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ON SOME PROPERTIES OF GLUING HYPERCONVEX SUBSETS

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ABSTRACT

In this paper we generalize gluing hyperconvex metric spaces with the properties of gluing along externally hyperconvex subsets and additionally generalize gluing along weakly externally hyperconvex subsets. Our results are the extensions of the results obtained by Benjamin Miesch [10] to the case of intersection property for externally hyperconvex subsets leads to hyperconvex spaces.

Keywords: *hyperconvex spaces; injective metric spaces; gluing hyperconvex; externally hyperconvex subsets.*

1. INTRODUCTION

A metric space (X, d) is called hyperconvex if every collection of closed balls $\{B(x_i, r_i)\}_{i\in I}$ with $d(x_i, x_j) \leq r_i + r_j$ has non-empty intersection $\bigcap_i B\,(x_i, r_i) \neq \emptyset$. Hyperconvex metric spaces appear in various contexts. They are introduced by N. Aronszajn and P. panitchpakdi [1] who proved that they are the same as absolute I-lipschitz retracts or injective metric spaces and later studied by J. lindenstrauss *et al.* [7] to receive extension results for linear operators between Banach spaces. Hyperconvex metric spaces also play an important role in metric fixed point theory. Since then many interesting results [4, 5, 6, 8] have been proved in hyperconvex spaces. Hyperconvexity is a very wide subject and hence research writings about it are very diverse. Baillon [2] proved that any intersection of hyperconvex spaces with a certain finite intersection property is a nonempty hyperconvex space.

A classical problem that arises in metric geometry is how to glue metric spaces such that their properties are preserved. Recently, B. piatek [12] proved that if we glue two hyperconvex metric spaces along a unique interval, then the resulting metric space is hyperconvex as well where it is shown that gluing along unique intervals preserves hyperconvexity.

In this paper we generalize and extend the work of Miesch [10] by considering some intersection properties for gluing hyperconvex subsets.

2. PRELIMINARIES

Let (G, d) be a metric space. We denote by
$$B(g_0, r) = \{g \in G: d(g, g_0) \leq r\}$$

The closed ball of radius r with centre in g_0. For any subset $M \subset G$ let
$$B(M, r) = \{g \in G: d(g, M), d(g, M^*) \coloneqq \min_{h\in M} d\,(g, h) \leq r\}$$

Be the closed r-neighborhood of M, M^*.

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Definition 2.1: let $(G_\lambda, d_\lambda)_{\lambda\in\Delta}$ be a family of metric spaces with closed subspaces $M_\lambda \subset G_\lambda$. Suppose that M_λ are isometric to some metric space M. For every $\lambda \in \Delta$ fix some isometry $\varphi_\lambda: M \to M_\lambda$. We define an equivalence relation on the disjoint union $\coprod_\lambda G_\lambda$ generated by $\varphi_\lambda(m) \sim \varphi_{\lambda^*}(m)$ for $m \in M$. The resulting space $G = \coprod_\lambda G_\lambda \backslash \sim$ is called the gluing of the M_λ along M, M^*.

Definition 2.2: A metric space (G, d) is injective if for every isometric embedding $\imath: M \to h$ of a metric spaces and every I-lipschitz map $f: M \to G$ there is some I-lipschitz map $\bar{f}: H \to G$ such that $f = \bar{f} \circ \imath$.

Definition 2.3: Let (X, d_X) and (Y, d_Y) be two given metric spaces, then a mapping $P: X \to Y$ is nonexpansive if it is lipschitzian with lipschitz constant I, that is, if

$$d_Y\big(P(x), P(y)\big) \le d_X(x, y), \text{ for any } x, y \in X.$$

Definition 2.4: A metric space (G, d) is (m-)hyperconvex if for any collection $\{B(x_i, r_i)\}_{i\in I}$ of closed balls with $d(x_i, x_j) \le r_i + r_j$ (and $|I| \le m$) we have

$$\bigcap_{i\in I} B(x_i, r_i) \neq \emptyset.$$

Proposition 2.5: let (G, d) be a gluing hyperconvex metric space and $\{M_i\}_{i\in I}$ a family of pairwise intersection, gluing along externally hyperconvex subsets such that at least one of them is bounded. Then we have

$$\bigcap_{i\in I} M_i \neq \emptyset.$$

Lemma 2.6: let (G, d) be a gluing hyperconvex and $M, M^* \varepsilon(G)$. Then also $B(M, r)\varepsilon(G)$ and $B(M^*, r)\varepsilon(G)$.

Proof: Let $\{B(x_i, r_i)\}_{i\in I}$ be a collection of closed balls with $d(x_i, x_j) \le r_i + r_j$ and $d\big(x_i, B(M, r)\big) \le r_i, d\big(x_i, B(M^*, r^*)\big) \le r_i$,. Then we also have $d(x_i, M) \le r_i + r$, and $d(x_i, M^*) \le r_i + r^*$. Since M, M^* is gluing externally hyperconvex there is some $h \in \bigcap B(x_i, r_i + r + r^*) \cap M \cap M^*$. Especially we have $d(x_i, h) \le r_i + r + r^*$ and therefore since G is gluing hyperconvex we get $\emptyset \neq \bigcap B(x_i, r_i) \cap B(h, r) \cap B(h, r^*) \subset \bigcap B(x_i, r_i) \cap B(M, r) \cap B(M^*, r^*)$.

3. GLUING ALONG EXTERNALLY HYPERCONVEX SUBSETS

In this section, we generalized and extend the result of Benjamin Miesch [10] to the gluing externally hyperconvex subsets. Let (G, d) be a metric space. A subset $M \subset G$ is called admissible if it can be written as a non-empty intersection of closed balls $M = \bigcap_{i\in I} B(x_i, r_i)$. We denote the family of all admissible subsets of G by $M(G)$. Furthermore, denote the set of gluing externally hyperconvex subsets of G by $\varepsilon(G)$.

Theorem 3.1: Let (G, d) be a gluing hyperconvex metric space. Let $M, M^*, M^{**} \in \varepsilon(G)$ with $h \in M \cap M^* \cap M^{**} \neq \emptyset$ and $g \in G$ with $d(g, M), d(g, M^*), d(g, M^{**}) \le r$.

Denote $d = d(g, h)$ and $s = d - r$. then we have

$$M \cap M^* \cap M^{**} \cap B(g, r) \cap B(h, s) \neq \emptyset,$$

given $s \ge 0$. Then the intersection $M \cap M^* \cap M^{**} \cap B(g, r)$ is non-empty.

Proof: For $s \le 0$ then we have $h \in M \cap M^* \cap M^{**} \cap B(g, r)$. Let us assume $s > 0$. For each $0 < p \le s$ there are $m \in M, m^* \in M^*, m^{**} \in M^{**}$ such that $d(m, m^*) \le p, d(m^*, m^{**}) \le p,$ and $d(m, m^{**}) \le p$ therefore $m, m^*, m^{**} \in B(g, r) \cap B(h, s)$. Next we choose

$$m_1 \in B(h, p) \cap B(g, d-p) \cap M$$
$$m_1^* \in B(h, p) \cap B(g, d-p) \cap B(m_1, p) \cap M^*$$

and

$$m_1^{**} \in B(h, p) \cap B(g, d-p) \cap B(m_1, p) \cap B(m_1^*, p) \cap M^* \cap M^{**}$$

Then we inductively take

$$m_v \in B(h, vp) \cap B(g, d-vp) \cap B(m_{v-1}^*, p) \cap M$$
$$m_v^* \in B(h, vp) \cap B(g, d-vp) \cap B(m_v, p) \cap M^*$$

and

$$m_v^{**} \in B(h, vp) \cap B(g, d-vp) \cap B(m_v, p) \cap B(m_v^*, p) \cap M^* \cap M^{**}$$

As long as $v \le \left[\frac{s}{p}\right] =: v_0$. Finally there are

$$m \in B(h, s) \cap B(g, r) \cap B(m_{v0}^*, p) \cap M$$
$$m^* \in B(h, s) \cap B(g, r) \cap B(m, p) \cap M^*$$

and

$$m^{**} \in B(h, s) \cap B(g, r) \cap B(m, p) \cap B(m^*, p) \cap M^* \cap M^{**}$$

Now we construct recursively two converging sequence $(m_v) \subset M$ and $(m_v^*) \subset M^*$ such that $m_v, m_v^* \in B(g, r) \cap B(h, s)$ with

$$d(m_v, m_v^*) \le \frac{1}{2^{v+1}} \text{ and } d(m_{v-1}, m_v), d(m_{v-1}^*, m_v^*) \le \frac{1}{2^v}$$

Now first we choose $m_0, m_0^*, m_0^{**} \in B(g,r) \cap B(h,s)$ with $d(m_0, m_0^*) \le \frac{1}{2}$ and $d(m_0^*, m_0^{**}) \le \frac{1}{2}$.
Given that $m_{v-1}, m_{v-1}^*, m_{v-1}^{**}$ with $d(m_{v-1}, m_{v-1}^*)$, $d(m_{v-1}, m_{v-1}^{**})$, $d(m_{v-1}^*, m_{v-1}^{**}) \le \frac{1}{2^v}$ there is some $g_v \in B\left(m_{v-1}, \frac{1}{2^{v+1}}\right) \cap B\left(m_{v-1}^*, \frac{1}{2^{v+1}}\right) \cap B\left(m_{v-1}^{**}, \frac{1}{2^{v+1}}\right) \cap B\left(g, r - \frac{1}{2^{v+1}}\right)$.

Now applying that g_v and h we find

$$m_v, m_v^*, m_v^{**} \in B(h,s) \cap B(m_v, \frac{1}{2^{v+1}}) \subset B(h,s) \cap B(g,r)$$

With $d(m_v, m_v^*)$, $d(m_v, m_v^{**})$, $d(m_v^*, m_v^{**}) \le \frac{1}{2^{v+1}}$. Moreover, we have

$$d(m_{v-1}, m_v) \le d(m_{v-1}, g_v) + d(g_v, m_v) \le \frac{1}{2^{v+1}} + \frac{1}{2^{v+1}} = \frac{1}{2^v}$$

For $u \ge v$ we get

$$d(m_v, m_u) \le \sum_{k=v+1}^{u} d(m_{k-1}, m_k) \le \sum_{k=v+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^v}$$

And similarly for (m_v^*). Hence the two sequences converge and since $d(m_v, m_v^*)$, $d(m_v, m_v^{**})$, $d(m_v^*, m_v^{**}) \to 0$ they have a common limit point $m \in B(h,s) \cap B(g,r) \cap M \cap M^* \cap M^{**}$. Hence proved.

4. GLUING ALONG WEAKLY EXTERNALLY HYPERCONVEX SUBSETS

In this section, we extend the following intersection property of gluing along weakly externally hyperconvex subsets.

Lemma 4.1: Let G be a gluing hyperconvex metric space and $M, M^* \varepsilon(G), H, H^* \in \mathcal{W}(G)$ such that $M \cap H \cap M^* \cap H^* \ne \emptyset$. Let $\{x_i\}_{i\in I} \subset H$ be a collection of points with $d(x_i, x_j) \le r_i + r_j$ and $d(x_i, M) \le r_i, d(x_i, M^*) \le r_i$. Then, for any $s > 0$, there are $m \in M \cap M^* \cap \bigcap_{i\in I} B(x_i, x_j)$ and $h \in H \cap H^* \cap \bigcap_{i\in I} B(x_i, x_j)$with $d(m,h) \le s, \delta d(m^*, h^*) \le s$.

Proof: Let $h_0, h_0^* \in M \cap H \cap M^* \cap H^*$ and $d = d\left(h_0, \bigcap_{i\in I} B(x_i, r_i)\right)$. Without loss of generality, we may assume that $s \le d$. Then, since $M, M^* \in \varepsilon(G), H, H^* \in \mathcal{W}(G)$, there are

$$m_1 \in \bigcap_{i\in I} B(x_i, r_i + d - s) \cap B(h_0, s) \cap M \cap M^*,$$
$$m_1 \in \bigcap_{i\in I} B(x_i, r_i + d - s) \cap B(m_1, s) \cap H \cap H^*$$

Proceeding this way, we can choose inductively

$$m_n \in \bigcap_{i\in I} B(x_i, r_i + d - ns) \cap B(h_{n-1}, s) \cap M \cap M^*,$$

$$m_n \in \bigcap_{i\in I} B(x_i, r_i + d - ns) \cap B(h_n, s) \cap H \cap H^*$$

For $n \le \left[\frac{d}{s}\right] = n_0$ and finally, there are

$$m \in \bigcap_{i\in I} B(x_i, r_i) \cap B(h_{n0}, s) \cap M \cap M^*$$
$$h \in \bigcap_{i\in I} B(x_i, r_i) \cap B(m, s) \cap H \cap H^*$$

as desired.

Theorem 4.2: let (G,d) be a gluing hyperconvex metric space and let $M, M^* \in \varepsilon(G)$, $H, H^* \in \mathcal{W}(G)$ such that $M \cap H \cap M^* \cap H^* \ne \emptyset$. Then we have $M \cap H \cap M^* \cap H^* \in \varepsilon(H)$ and therefore $M \cap H \cap M^* \cap H^* \in \mathcal{W}(G)$

Proof: Let $\{x_i\}_{i\in I} \subset H$ with $d(x_i, x_j) \le r_i + r_j$ and $d(x_i, M) \le r_i, d(x_i, M^*) \le r_i$. We now construct inductively two converging sequences $(m_n)_{n\in\mathbb{N}} \subset \bigcap_{i\in I} B(x_i, r_i) \cap M \cap M^*$ and $(h_n)_{n\in\mathbb{N}} \subset \bigcap_{i\in I} B(x_i, r_i) \cap H \cap H^*$ with $d(m_n, h_n) \le \frac{1}{2^{n+1}}$ as follows: Now we may choose

$$m_0 \in \bigcap_{i\in I} B(x_i, r_i) \cap M \cap M^*,$$
$$h_0 \in \bigcap_{i\in I} B(x_i, r_i) \cap H \cap H^*,$$

with $d(m_o, h_o) \le \frac{1}{2}$. Assume now, that we have

$$m_n \in \bigcap_{i\in I} B(x_i, r_i) \cap M \cap M^*$$
$$h_n \in \bigcap_{i\in I} B(x_i, r_i) \cap H \cap H^*$$

with $d(m_n, h_n) \le \frac{1}{2^{n+1}}$. Then, applying lemma 3.8 for the collection of balls $\{B(x_i, r_i)\}_{i\in I} \cup \left\{B\left(h_n, \frac{1}{2^{n+1}}\right)\right\}$, we find

$$m_{n+1} \in \bigcap_{i\in I} B(x_i, r_i) \cap B(h_n, \frac{1}{2^{n+1}}) \cap M \cap M^*$$
$$h_{n+1} \in \bigcap_{i\in I} B(x_i, r_i) \cap B(h_n, \frac{1}{2^{n+1}}) \cap H \cap H^*$$

with $d(m_{n+1}, h_{n+1}) \le \frac{1}{2^{n+1}}$. Especially, we have $d(h_{n+1}, h_n) \le \frac{1}{2^{n+1}}$ and

$$d(m_{n+1}, m_n) \le d(m_{n+1}, h_n) + d(h_n, m_n) \le \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}.$$

Hence the two sequences are Cauchy and therefore converges to some common limit point $z \in M \cap H \cap M^* \cap H^* \cap \bigcap_{i\in I} B(x_i, r_i) \neq \emptyset$. Finally, we have $M \cap H \cap M^* \cap H^* \in \mathcal{W}(G)$.

Corollary 4.3: Let (G, d) be a gluing hyperconvex metric space and let $M, M^* \in \varepsilon(G)$, $H, H^* \in \mathcal{W}(G)$ such that $M \cap H \cap M^* \cap H^* \neq \emptyset$. Then, for all $g \in H$, we have

$$d(g, M) = d(g, M \cap H), \text{ and } d(h, M^*) = d(h, M^* \cap H^*).$$

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