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SUBCLASS OF GENERALIZED SAKAGUCHI TYPE FUNCTIONS WITH RESPECT TO SYMMETRIC CONJUGATE POINTS

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ABSTRACT

Let $S_{sc}^{*}(A,B,s,t)$ *denote the class of concerning with Generalized Sakaguchi functions which are analytic in an open unit disc* $\Delta=\{z:|z|<1\}$ *and satisfying the condition* $\left\{\frac{(s-t)zf'(z)}{f(sz)-f(zt)}\right\}\prec\frac{1+Az}{1+Bz};\ s,t\in C;t\neq s,-1\le B<A\le 1,\ z\in\Delta.$ *In this paper, we obtain some properties of functions* $f\in S_{sc}^{*}(A,B,s,t)$.

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1. INTRODUCTION

Let A be the class of analytic functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad \left(z\in\Delta:=\{z\in C:|z|<1\}\right) \tag{1.1}$$

and S_s^{*} be the subclass of A consisting of univalent functions. For two functions $f,g\in A,$ we say that the function f(z) is subordinate to g(z) in Δ and write $f\prec g$ or $f(z)\prec g(z),$ if there exists an analytic function w(z) with w(0)=0 and $|w(z)|<1\ (z\in\Delta),$ such that $f(z)=g(w(z)),(z\in\Delta).$ In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0)=g(0)$ and $f(\Delta)\subset g(\Delta).$

Here we studied a Generalized Sakaguchi type class $S_s^{*}(\alpha,s,t)$.The function $f(z)\in A$ is said to be in the class $S_s^{*}(\alpha,s,t)$ if it satisfies,

$$\left\{\frac{(s-t)zf'(z)}{f(sz)-f(zt)}\right\}>\alpha;\ s,t\in C,t\neq s,0\le\alpha<1,\ z\in\Delta. \tag{1.2}$$

for some $0\le\alpha<1,\ s,t\in C$ with $t\neq s,$ and for all $z\in\Delta.$ The class $S_s^{*}(\alpha,1,t)$ was introduced and studied by Owa *et al.* [7], and taking $t=-1$ in above class, the class reduces in to $S_s^{*}(\alpha,1,-1)=S_s^{*}(\alpha)$ which was introduced by Sakaguchi [4] and is called Sakaguchi Function of order α [8,9], where as $S_s^{*}(0)=S_s^{*}$ of Starlike Functions with respect to symmetric point in Δ.

Let $S_s^{*}(A,B,s,t),$ denote the class of functions of the form (1.1) and satisfying the condition,

$$\frac{(s-t)zf'(z)}{f(sz)-f(zt)}\prec(1+Az)/(1+Bz);\ \textit{for } s,t\in C \text{ with } t\neq s, \text{where}\left(-1\le B<A\le 1,\ z\in\Delta\right). \tag{1.3}$$

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In this paper, we consider the class $S_{sc}^{*}(A,B,s,t),$ of starlike functions with respect to the symmetric conjugate point of the form (1.1) and satisfying the condition

$$\frac{(s-\mathrm{t})\mathrm{z}f'(\mathrm{z})}{\overline{f(\mathrm{sz})-\overline{f(\overline{\mathrm{z}}\mathrm{t})}}} \prec \frac{(1+\mathrm{Az})}{(1+\mathrm{Bz})};\ (\mathrm{s},\mathrm{t}\in C,\mathrm{t}\neq s,-1\leq \mathrm{B}<\mathrm{A}\leq 1, z\in\Delta). \tag{1.4}$$

By definition of subordination it follows that $f\in S_{sc}^{*}(A,B,s,t)$) if and only if

$$\frac{(s-\mathrm{t})\mathrm{z}f'(\mathrm{z})}{\overline{f(\mathrm{sz})-\overline{f(\overline{\mathrm{zt}})}}}=\frac{1+\mathrm{Aw(z)}}{1+B\mathrm{w(z)}}=P(\mathrm{z});\ \left(s,t\in C,\text{with }\mathrm{t}\neq s,|w(z)|<1,w\in\Delta\right) \tag{1.5}$$

where

$$P(\mathrm{z})\ =1+\sum_{n=1}^{\infty}p_n z^n. \tag{1.6}$$

In the present paper, we study the class $S_{sc}^{*}(A,B,s,t)$ to obtain the inequality for the coefficient estimate and other result.

2. PRELIMINARY RESULT

Lemma 2.1: ([3]) If P (z) is given by (1.6) then

$$|p_n|\leq (A-B). \tag{2.1}$$

Lemma 2.2: Let N(z) be analytic and M(z) starlike in Δ and N(0)=M(0)=0.Then

$$\frac{\left|\left(\frac{N'(z)}{M'(z)}-1\right)\right|}{\left|\left(A-B\frac{N'(z)}{M'(z)}\right)\right|}<1 \tag{2.2}$$

Implies

$$\frac{\left|\left(\frac{N(z)}{M(z)}-1\right)\right|}{\left|\left(A-B\frac{N(z)}{M(z)}\right)\right|}<1,\ \ z\in\Delta. \tag{2.3}$$

3. MAIN RESULT

We give the coefficients inequality for the class $S_{sc}^{*}(A,B,s,t)$.

Theorem 3.1: *Let* $f\in S_{sc}^{*}(A,B,s,t)$, *then for* $n\geq 1,$

$$|a_n|\leq \frac{\alpha}{|(n-u_n)|}\left[1+\alpha\sum_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}+\alpha^2\sum_{i_2>i_1}^{n-1}\sum_{i_1=2}^{n-2}\frac{|u_{i_1}u_{i_2}|}{|(i_1-u_{i_1})||(i_2-u_{i_2})|}+\ldots+\alpha^{n-2}\prod_{i=2}^{n-1}\frac{|u_i|}{|(i-u_i)|}\right],$$

where $\alpha=A-B$ and $u_i=\dfrac{(s^i-t^i)}{(s-t)}$.

Proof:

Since $f\in S_{sc}^{*}(A,B,s,t),$ this implies that

$$(s-t)zf'(z)=[f(sz)-\overline{f(\overline{tz})}]P(z)$$

for $z\in\Delta$,with $\mathrm{Re}\{P(z)\}>0$, where $P(z)=1+p_1z+p_2z^2+p_3z^3+\ldots$

$$\left(s-t\right)\left[z+2a_2z^2+3a_3z^3+4a_4z^4+...\right]$$
$$=\left[z\left(s-t\right)+a_2\left(s^2-t^2\right)z^2+a_3\left(s^3-t^3\right)z^3+a_4\left(s^4-t^4\right)z^4+...\right]\left[1+p_1z+p_2z^2+p_3z^3+...\right]$$

Or

$$\left(z+2a_2z^2+3a_3z^3+4a_4z^4+...\right)$$
$$=\left[z+a_2\frac{\left(s^2-t^2\right)}{\left(s+t\right)}z^2+a_3\frac{\left(s^3-t^3\right)}{\left(s+t\right)}z^3+a_4\frac{\left(s^4-t^4\right)}{\left(s+t\right)}z^4+...\right]\left[1+p_1z+p_2z^2+p_3z^3+...\right].$$

Put $u_i=\dfrac{\left(s_i-t_i\right)}{\left(s-t\right)}$ in above equation, we get

$$\left(z+2a_2z^2+3a_3z^3+4a_4z^4+...\right)$$
$$=\left[z+a_2u_2z^2+a_3u_3z^3+a_4u_4z^4+...\right]\left[1+p_1z+p_2z^2+p_3z^3+...\right].$$

On equating coefficients of various degree terms, from above equation,

We have

$$a_2(2-u_2)=p_1 \tag{3.1}$$
$$a_3(3-u_3)=p_2+p_1a_2u_2 \tag{3.2}$$
$$a_4(4-u_4)=p_3+p_2a_2u_2+p_1a_3u_3 \tag{3.3}$$
$$a_5(5-u_5)=p_4+p_3a_2u_2+p_2a_3u_3+p_1a_4u_4 \tag{3.4}$$

Similarly

$$a_n(n-u_n)=p_{n-1}+p_{n-2}a_2u_2+p_{n-3}a_3u_3+...+p_1a_{n-1}u_{n-1}. \tag{3.5}$$

Using Lemma 2.1 in equation (3.1) and (3.2) respectively, we get

$$|a_2|\le\frac{\alpha}{|\left(2-u_2\right)|}, \tag{3.6}$$

$$|a_3|\le\frac{\alpha}{|\left(3-u_3\right)|}\left[1+\alpha\frac{|u_2|}{|\left(2-u_2\right)|}\right], \tag{3.7}$$

On the same manner using Lemma 2.1 in (3.3) and (3.4) respectively, we get

$$|a_4|\le\frac{\alpha}{|\left(4-u_4\right)|}\left[1+\alpha\left(\frac{|u_2|}{|\left(2-u_2\right)|}+\frac{|u_3|}{|\left(3-u_3\right)|}\right)+\alpha^2\left(\frac{|u_2u_3|}{|\left(2-u_2\right)||\left(3-u_3\right)|}\right)\right] \tag{3.8}$$

$$|a_5|\le\frac{\alpha}{|\left(5-u_5\right)|}\left[\begin{array}{l}1+\alpha\left(\dfrac{|u_2|}{|\left(2-u_2\right)|}+\dfrac{|u_3|}{|\left(3-u_3\right)|}+\dfrac{|u_4|}{|\left(4-u_4\right)|}\right)\\ +\alpha^2\left(\dfrac{|u_2u_3|}{|\left(2-u_2\right)||\left(3-u_3\right)|}+\dfrac{|u_2u_4|}{|\left(2-u_2\right)||\left(4-u_4\right)|}+\dfrac{|u_3u_4|}{|\left(3-u_3\right)||\left(4-u_4\right)|}\right)\\ +\alpha^3\dfrac{|u_2u_3u_4|}{|\left(2-u_2\right)||\left(3-u_3\right)||\left(4-u_4\right)|}\end{array}\right]. \tag{3.9}$$

It follows that from above equations Theorem 3.1 holds for n = 2, 3, 4 and 5. Now by Mathematical Induction, we want to prove Theorem.3.1.

Corollary 3.2: If we take $|P_n| \le 2$ then the result reduces into

$$|a_2| \le \frac{2}{|(2-u_2)|}, \tag{3.10}$$

$$|a_3| \le \frac{2}{|(3-u_3)|}\left[1+2\frac{|u_2|}{|(2-u_2)|}\right], \tag{3.11}$$

$$|a_4| \le \frac{2}{|(4-u_4)|}\left[1+2\left(\frac{|u_2|}{|(2-u_2)|}+\frac{|u_3|}{|(3-u_3)|}\right)+2^2\left(\frac{|u_2u_3|}{|(2-u_2)||(3-u_3)|}\right)\right] \tag{3.12}$$

$$|a_5| \le \frac{2}{|(5-u_5)|}\left[\begin{array}{l} 1+2\left(\frac{|u_2|}{|(2-u_2)|}+\frac{|u_3|}{|(3-u_3)|}+\frac{|u_4|}{|(4-u_4)|}\right) \\ +2^2\left(\frac{|u_2u_3|}{|(2-u_2)||(3-u_3)|}+\frac{|u_2u_4|}{|(2-u_2)||(4-u_4)|}+\frac{|u_3u_4|}{|(3-u_3)||(4-u_4)|}\right) \\ +2^3\frac{|u_2u_3u_4|}{|(2-u_2)||(3-u_3)||(4-u_4)|} \end{array}\right]. \tag{3.13}$$

It follows that from above equations Theorem holds for n = 2, 3, 4 and 5. Now by Mathematical Induction, we want to prove Corollary 3.2.

From the above relation we get the desired result.

Theorem 3.2: If $f \in S^*_{sc}(A,B,s,t)$, then the function $F \in S^*_{sc}$, where

$$F(z) = \frac{(s-t)}{z}\int_0^z f(\theta)d\theta. \tag{3.14}$$

Proof:
With the above equation (3.14) defined function F, consider

$$F(z) = \frac{(s-t)}{z}\int_0^z f(\theta)d\theta,$$

Or

$$zF(z) = (s-t)\int_0^z f(\theta)d\theta$$

Using above equation, we can easily say that

$$\frac{(s-t)zF'(z)}{F(sz)-\overline{F(t\overline{z})}} = \frac{\left[zf(z)-\int\limits_0^z f(\theta)d\theta\right]}{\frac{1}{(s-t)}[\int\limits_0^z f(\theta s)d\theta-\int\limits_0^z \overline{f(\overline{\theta}t)}d\theta]}. \tag{3.15}$$

Let us consider,

$$M(z) = \frac{1}{(s-t)}\left[\int\limits_0^z f(\theta s)d\theta-\int\limits_0^z \overline{f(\overline{\theta}t)}d\theta\right],, \tag{3.16}$$

and

$$N(z) = \left[zf(z)-\int\limits_0^z f(\theta)d\theta\right], \tag{3.17}$$

where N(z) and M(z) be the numerator and denominator of the above function by equation (3.15) respectively. Here the function M(z) is starlike.

Now on solving we get

$$\frac{N'(z)}{M'(z)}=\frac{(s-t)zf'(z)}{f(sz)-\overline{f(t\overline{z})}}, \tag{3.18}$$

where $f \in S^*_{sc}(A,B,s,t)$ .Thus we can say that

$$\frac{N'(z)}{M'(z)}=\frac{1+Aw(z)}{1+Bw(z)}, w\in U. \tag{3.19}$$

On the above relation we get,

$$\frac{\left|\left(\frac{N'(z)}{M'(z)}-1\right)\right|}{\left|\left(A-B\frac{N'(z)}{M'(z)}\right)\right|}<1. \tag{3.20}$$

Hence by using Lemma 2.2, we get,

$$\frac{\left|\left(\frac{N(z)}{M(z)}-1\right)\right|}{\left|\left(A-B\frac{N(z)}{M(z)}\right)\right|}<1, z\in\Delta, \tag{3.21}$$

Or, we can say that

$$\frac{N(z)}{M(z)}=\frac{1+Aw_1(z)}{1+Bw_1(z)},\quad w_1\in\Delta. \tag{3.22}$$

Hence by above result, we get $F \in S^*_{sc}(A,B,s,t)$.

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