t))} \varphi(s)ds \geq 0$$

Which implies $M(Au,z,t) = 1$ and so $Au = z$. Being $T(X)$ a closed subset of X, repeating the same argument, we deduce that there exists a point $w \in X$ such that $Bw = Tw$.

Since the pair (A,S) is weakly compatible and $Au = Su$, we deduce that $Az = ASu = SAu = Sz$.

Now we assert that z is a common fixed point of the pair (A,S). Suppose that $Az \neq z$. then by using $(2.3.2)$ with $x = z$ and $y = w$, we have

$$\int_0^{\phi(M(Az,z,t),M(Az,z,t),M(Az,z,t),1,1,M(Az,z,t))} \varphi(s)ds \geq 0$$

That implies $M(Az,z,t) = 1$. hence $Az = z$. Similarly we prove that, $Bz = Tz = z$, and so z is a common fixed point of A,B,S,T. Uniqueness of z, is a consequence of condition $(2.3.2)$.

This complete proof of the theorem.

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