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COMMON FIXED POINTS FOR FOUR MAPS USING $\alpha-$ ADMISSIBLE FUNCTIONS IN METRIC SPACES

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ABSTRACT

In this paper, we introduce $\alpha-$admissible function associated with four maps and obtain a unique common fixed point theorem. We also give an example to illustrate our main theorem.

Keywords: *Complete metric spaces, $\alpha-$admissible functions, Compatible mappings*

Mathematics Subject Classification: *54H25, 47H10.*

1. INTRODUCTION AND PRELIMINARIES

In 1973, Geraghty [3] introduced an interesting class of auxiliary functions to refine the Banach contraction mapping principle. Let $\mathcal{F}$ denote a set of all functions $\beta:[0,\infty)\to[0,1)$ satisfying the condition

$$\lim_{n\to\infty}\beta(t_n)=1 \quad \text{implies} \quad \lim_{n\to\infty}t_n=0.$$

By using the function $\beta\in\mathcal{F}$, Geraghty [3] proved the following remarkable theorem.

Theorem 1.1 [3]: Let (X,d) be a complete metric space and $T:X\to X$ be an operation. If T satisfies $d(Tx,Ty)\le\beta(d(x,y))d(x,y)$ *for all* $x,y\in X$, *where* $\beta\in\mathcal{F}$, then T has a unique fixed point in X.

Definition 1.2: Let Ψ denote the class of all functions $\psi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions

(a) ψ is non-decreasing and continuous,

(b) $\psi(t)=0\Leftrightarrow t=0$.

Definition 1.3 [4]: Let f *and* g be self mappings on a metric space (X,d). The pair (f,g) is said to be compatible if $d(fgx_n,gfx_n)\to 0$ whenever there exists a sequence $\{x_n\}$ in X such that $fx_n\to z$ *and* $gx_n\to z$ *for some* $z\in X$.

Samet *et.al* [6] introduced the notion of $\alpha-$admissible mappings as follows

Definition 1.4 [6]: Let X be a non empty set, $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$ be mappings. Then T is called $\alpha-$admissible if for all $x,y\in X$, we have $\alpha(x,y)\ge 1$ *implies* $\alpha(Tx,Ty)\ge 1$.

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Some interesting examples of such mappings are given in [6]. Actually, they proved the following

Theorem 1.5 [6]: Let (X,d) be a complete metric space. Suppose that $\alpha: X\times X\to[0,\infty)$ and $\varphi:[0,\infty)\to[0,\infty)$, where ϕ is non-decreasing and $\sum\phi^n(t)<\infty$ for each $t>0$. Suppose that $T: X\to X$ satisfies $\alpha(x,y)d(Tx,Ty)\le\phi(d(x,y))$ for all $x,y\in X$.

Assume the following

(i) T is $\alpha-$admissible,

(ii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$,

(iii) either T is continuous or if $\{x_n\}$ is a sequence in X with $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathcal{N}$ (the set of all natural numbers) $and\ x_n\to x\ as\ n\to\infty$, then $\alpha(x_n,x)\ge 1$ for all $n\in\mathcal{N}$.

Then T has a fixed point in X.

Further, if for any $x,y\in X$, there exists $z\in X$ such that $\alpha(x,z)\ge 1$ and $\alpha(y,z)\ge 1$ then T has a unique fixed point in X.

Recently. Karapinar *et.al* [5] defined the notion of triangular $\alpha-$admissible mappings as follows

Definition 1.6 [5]: Let X be a non empty set, $T: X\to X$ and $\alpha: X\times X\to[0,\infty)$. Then T is called triangular $\alpha-$admissible if

(i) $x,y\in X,\ \alpha(x,y)\ge 1\Rightarrow\alpha(Tx,Ty)\ge 1$

(ii) $x,y,z\in X,\ \alpha(x,z)\ge 1\ and\ \alpha(z,y)\ge 1\Rightarrow\alpha(x,y)\ge 1.$

Later Shahi *et.al* [7] and Abdeljawad [1] defined the following

Definition 1.7 [7]: Let X be a non empty set, $f,g: X\to X$ and $\alpha: X\times X\to[0,\infty)$. Then f is said to be $\alpha-$admissible with respect to g if $\alpha(gx,gy)\ge 1\ implies\ \alpha(fx,fy)\ge 1\ for\ all\ x,y\in X.$

Definition 1.8 [1]: Let X be a non empty set, $f,g: X\to X$ and $\alpha: X\times X\to[0,\infty)$. Then the pair (f,g) is said to be $\alpha-$admissible if $\alpha(x,y)\ge 1\ implies\ \alpha(fx,gy)\ge 1\ and\ \alpha(gx,fy)\ge 1\ for\ all\ x,y\in X.$

Using these definitions, we introduce the following

Definition 1.9: Let X be a non empty set and $f,g,S,T: X\to X$ and $\alpha: X\times X\to[0,\infty)$. The pair (f,g) is said to be $\alpha-$admissible w.r.to the pair (S,T)

$if\ \alpha(Sx,Ty)\ge 1\ implies\ \alpha(fx,gy)\ge 1\ and\ \alpha(Tx,Sy)\ge 1\ implies\ \alpha(gx,fy)\ge 1.$

Definition 1.10: (f,g) is called triangular $\alpha-$admissible w.r.to (S,T) if

(i) (f,g) is $\alpha-$admissible w.r.to (S,T) and

(ii) $\alpha(x,y)\ge 1\ and\ \alpha(y,z)\ge 1\Rightarrow\alpha(x,z)\ge 1\ for\ all\ x,y,z\in X.$

Recently Shahi *et.al* [7] and Cho *et.al* [2] proved the following

Theorem 1.11 (Theorem 3.1, [7]): Let (X,d) be a complete metric space and $f,g: X\to X$ be such that $f(X)\subseteq g(X)$. Assume the following

(1.12.1) f is $\alpha-$admissible with respect to g,

(1.12.2) $\alpha(gx,gy)d(fx,fy)\le\psi(M(gx,gy))$, $where$

$$M(gx,gy)=\max\left\{d(gx,gy),\frac{d(gx,fx)+d(gy,fy)}{2},\frac{d(gx,fy)+d(gy,fx)}{2}\right\}\ and$$

$\psi:[0,\infty)\to[0,\infty)$ is continuous, nondecreasing and $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ *for all* $t>0$,

(1.12.3) there exists $x_0\in X$ such that $\alpha(gx_0, fx_0)\geq 1$,

(1.12.4) if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1})\geq 1$, *for all n and*
$gx_n\to gz\in g(X)$, then there exists a subsequence $\{gx_{n_k}\}$ *of* $\{gx_n\}$ such that $\alpha(gx_{n_k}, gz)\geq 1$, *for all k*.

Also suppose that $g(X)$ is closed.

Then f *and* g have a coincidence point.

Theorem 1.12 (Theorem 2.1, [2]): Let (X,d) be a complete metric space $\alpha: X\times X\to[0,\infty)$ be a function and $T: X\to X$. suppose that the following conditions are satisfied

(1.13.1) $\alpha(x,y)d(Tx,Ty)\leq\beta(M(x,y))M(x,y)$ *for all* $x,y\in X$, *where* $\beta\in\mathcal{F}$ and
$M(x,y)=\max\{d(x,y), d(x,Tx), d(y,Ty)\}$,

(1.13.2) T is triangular $\alpha-$admissible,

(1.13.3) there exists $x_1\in X$ such that $\alpha(x_1, Tx_1)\geq 1$,

(1.13.4) T is continuous.

Then T has a fixed point.

Now we prove our main result to generalize Theorems 1.11 and 1.12.

2. MAIN RESULT

Theorem 2.1: Let (X,d) be a complete metric space and $\alpha: X\times X\to[0,\infty)$ be a function. Let f,g,S *and* T *be self mappings on* X satisfying

(2.1.1) $f(X)\subseteq T(X)$, $g(X)\subseteq S(X)$,

(2.1.2) $\alpha(Sx,Ty)\psi(d(fx,gy))\leq\beta(\psi(M(x,y)))\psi(M(x,y))$ *for all* $x,y\in X$
where $\beta\in\mathcal{F}$, $\psi\in\Psi$ and

$$M(x,y)=\max\left\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{1}{2}[d(Sx,gy)+d(Ty,fx)]\right\},$$

(2.1.3) the pairs (f,S) *and* (g,T) are compatible and S and T are continuous on X,

(2.1.4) (f,g) is triangular $\alpha-$admissible w.r.to (S,T),

(2.1.5) $\alpha(Sx_1, fx_1)\geq 1$ and $\alpha(fx_1, Sx_1)\geq 1$ for some $x_1\in X$,

(2.1.6) Assume that $\alpha(Sy_{2n}, \mathrm{y}_{2n-1})\geq 1$, $\alpha(y_{2n}, \mathrm{Ty}_{2n+1})\geq 1$, $\alpha(\mathrm{z}, \mathrm{y}_{2n-1})\geq 1$ *and* $\alpha(z,z)\geq 1$
whenever there exists a sequence
$\{y_n\}$ *in* X *such that* $\alpha(y_n, y_{n+1})\geq 1$ *for* $n=1,2,3,......$ *and* $y_n\to z$ *for some* $z\in X$.

Then f,g,S *and* T have a common fixed point.

(2.1.7) Further if $\alpha(u,\mathrm{v})\geq 1$ whenever u and v are common fixed points of f,g,S *and* T
then f,g,S *and* T have unique common fixed point in X.

Proof: From (2.1.5), we have $\alpha(Sx_1, fx_1)\geq 1$ *for some* $x_1\in X$.

From (2.1.1), define the sequences $\{x_n\}$ *and* $\{y_n\}$ *as follows*:

$$y_1=fx_1=Tx_2,\ y_2=gx_2=Sx_3,\ y_3=fx_3=Tx_4,\ y_4=gx_4=Sx_5,....$$
$$y_{2n+1}=fx_{2n+1}=Tx_{2n+2},\ y_{2n+2}=gx_{2n+2}=Sx_{2n+3},\ n=0,1,2,....$$

Now

$$\alpha\left(Sx_1, fx_1\right)\geq 1 \;\Rightarrow \alpha\left(Sx_1,Tx_2\right)\geq 1$$
$$\Rightarrow \alpha\left(fx_1,\mathrm{g}\,x_2\right)\geq 1,\; from\ (2.1.4),\ i.e\ \ \alpha\left(y_1,\mathrm{y}_2\right)\geq 1$$
$$\Rightarrow \alpha\left(Tx_2,Sx_3\right)\geq 1$$
$$\Rightarrow \alpha\left(\mathrm{g}\,x_2, fx_3\right)\geq 1,\; from\ (2.1.4),\ i.e\ \ \alpha\left(y_2,\mathrm{y}_3\right)\geq 1$$
$$\Rightarrow \alpha\left(Sx_3,Tx_4\right)\geq 1$$
$$\Rightarrow \alpha\left(fx_3,\mathrm{g}\,x_4\right)\geq 1,\; from\ (2.1.4),\ i.e\ \ \alpha\left(y_3,\mathrm{y}_4\right)\geq 1$$

Continuing in this way, we have

$$\alpha\left(y_n,\mathrm{y}_{n+1}\right)\geq 1\;\; for\;\; n=1,2,3,... \tag{1}$$

Similarly using $\alpha(fx_1,Sx_1)\geq 1$, we have $\alpha(y_{n+1},y_n)\geq 1$ for n=1,2,……. (1)[1]

By (2.1.4), using triangular property, we have

$$\alpha\left(y_m,\mathrm{y}_n\right)\geq 1\;\; for\;\; m<n. \tag{2}$$

Case-(a): Suppose $y_{2m}=y_{2m+1}$.

Then $\alpha\left(Sx_{2m+1},\mathrm{T}\,x_{2m+2}\right)=\alpha\left(y_{2m},\mathrm{y}_{2m+1}\right)\geq 1\;\; from\;\; (1).$

Now

$$\begin{aligned}\psi\left(d\left(y_{2m+1},\mathrm{y}_{2m+2}\right)\right)&=\psi\left(d\left(fx_{2m+1},gx_{2m+2}\right)\right)\\ &\leq \alpha\left(Sx_{2m+1},\mathrm{T}\,x_{2m+2}\right)\psi\left(d\left(fx_{2m+1},gx_{2m+2}\right)\right)\\ &\leq \beta\left(\psi\left(M\left(x_{2m+1},x_{2m+2}\right)\right)\right)\psi\left(M\left(x_{2m+1},x_{2m+2}\right)\right),\end{aligned}$$

where

$$\begin{aligned}M\left(x_{2m+1},x_{2m+2}\right)&=\max\left\{\begin{array}{c}d\left(y_{2m},y_{2m+1}\right),d\left(y_{2m},y_{2m+1}\right),d\left(y_{2m+1},y_{2m+2}\right),\\ \frac{1}{2}\left[d\left(y_{2m},y_{2m+2}\right)+d\left(y_{2m+1},y_{2m+1}\right)\right]\end{array}\right\}\\ &=d\left(y_{2m+1},y_{2m+2}\right)\end{aligned}$$

$$Thus\;\;\psi\left(d\left(y_{2m+1},\mathrm{y}_{2m+2}\right)\right)\leq \beta\left(\psi\left(d\left(y_{2m+1},y_{2m+2}\right)\right)\right)\psi\left(d\left(y_{2m+1},y_{2m+2}\right)\right)$$

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$\left[1-\beta\left(\psi\left(d\left(y_{2m+1},y_{2m+2}\right)\right)\right)\right]\psi\left(d\left(y_{2m+1},y_{2m+2}\right)\right)\leq 0$ which in turn yields that $\psi\left(d\left(y_{2m+1},y_{2m+2}\right)\right)=0$ so that $y_{2m+1}=y_{2m+2}$.

Continuing in this way we get $y_{2m}=y_{2m+1}=y_{2m+2}=....$

Hence $\left\{y_n\right\}$ is Cauchy.

Case-(b): Suppose $y_n\neq y_{n+1}\;\; for\;\; all\;\; n.$

As in Case (a), we have

$$\psi\left(d\left(y_{2n+1},\mathrm{y}_{2n+2}\right)\right)\leq \beta\left(\psi\left(M\left(x_{2n+1},x_{2n+2}\right)\right)\right)\psi\left(M\left(x_{2n+1},x_{2n+2}\right)\right) \tag{3}$$

where

$$M(x_{2n+1},x_{2n+2})=\max\{d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n+2})\}$$

$$\text{If } M(x_{2n+1},x_{2n+2})=d(y_{2n+1},y_{2n+2}) \text{ then we get}$$

$$\psi(d(y_{2n+1},y_{2n+2}))\le\beta(\psi(d(y_{2n+1},y_{2n+2})))\psi(d(y_{2n+1},y_{2n+2}))$$
$$<\psi(d(y_{2n+1},y_{2n+2}))$$

It is a contradiction. Hence

$$\psi(d(y_{2n+1},y_{2n+2}))\le\beta(\psi(d(y_{2n},y_{2n+1})))\psi(d(y_{2n},y_{2n+1}))$$
$$<\psi(d(y_{2n},y_{2n+1}))$$

which in turn yields that $d(y_{2n+1},y_{2n+2})<d(y_{2n},y_{2n+1})$.

Similarly using (2.1.2) and (1)[1], we can show that $d(y_{2n},y_{2n+1})<d(y_{2n-1},y_{2n})$.

Thus $\{d(y_n,y_{n+1})\}$ is a decreasing sequence of non-negative real numbers.

Hence it converges to some real number $r\ge 0$ such that

$$\lim_{n\to\infty}d(y_n,y_{n+1})=r.$$

Suppose $r>0$.

From (3), $\dfrac{\psi(d(y_{2n+1},y_{2n+2}))}{\psi(M(x_{2n+1},x_{2n+2}))}\le\beta(\psi(M(x_{2n+1},x_{2n+2})))<1.$

Letting $n\to\infty$ and using the continuity of ψ, we get

$$1\le\lim_{n\to\infty}\beta(\psi(M(x_{2n+1},x_{2n+2})))\le 1$$

so that $\lim_{n\to\infty}\psi(M(x_{2n+1},x_{2n+2}))=0$ which in turn yields that $\psi(r)=0$ *and hence* $r=0$.

It is a contradiction. Thus

$$\lim_{n\to\infty}d(y_n,y_{n+1})=0 \tag{4}$$

Now we prove that $\{y_n\}$ is a Cauchy sequence. In view of (4), it is sufficient to show that $\{y_{2n}\}$ is Cauchy.

Assume on the contrary that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\in>0$ for which we can find two subsequences $\{y_{2m_k}\}$ *and* $\{y_{2n_k}\}$ *of* $\{y_{2n}\}$ so that n_k is the smallest positive integer such that $2n_k>2m_k>k$,

$$d(y_{2m_k},y_{2n_k})\ge\in \tag{5}$$

$$d(y_{2m_k},y_{2n_k-2})<\in \tag{6}$$

Now from (5) and (6), we have

$$\in\le d(y_{2m_k},y_{2n_k})\le d(y_{2m_k},y_{2n_k-2})+d(y_{2n_k-2},y_{2n_k-1})+d(y_{2n_k-1},y_{2n_k})$$
$$<\in+d(y_{2n_k-2},y_{2n_k-1})+d(y_{2n_k-1},y_{2n_k})$$

Letting $k\to\infty$ and using (4), we get $\in\le\lim_{k\to\infty} d\left(y_{2m_k}, y_{2n_k}\right)\le\in$ *so that*

$$\lim_{k\to\infty} d\left(y_{2m_k}, y_{2n_k}\right)=\in \tag{7}$$

We have $\left|d\left(y_{2m_k+1}, y_{2n_k}\right)-d\left(y_{2m_k}, y_{2n_k}\right)\right|\le d\left(y_{2m_k+1}, y_{2m_k}\right)$

Letting $k\to\infty$ and using (4) and (7), we get

$$\lim_{k\to\infty} d\left(y_{2m_k+1}, y_{2n_k}\right)=\in \tag{8}$$

We have $\left|d\left(y_{2m_k}, y_{2n_k-1}\right)-d\left(y_{2m_k}, y_{2n_k}\right)\right|\le d\left(y_{2n_k-1}, y_{2n_k}\right)$

Letting $k\to\infty$ and using (4) and (7), we get

$$\lim_{k\to\infty} d\left(y_{2m_k}, y_{2n_k-1}\right)=\in \tag{9}$$

We have $\left|d\left(y_{2n_k-1}, y_{2m_k+1}\right)-d\left(y_{2m_k}, y_{2n_k}\right)\right|\le d\left(y_{2n_k-1}, y_{2n_k}\right)+d\left(y_{2m_k}, y_{2m_k+1}\right)$

Letting $k\to\infty$ and using (4) and (7), we get

$$\lim_{k\to\infty} d\left(y_{2n_k-1}, y_{2m_k+1}\right)=\in \tag{10}$$

$$\alpha\left(Sx_{2m_k+1}, Tx_{2n_k}\right)=\alpha\left(y_{2m_k}, y_{2n_k-1}\right)\ge 1 \text{ from (2)}$$

$$\begin{aligned}\psi\left(d\left(y_{2m_k+1}, y_{2n_k}\right)\right)&=\psi\left(d\left(fx_{2m_k+1}, gx_{2n_k}\right)\right)\\ &\le\alpha\left(Sx_{2m_k+1}, Tx_{2n_k}\right)\psi\left(d\left(fx_{2m_k+1}, gx_{2n_k}\right)\right)\\ &\le\beta\left(\psi\left(M\left(x_{2m_k+1}, x_{2n_k}\right)\right)\right)\psi\left(M\left(x_{2m_k+1}, x_{2n_k}\right)\right)\end{aligned}$$

where

$$M\left(x_{2m_k+1}, x_{2n_k}\right)=\max\left\{\begin{matrix}d\left(y_{2m_k}, y_{2n_k-1}\right), d\left(y_{2m_k}, y_{2m_k+1}\right), d\left(y_{2n_k-1}, y_{2n_k}\right),\\ \frac{1}{2}\left[d\left(y_{2m_k}, y_{2n_k}\right)+d\left(y_{2n_k-1}, y_{2m_k+1}\right)\right]\end{matrix}\right\}$$

$$\to\in \text{ as } k\to\infty, \text{ from (9), (4), (7), (10).}$$

From (11), we get

$$\frac{\psi\left(d\left(y_{2m_k+1}, y_{2n_k}\right)\right)}{\psi\left(M\left(x_{2m_k+1}, x_{2n_k}\right)\right)}\le\beta\left(\psi\left(M\left(x_{2m_k+1}, x_{2n_k}\right)\right)\right)<1.$$

Letting $k\to\infty$ and using (8) and the continuity of ψ, we get

$$1\le\lim_{k\to\infty}\beta\left(\psi\left(M\left(x_{2m_k+1}, x_{2n_k}\right)\right)\right)\le 1$$

so that $\lim_{k\to\infty}\psi\left(M\left(x_{2m_k+1}, x_{2n_k}\right)\right)=0$ and hence $\psi(\in)=0$. *Thus* $\in=0$.

It is a contradiction.

Hence $\{y_{2n}\}$ is a Cauchy sequence. From (4), $\{y_{2n+1}\}$ is also Cauchy.

Since X is complete, there exists $z \in X$ such that $y_n \to z$ and hence

$$\lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = \lim_{n\to\infty} gx_{2n+2} = \lim_{n\to\infty} Sx_{2n+1} = z.$$

Since *S* is Continuous, we have

$$\lim_{n\to\infty} S^2 x_{2n+1} = Sz \text{ and } \lim_{n\to\infty} Sfx_{2n+1} = Sz.$$

Since the pair (f,S) is compatible, we have $\lim_{n\to\infty} d(fSx_{2n+1}, Sfx_{2n+1}) = 0.$

Hence $\lim_{n\to\infty} fSx_{2n+1} = Sz.$

Now

$$\alpha(SSx_{2n+1}, Tx_{2n}) = \alpha(Sy_{2n}, y_{2n-1}) \geq 1 \quad \text{from (2.1.6)}$$

$$\psi(d(fSx_{2n+1}, gx_{2n})) \leq \alpha(SSx_{2n+1}, Tx_{2n})\psi(d(fSx_{2n+1}, gx_{2n}))$$
$$\leq \beta(\psi(M(Sx_{2n+1}, x_{2n})))\psi(M(Sx_{2n+1}, x_{2n})),$$

where

$$M(Sx_{2n+1}, x_{2n}) = \max\left\{\begin{array}{l} d(SSx_{2n+1}, Tx_{2n}), d(SSx_{2n+1}, fSx_{2n+1}), d(Tx_{2n}, gx_{2n}), \\ \frac{1}{2}\left[d(SSx_{2n+1}, gx_{2n}) + d(Tx_{2n}, fSx_{2n+1})\right] \end{array}\right\}$$
$$\to d(Sz, z) \text{ as } n \to \infty.$$

From (12), we get

$$\frac{\psi(d(fSx_{2n+1}, gx_{2n}))}{\psi(M(Sx_{2n+1}, x_{2n}))} \leq \beta(\psi(M(Sx_{2n+1}, x_{2n}))) < 1.$$

Letting $n \to \infty$ and using the continuity of ψ, we get

$$1 \leq \lim_{n\to\infty} \beta(\psi(M(Sx_{2n+1}, x_{2n}))) \leq 1$$

which in turn yields that $\lim_{n\to\infty} \psi(M(Sx_{2n+1}, x_{2n})) = 0$ so that $\psi(d(Sz, z)) = 0.$ Hence $Sz = z.$

Since *T* is continuous, we have

$$\lim_{n\to\infty} T^2 x_{2n+2} = Tz \text{ and } \lim_{n\to\infty} Tgx_{2n+2} = Tz.$$

Since the pair (g,T) is compatible, we have $\lim_{n\to\infty} d(Tgx_{2n+2}, gTx_{2n+2}) = 0.$

Hence $\lim_{n\to\infty} gTx_{2n+2} = Tz.$

Now

$$\alpha(Sx_{2n+1}, TTx_{2n+2}) = \alpha(y_{2n}, Ty_{2n+1}) \geq 1 \quad \text{from (2.1.6)}$$

$$\psi(d(fx_{2n+1}, gTx_{2n+2})) \leq \alpha(Sx_{2n+1}, TTx_{2n+2})\psi(d(fx_{2n+1}, gTx_{2n+2}))$$
$$\leq \beta(\psi(M(x_{2n+1}, Tx_{2n+2})))\psi(M(x_{2n+1}, Tx_{2n+2})) \quad (13)$$

where

$$M(x_{2n+1}, Tx_{2n+2}) = \max\left\{\begin{array}{l} d(Sx_{2n+1}, TTx_{2n+2}), d(Sx_{2n+1}, fx_{2n+1}), d(TTx_{2n+2}, gTx_{2n+2}), \\ \frac{1}{2}\left[d(Sx_{2n+1}, gTx_{2n+2}) + d(TTx_{2n+2}, fx_{2n+1})\right] \end{array}\right\}$$
$$\to d(z, Tz) \text{ a sn} \to \infty.$$

From (13), we get

$$\frac{\psi(d(fx_{2n+1}, gTx_{2n+2}))}{\psi(M(x_{2n+1}, Tx_{2n+2}))} \leq \beta(\psi(M(x_{2n+1}, Tx_{2n+2}))) < 1.$$

Letting $n \to \infty$ and using the continuity of ψ, we get

$$1 \leq \lim_{n\to\infty} \beta(\psi(M(x_{2n+1}, Tx_{2n+2}))) \leq 1$$

which in turn yields that $\lim_{n\to\infty} \psi(M(x_{2n+1}, Tx_{2n+2})) = 0$ so that $\psi(d(z, Tz)) = 0.$ *Hence* $Tz = z.$

Now

$$\alpha(Sz, Tx_{2n}) = \alpha(z, y_{2n-1}) \geq 1 \quad \text{from (2.1.6)}$$

$$\begin{aligned} \psi(d(fz, gx_{2n})) &\leq \alpha(Sz, Tx_{2n})\psi(d(fz, gx_{2n})) \\ &\leq \beta(\psi(M(z, x_{2n})))\psi(M(z, x_{2n})), \end{aligned} \tag{14}$$

where

$$M(z, x_{2n}) = \max\begin{Bmatrix} d(Sz, Tx_{2n}), d(Sz, fz), d(Tx_{2n}, gx_{2n}), \\ \frac{1}{2}[d(Sz, gx_{2n}) + d(Tx_{2n}, fz)] \end{Bmatrix}$$

$$\to d(z, fz) \ as \ n \to \infty.$$

$$\frac{\psi(d(fz, gx_{2n}))}{\psi(M(z, x_{2n}))} \leq \beta(\psi(M(z, x_{2n}))) < 1.$$

Letting $n \to \infty$ and using the continuity of ψ, we get

$$1 \leq \lim_{n\to\infty} \beta(\psi(M(z, x_{2n}))) \leq 1$$

which in turn yields that $\lim_{n\to\infty} \psi(M(z, x_{2n})) = 0$ so that $\psi(d(z, fz)) = 0.$ *Hence* $fz = z.$

Now

$$\alpha(Sz, Tz) = \alpha(z, z) \geq 1 \quad \text{from (2.1.6)}$$

$$\begin{aligned} \psi(d(z, gz)) &= \psi(d(fz, gz)) \\ &\leq \alpha(Sz, Tz)\psi(d(fz, gz)) \\ &\leq \beta(\psi(M(z, z)))\psi(M(z, z)), \end{aligned} \tag{15}$$

where

$$M(z, z) = \max\begin{Bmatrix} d(Sz, Tz), d(Sz, fz), d(Tz, gz), \\ \frac{1}{2}[d(Sz, gz) + d(Tz, fz)] \end{Bmatrix}$$

$$\to d(z, gz) \quad as \quad n \to \infty.$$

From (15), we have

$$[1 - \beta(\psi(d(z, gz)))]\psi(d(z, gz)) \leq 0$$

which in turn yields that $\psi(d(z, gz)) = 0$.

Thus gz = z.
Hence z is a common fixed point of f, g, S and T

Suppose u and v are two common fixed points of f, g, S and T

Then $\alpha(Su,Tv)=\alpha(u,v)\geq 1$ from (2.1.7)

$$\begin{aligned}\psi(d(u,v)) &= \psi(d(fu,gv)) \\ &\leq \alpha(Su,Tv)\psi(d(fu,gv)) \\ &\leq \beta(\psi(M(u,v)))\psi(M(u,v)),\end{aligned} \tag{16}$$

where

$$\begin{aligned}M(u,v) &= \max\left\{d(u,v), d(u,u), d(v,v), \frac{1}{2}[d(u,v)+d(v,u)]\right\} \\ &= d(u,v).\end{aligned}$$

From (16), we have

$$[1-\beta(\psi(d(u,v)))]\psi(d(u,v))\leq 0$$

which in turn yields that so that $\psi(d(u,v))=0$. Hence $u=v.$

Thus f, g, S *and* T have a unique common fixed point.

Now, we give an example to illustrate Theorem 2.1.

Example 2.2: Let $X=[0,\infty)$ be endowed with the metric $d(x,y)=|x-y|$ *for all* $x,y\in X$.

Define $f,g,S,T:X\to X$ *by* $fx=\frac{x}{18},\ Sx=\frac{x}{2}, gx=\frac{x^2}{27}$ *and* $Tx=\frac{x^2}{3}$ *for all* $x,y\in X$.

Define $\alpha: X\times X\to[0,\infty)$ *by* $\alpha(x,y)=\begin{cases}1, \text{ if } x,y\in[0,1]\\ 0, \text{ otherwise}\end{cases}$

Define $\psi:[0,\infty)\to[0,\infty)$ *by* $\psi(t)=\frac{t}{2}$ *for all* $t\in[0,\infty)$ *and* $\beta:[0,\infty)\to[0,1)$ *by*

$\beta(t)=\frac{1}{9}$ *for all* $t\in[0,\infty)$.

Clearly the conditions $(2.1.1), (2.1.3), (2.1.4)$ and condition $(2.1.5)$ with $x_1=0$ are satisfied.

If $Sx=\frac{x}{2}\in[0,1]$ *and* $Ty=\frac{y^2}{3}\in[0,1]$ *then* $\alpha(Sx,Ty)=1.$

$$\begin{aligned}\alpha(Sx,Ty)\psi(d(fx,gy)) &= \frac{1}{2}\left|\frac{x}{18}-\frac{y^2}{27}\right| \\ &= \frac{1}{9}\left|\frac{x}{4}-\frac{y^2}{6}\right| \\ &= \frac{1}{9}\psi(d(Sx,Ty)) \\ &\leq \frac{1}{9}\psi(M(x,y)) \\ &= \beta(\psi(M(x,y)))\psi(M(x,y))\end{aligned}$$

If $Sx=\frac{x}{2}\notin[0,1]$ *or* $Ty=\frac{y^2}{3}\notin[0,1]$ *then* $\alpha(Sx,Ty)=0.$

Thus $(2.1.2)$ is satisfied.

Clearly $(2.1.6)$ is satisfied if we take $y_n = \frac{1}{n}$ *for all n.* Clearly '0' is a common fixed point of f, g, S *and* T. The condition $(2.1.7)$ is clear and '0' is unique common fixed point of f, g, S *and* T.

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