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AN ENERGY RELATIONSHIP IN MAGNETOHYDRODYNAMIC MULTICOMPONENT CONVECTION PROBLEM ANALOGOUS TO STERN TYPE

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ABSTRACT

In the present paper the relationship between various energies in Magnetohydrodynamic multicomponent convection problem has been established, the configuration being considered analogous to thermohaline convection of Stern (Stern, M.E., Tellus. 12 [13], 172-175) type. The established relationship proves that the total kinetic energy associated with a disturbance exceeds the sum of its total magnetic and thermal energies in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{|R|\sigma}{\pi^4}\leq 1$, *where* Q, σ, σ_1 *and R represent the Chandrasekhar number, the thermal Prandtl number, the magnetic Prandtl number and the Rayleigh number respectively. Further, this result is valid for the quite general nature of the bounding surfaces.*

Keywords: *Multicomponent convection; Chandrasekhar number; Lewis number; Prandtl number; Rayleigh number.*

INTRODUCTION

Convective phenomena which are driven by the differential diffusion of two properties such as heat and salt is named as thermosolutal convection or more generally double diffusive convection. Double diffusive convection has matured into a subject possessing fundamental departure from its counterpart, namely Rayleigh Benard convection and its studies have importance in the fields of limnology, oceanography, geophysics, chemical engineering and astrophysics etc.

For reviews of this subject one may be referred to Turner [14], Huppert and Moore [7], Griffiths [5], Huppert and Turner [8], Griffiths [6], Krishnamurti [9], Turner [15], Brandt and Fernando [3], Radko [11]. Two fundamental configurations have been studied in the context of thermosolutal convection problem, one of Veronis [16], wherein the temperature gradient is destabilizing and the concentration gradient is stabilizing and another by Stern [13], wherein the temperature gradient is stabilizing and concentration gradient is destabilizing. The main results derived by Veronis and Stern for their respective configurations are that both allow the occurrence of a stationary convection or an oscillatory convection of growing amplitude, provided the destabilizing temperature gradient or concentration gradient is sufficiently large. More interesting double diffusive phenomenon appears if the destabilizing thermal or concentration gradient is opposed by the effect of vertical magnetic field.

Chandrasekhar [4] in his investigation of the hydromagnetic Rayleigh-Benard convection problem, sought unsuccessfully the regime in terms of the parameters of the system alone in which total kinetic energy associated with a disturbance exceeds the total magnetic energy associated with it, since these considerations are of decisive significance in deciding the validity of the 'principle of the exchange of stabilities' (Banerjee *et al.* [1]). Banerjee and Gupta [2] showed that in the parameter regime $\frac{Q\sigma_1}{\pi^2}\leq 1$, the total kinetic energy associated with a perturbation is greater than the total magnetic energy associated with it. Banerjee *et al.* [1] also extended these energy considerations to a more general problem, namely, magnetothermohaline convection problem of Stern type and proved that in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{|R|\sigma}{\pi^4}\leq 1$, the total kinetic energy associated with a perturbation exceeds the sum of its total magnetic and thermal energy. The present analysis extends these energy considerations to another more complex problem, namely,

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magnetohydrodynamic multicomponent convection problem (analogous to thermohaline convection of the Stern [13] type) wherein one stabilizing heat component and $(n-1)$ destabilizing concentration components have been considered. We establish that in the parameter regime $\frac{Q\sigma_1}{\pi^2}+\frac{|R|\sigma}{\pi^4}\leq 1$, the total kinetic energy associated with a disturbance exceeds the sum of its total magnetic and thermal energies. Further, this result is valid for quite general nature of the bounding surfaces. Furthermore result of Banerjee *et al.* [1] follows as a consequence.

MATHEMATICAL FORMULATION AND ANALYSIS

Consider a viscous finitely heat conducting Boussinesq fluid layer of infinite horizontal extension statically confined between two horizontal boundaries $z=0$ and $z=d$ which are respectively maintained at uniform temperatures T_0 and $T_1(>T_0)$ and uniform concentrations $S_{10},\ S_{20},\ldots,S_{(n-1)0}$ and $S_{11}(>S_{10}),S_{21}(>S_{20}),\ldots,S_{(n-1)1}(>S_{(n-1)0})$ in the presence of a uniform vertical magnetic field, $\vec{H}$ (see Fig 1).

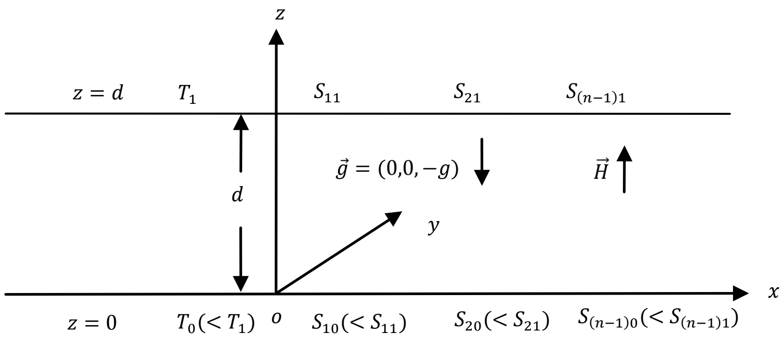

Figure-1: Physical Configuration

The governing equations and boundary conditions for magnetohydrodynamic multicomponent convection problem, when a uniform vertical magnetic field opposite to gravity is impressed upon the system, in their non-dimensional form are given by (Prakash *et al.* [10], Turner [15])

$$(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w=-|R|a^2\theta+|R_1|a^2\phi_1+|R_2|a^2\phi_2+\cdots+|R_{n-1}|a^2\phi_{n-1}-QD(D^2-a^2)h_z \quad (1)$$

$$(D^2-a^2-p)\theta=-w, \quad (2)$$

$$\left(D^2-a^2-\frac{p}{\tau_1}\right)\phi_1=-\frac{w}{\tau_1}, \quad (3)$$

$$\left(D^2-a^2-\frac{p}{\tau_2}\right)\phi_2=-\frac{w}{\tau_2}, \quad (4)$$

....

$$\left(D^2-a^2-\frac{p}{\tau_{n-1}}\right)\phi_{n-1}=-\frac{w}{\tau_{n-1}}, \quad (5)$$

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)h_z=-Dw, \quad (6)$$

together with the following boundary conditions

$$\left.\begin{array}{ll} w=0=\theta=\phi_1=\phi_2=\ldots=\phi_{n-1} & \text{on both the boundaries,}\\ D^2w=0 & \text{on a dynamically free boundary,}\\ Dw=0 & \text{on a rigid boundary,}\\ h_z=0 & \text{on both the boundaries if the regions outside the fluid are perfectly conducting,}\\ \left.\begin{array}{l} Dh_z=-ah_z\ \ at\ z=1\\ Dh_z=\ ah_z\ \ at\ z=0\end{array}\right\} & \text{if the regions outside the fluid are insulating}\end{array}\right\} \quad (7)$$

where z is the real independent variable such that $0\leq z\leq 1$. $D=\frac{d}{dz}$ is the differentiation with respect to z, $a^2>0$ is a constant, $\sigma>0$ is a constant, $\sigma_1>0$ is a constant, $\tau_1>0$, $\tau_2>0,\ldots,\tau_{n-1}>0$ are constants, $R<0$, $R_1<0, R_2<0,\ldots,R_{n-1}<0$ are constants, $Q>0$ is a constant, $p=p_r+ip_i$ is a complex constant such that p_r and p_i are real constants and as a consequence the dependent variables $w(z)=w_r(z)+iw_i(z)$, $\theta(z)=\theta_r(z)+i\theta_i(z)$, $\phi_1(z)=\phi_{1r}(z)+i\phi_{1i}(z)$, $\phi_2(z)=\phi_{2r}(z)+i\phi_{2i}(z)$,$\phi_{n-1}(z)=\phi_{(n-1)r}(z)+i\phi_{(n-1)i}(z)$ $h_z(z)=h_{zr}(z)+ih_{zi}(z)$ are complex valued functions of the real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_{1r}(z)$, $\phi_{1i}(z)$, $\phi_{2r}(z)$, $\phi_{2i}(z)$, $\phi_{(n-1)r}(z)$, $\phi_{(n-1)i}(z)$, h_{zr} and h_{zi} are real valued functions of the real variable z. The meaning of the symbols from the physical point of view are as follows: z is the vertical coordinate, $a^2>0$ is square of the wave number, $\sigma=\frac{\nu}{\kappa}$ is the Prandtl number, $\tau_1=\frac{\kappa_1}{\kappa}$, $\tau_2=\frac{\kappa_2}{\kappa},\ldots,\tau_{n-1}=\frac{\kappa_{n-1}}{\kappa}$ are the Lewis numbers for the (n-1) concentration components with mass diffusivities $\kappa_1,\kappa_2,\ldots,\kappa_{n-1}$ respectively and κ is thermal diffusivity, R is the Rayleigh number, $R_1, R_2,\ldots,R_{n-1}$ are concentration Rayleigh numbers for the (n-1) concentration components, Q is the chandershekhar number, p is the complex growth rate, w is the vertical velocity, θ is the temperature, $\phi_1, \phi_2, \ldots, \phi_{n-1}$ are the (n-1) concentrations and h_z is the magnetic field. It may further be noted that Eqs. (1) – (7) describe an eigenvalue problem for p and govern magnetohydrodynamic multicomponent convection.

We now prove the following theorem:

Theorem 1: If $(p, w, \theta, \phi_1, \phi_2, \ldots, \phi_{n-1}, h_z), R < 0, R_1 < 0, R_2 < 0, \ldots, R_{n-1} < 0, p = p_r + ip_i, p_r \geq 0$ is a solution of Eqs. (1) –(7) and $\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{\pi^4} \leq 1$, then

$$\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz > Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2) dz + |R|\, a^2 \sigma \int_0^1 |\theta|^2\, dz. \quad (8)$$

Proof: Multiplying Eq. (6) by h_z^*(the superscript $*$ denotes the complex conjugation) throughout, integrating the resulting equation, by parts, over the vertical range of z, and making use of boundary conditions (7), we get

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2|h_z|^2) dz + \frac{p\sigma_1}{\sigma} \int_0^1 |h_z|^2 dz = -\int_0^1 wDh_z^*\, dz. \quad (9)$$

Equating the real parts of Eq. (9), we obtain

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2|h_z|^2) dz + \frac{p_r\sigma_1}{\sigma} \int_0^1 |h_z|^2 dz = \text{Real part of } \left(-\int_0^1 wDh_z^*\, dz\right)$$
$$\leq \left|\int_0^1 wDh_z^* dz\right| \leq \int_0^1 |w||Dh_z|\, dz$$
$$\leq \left(\int_0^1 |w|^2\, dz\right)^{1/2} \left(\int_0^1 |Dh_z|^2\, dz\right)^{1/2}. \text{ (Using Schwartz inequality)} \quad (10)$$

Since $p_r \geq 0$, therefore we have from inequality (10), that

$$\int_0^1 |Dh_z|^2\, dz < \left(\int_0^1 |w|^2\, dz\right)^{1/2} \left(\int_0^1 |Dh_z|^2\, dz\right)^{1/2}$$
$$\text{or } \left(\int_0^1 |Dh_z|^2\, dz\right)^{1/2} < \left(\int_0^1 |w|^2\, dz\right)^{1/2}. \quad (11)$$

Utilizing inequality (11) in inequality (10), we have

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2) dz < \int_0^1 |w|^2\, dz. \quad (12)$$

Since $w(0) = 0 = w(1)$, therefore using Rayleigh Ritz inequality (Schultz [12]), we obtain

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz. \quad (13)$$

It follows from inequality (12) and (13) that

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2) dz < \frac{1}{\pi^2} \int_0^1 |Dw|^2\, dz < \frac{1}{\pi^2} \int_0^1 (|Dw|^2 + a^2|w|^2) dz. \quad (14)$$

Now multiplying Eq. (2) by θ^* and integrating the resulting equation by parts for a suitable number of times and making use of the boundary conditions (7) and equating the real parts of resulting equation, we have

$$\int_0^1 (|D\theta|^2 + a^2|\theta|^2) dz + p_r \int_0^1 |\theta|^2 dz = \text{ Real part of } \left(\int_0^1 \theta^* w dz\right)$$
$$\leq \left|\int_0^1 \theta^* w dz\right| \leq \int_0^1 |\theta||w|\, dz$$
$$\leq \left(\int_0^1 |\theta|^2\, dz\right)^{1/2} \left(\int_0^1 |w|^2\, dz\right)^{1/2}. \text{ (using Schwartz inequality)} \quad (15)$$

Since $p_r \geq 0$, therefore we have from inequality (15), we obtain

$$\int_0^1 |D\theta|^2\, dz < \left(\int_0^1 |\theta|^2\, dz\right)^{1/2} \left(\int_0^1 |w|^2\, dz\right)^{1/2}. \quad (16)$$

Since $\theta(0) = 0 = \theta(1)$, therefore using Rayleigh Ritz inequality (Schultz [12]), we obtain

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz. \quad (17)$$

Combining inequalities (16) and (17), we have

$$\int_0^1 |\theta|^2 dz < \frac{1}{\pi^4} \int_0^1 |w|^2\, dz. \quad (18)$$

From inequalities (15) and (18), we get

$$a^2 \int_0^1 |\theta|^2 dz < \frac{1}{\pi^2} \int_0^1 |w|^2\, dz. \quad (19)$$

Combining inequalities (13) and (19), we have

$$a^2 \int_0^1 |\theta|^2 dz < \frac{1}{\pi^4} \int_0^1 |Dw|^2\, dz < \frac{1}{\pi^4} \int_0^1 (|Dw|^2 + a^2|w|^2) dz. \quad (20)$$

Finally from inequalities (14) and (20), we obtain

$$Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz + |R|a^2\sigma \int_0^1 |\theta|^2 dz < \left(\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{\pi^4}\right) \int_0^1 (|Dw|^2 + a^2|w|^2)dz. \quad (21)$$

Thus, if $\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{\pi^4} \leq 1$, then inequality (21) yields

$$\int_0^1 (|Dw|^2 + a^2|w|^2)dz > Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)dz + |R|a^2\sigma \int_0^1 |\theta|^2 dz \quad (22)$$

which completes the proof of the theorem.

It is clear from inequality (22), that left hand side represents total kinetic energy associated with a perturbation while the right hand side represents the sum of its total magnetic and thermal energies and thus theorem 1 may be stated in equivalent form as: ‘At the neutral or unstable state in the magnetohydrodynamic multicomponent convection problem of the Stern type, the total kinetic energy associated with a disturbance is greater than the sum of its total magnetic and thermal energies in the parameter regime $\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{\pi^4} \leq 1$, and this result is uniformly valid for any combination of a dynamically free or a rigid boundary that are either perfectly conducting or insulating’.

Note: If we put $R_2 = R_3 = \cdots = R_{n-1} = 0$, we obtain the result of Banerjee *et al.* [1].

CONCLUSION

Linear stability of multicomponent configuration has been analyzed in the presence of a uniform vertical magnetic field. An energy relationship has been derived for this configuration which proves that the total kinetic energy associated with the perturbation exceeds the sum of its total magnetic and thermal energies in the parameter regime $\frac{Q\sigma_1}{\pi^2} + \frac{|R|\sigma}{\pi^4} \leq 1$. The result derived herein is uniformly valid for the quite general nature of the bounding surfaces.

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