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THE STUDY ON ALMOST KAEHLERIAN CONFORMAL RECURRENT AND SYMMETRIC MANIFOLDS

U. S. Negi* and Ashish Semwal

Department of Mathematics, H. N. B. Garhwal University, Campus Badshahi Thaul, Tehri Garhwal – 249 199, Uttarakhand, India

E-mail: usnegi7@gmail.com

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ABSTRACT

Tachibana (1959) has studied the almost analytic vectors in certain almost Hermitian manifolds. Prasad (1973) has defined and studied certain properties of recurrent and Ricci-recurrent almost Hermite spaces and almost Tachibana spaces. Further, Singh and Samyal (2004) have studied on a Tachibana space with parallel Bochner curvature tensor. Also, Negi and Rawat (2009) have studied some theorems on almost Kaehlerian spaces with recurrent and symmetric projective curvature tensors.

In this paper, we have defined and studied almost Kaehlerian Conformal recurrent and symmetric manifolds and several theorems have been established.

Key word: *Kaehlerian, Conformal, Recurrent, Symmetric, manifold;*

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1. INTRODUCTION

An almost Kaehler manifold is first of all an almost complex manifold, that is, a 2n-dimensional space with an almost complex structure $F^h{}_i$:

$$F^i{}_j\, F^h{}_i = -\delta^h{}_j, \tag{1.1}$$

And always admits a positive definite Riemannian metric tensor g_{ji} satisfying:

$$F^a{}_j\, F^b{}_i\, g_{ab} = g_{ji}, \tag{1.2}$$

From which

$$F_{ji} = -F_{ij}, \tag{1.3}$$

Where

$$F_{ji} \overset{\text{def}}{=} F^a{}_j\, g_{ai} \tag{1.4}$$

And finally has the property that the differential form

$F_{ji}\, d\xi^j \wedge d\xi^i$ is closed, that is,

$$F_{jih} \overset{\text{def}}{=} \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0 \tag{1.5}$$

And the identity

$$F^i{}_j\, F_i = \tfrac{1}{2} F_{jih}\, F^{ih}, \tag{1.6}$$

Where

$$F_i = -\nabla_j F^h{}_i \tag{1.7}$$

**Corresponding author: U. S. Negi*, *E-mail: usnegi7@ gmail.com*

And ∇ denotes the operation of covariant differentiation with respect to the Riemannian connection $\{{}_j{}^i{}_k\}$.

If in an almost Kaehler manifold, the Nijenhuis tensor satisfies the condition

$$N_{jih} + N_{jhi} = 0,$$

Then we deduce from it $G_{jih} = 0$, i.e.

$$\nabla_j F^h{}_i + \nabla_i F^h{}_j = 0 \tag{1.8}$$

And the space is an almost Tachibana manifold. Thus, we have

$$3\nabla_j F_{ih} = F_{jih} = 0.$$

Consequently, the space is a Kaehler space i.e. an almost Kaehler manifold is a Kaehler manifold, if and only if the Nijenhuis tensor equation is satisfied.

A contravarient almost analytic vector field is defined as a vector field v^i, satisfying (Tachibana (1959) :

$$£_v F^h{}_i \equiv v^j \partial_j F^h{}_i - F^j{}_i \partial_j v^h + F^h{}_j \partial_i v^j = 0, \tag{1.9}$$

Where $£_v$ stands for the Lie-derivative with respect to v^i.

2. ALMOST KAEHLERIAN CONFORMAL RECURRENT MANIFOLD

Tachibana (1967) has shown that with respect to real co-ordinates a tensor,

$$K_{ijkm} == K^h{}_{ijk}\, g_{hm}\,,$$

Defined by

$$K^h{}_{ijk} = R^h{}_{ijk} + \frac{1}{n+4}\left(R_{ik}\,\delta^h{}_j - R_{jk}\,\delta^h{}_i + g_{ik} R^h{}_j - g_{jk} R^h{}_i + S_{ik} F^h{}_j - S_{jk} F^h{}_i + F_{ik} S^h{}_j - F_{jk} S^h{}_i + 2S_{ij} F^h{}_k + 2F_{ij} S^h{}_k\right)$$
$$- \frac{R}{(n+2)(n-2)}\left(g_{ik} S^h{}_j - g_{jk} S^h{}_i + F_{ik} F^h{}_j - F_{jk} F^h{}_i + 2 F_{ij} F^h{}_k\right).$$

Or equivalently,

$$K_{ijkm} = R_{ijkm} + \frac{1}{n+4}\left(R_{ik} g_{jm} - R_{jk}\, g_{im} + g_{ik} R_{jm} - g_{jk} R_{im} + S_{ik} F_{jm} - S_{jk} F_{im} + F_{ik} S_{jm} - F_{jk} S_{im} + 2S_{ij} F_{km} + 2F_{ij} S_{km}\right)$$
$$- \frac{R}{(n+2)(n+4)}\left(g_{ik}\, g_{jm} - g_{jk}\, g_{im} + F_{ik} F_{jm} - F_{jk} F_{im} + 2F_{ij} F_{km}\right),$$

has components of the tensor called the Conformal curvature tensor. Here $R^h{}_{ijk}$ and $R_{ij} = R^a{}_{ija}$ are the Riemannian curvature and Ricci-tensor respectively. The tensor which is defined by

$$S_{ij} = R^r{}_i\, g_{rj}\,,$$

Satisfies

$$S_{ij} = -S_{ji}.$$

We know that a manifold for which we have at every point

$$\nabla_a K_{ijkm} - \lambda_a K_{ijkm} = 0, \tag{2.1}$$

is called an almost Kaehlerian Conformal recurrent manifold.

Definition (2.1): An almost Kaehler manifold, for which at every point, we have

$$\nabla_a (F^i{}_h K_{ijkm}) - \lambda_a F^i{}_h K_{ijkm} = 0, \tag{2.2}$$

will be called an almost Kaehlerian Conformal recurrent manifold of the first order and first kind.

Definition (2.2): An almost Kaehler manifold, for which at every point, we have

$$\nabla_a (F^i_h F^j_t K_{ijkm}) - \lambda_a F^i_h F^j_t K_{ijkm} = 0, \tag{2.3}$$

will be called an almost Kaehlerian Conformal recurrent manifold of the second order and first kind.

Definition (2.3): An almost Kaehler manifold, for which at every point, we have

$$\nabla_a (F^i_h F^k_s K_{ijkm}) - \lambda_a F^i_h F^k_s K_{ijkm} = 0, \tag{2.4}$$

will be called an almost Kaehlerian Conformal recurrent manifold of the second order and second kind.

Definition (2.4): An almost Kaehler manifold, for which at every point, we have

$$\nabla_a (F^i_h F^j_t F^k_s K_{ijkm}) - \lambda_a F^i_h F^j_t F^k_s K_{ijkm} = 0, \tag{2.5}$$

will be called an almost Kaehlerian Conformal recurrent manifold of the third order and first kind.

Definition (2.5): An almost Kaehler manifold, for which at every point, we have

$$\nabla_a (F^i_h F^j_t F^k_s F^m_n K_{ijkm}) - \lambda_a F^i_h F^j_t F^k_s F^m_n K_{ijkm} = 0, \tag{2.6}$$

will be called an almost Kaehlerian Conformal recurrent manifold of the fourth order and first kind.

We, now, have the following:

Theorem (2.1): The condition that an almost Tachibana manifold be a Conformal recurrent manifold of the first order is

$$\nabla_a K_{rjkm} - \lambda_a K_{rjkm} = F^i_a F^h_r (\nabla_h K_{ijkm} - \lambda_h K_{ijkm}). \tag{2.7}$$

Proof: Equation (2.2) is equivalent to

$$(\nabla_a F^i_h) K_{ijkm} + F^i_h \nabla_a K_{ijkm} - \lambda_a F^i_h K_{ijkm} = 0. \tag{2.8}$$

Interchanging the indices **a** and **h** in equation (2,8) and adding the result thus obtained in the above equation, we get after using (1.8),

$$(\nabla_a K_{ijkm} - \lambda_a K_{ijkm}) F^i_h + (\nabla_h K_{ijkm} - \lambda_h K_{ijkm}) F^i_a = 0, \tag{2.9}$$

Transvecting the above equation with F^h_r and using (1.1) we get the required condition (2.7).

Theorem (2.2): The condition that an almost Tachibana manifold be a Conformal recurrent manifold of the second order and first kind is

$$\nabla_a (F^j_t K_{rjkm}) - \lambda_a F^j_t K_{rjkm} = [\nabla_h (F^j_t K_{ijkm}) - \lambda_h F^j_t K_{ijkm}] F^i_a F^h_r \tag{2.10}$$

Proof: Equation (2.3) is equivalent to

$$(\nabla_a F^i_h) F^j_t K_{ijkm} + F^i_h \nabla_a (F^j_t K_{ijkm}) - \lambda_a F^i_h F^j_t K_{ijkm} = 0. \tag{2.11}$$

Interchanging the indices **a** and **h** in the above equation and adding the result thus obtained in (2.11), we get after using (1.8),

$$F^i_h [\nabla_a (F^j_t K_{ijkm}) - \lambda_a F^j_t K_{ijkm}] + F^i_a [\nabla_h (F^j_t K_{ijkm}) - \lambda_h F^j_t K_{ijkm}] = 0 \tag{2.12}$$

Transvecting the above equation by F^h_r and using (1.1) we get the required condition (2.10).

Theorem (2.3): The condition that an almost Tachibana manifold be a Conformal recurrent manifold of the second order and second kind is

$$\nabla_a (F^k_s K_{rjkm}) - \lambda_a F^k_s K_{rjkm} = [\nabla_h (F^k_s K_{ijkm}) - \lambda_h F^k_s K_{ijkm}] F^i_a F^h_r \tag{2.13}$$

The proof is similar to the proof of theorem (2.2).

Theorem (2.4): The conditions that an almost Tachibana manifold be a Conformal recurrent manifold of the third and fourth order are

$$\nabla_a (F^j_t F^k_s K_{rjkm}) - \lambda_a F^j_t F^k_s K_{rjkm} = [\nabla_h (F^j_t F^k_s K_{ijkm}) - \lambda_h F^j_t F^k_s K_{ijkm}] F^i_a F^h_r \tag{2.14}$$

and

$$\nabla_a (F^j_t F^k_s F^m_n K_{rjkm}) - \lambda_a F^j_t F^k_s F^m_n K_{rjkm} =$$

$$= [\nabla_h (F^j_t F^k_s F^m_n K_{ijkm}) - \lambda_h F^j_t F^k_s F^m_n K_{ijkm}] F^i_a F^h_r. \tag{2.15}$$

Proof: The equation (2.5) is equivalent to

$$(\nabla_a F^i_h) F^j_t F^k_s K_{ijkm} + F^i_h \nabla_a (F^j_t F^k_s K_{ijkm}) - \lambda_a F^i_h F^j_t F^k_s K_{ijkm} = 0 \tag{2.16}$$

Interchanging the indices **a** and **h** in the above equation and adding the result, thus obtained in (2.16), we get after using (1.8),

$$F^i_h [\nabla_a (F^j_t F^k_s K_{ijkm}) - \lambda_a F^j_t F^k_s K_{ijkm}] + F_a^{\ i} [\nabla_h (F^j_t F^k_s K_{ijkm}) - \lambda_h F^j_t F^k_s K_{ijkm}] = 0 \tag{2.17}$$

Transvecting the above equation by F^h_r and using (1.1), we get the condition (2.14). The proof of the condition (2.15) is similar to the proof of the condition (2.14).

3. ALMOST KAEHLERIAN CONFORMAL SYMMETRIC MANIFOLD

Definition (3.1): An almost Kaehler manifold, for which the Conformal curvature tensor K^h_{ijk}, satisfies

$$\nabla_a K^h_{ijk} = 0, \quad \text{or} \quad \text{equivalently} \quad \nabla_a K_{ijkm} = 0, \tag{3.1}$$

Will be called an almost Kaehlerian Conformal symmetric manifold.

Definition (3.2): An almost Kaehler manifold, for which the Conformal curvature tensor K^h_{ijk}, satisfies

$$\nabla_a (F^i_h K_{ijkm}) = 0, \tag{3.2}$$

Will be called an almost Kaehlerian Conformal symmetric manifold of the first order and first kind.

Definition (3.3): An almost Kaehler manifold, for which the Conformal curvature tensor K^h_{ijk}, satisfies

$$\nabla_a (F^i_h F^j_t K_{ijkm}) = 0, \tag{3.3}$$

Will be called an almost Kaehlerian Conformal symmetric manifold of the second order and first kind.

Definition (3.4): An almost Kaehler manifold, for which the Conformal curvature tensor K^h_{ijk}, satisfies

$$\nabla_a (F^i_h F^k_s K_{ijkm}) = 0, \tag{3.4}$$

Will be called an almost Kaehlerian Conformal symmetric manifold of the second order and second kind.

Definition (3.5): An almost Kaehler manifold, for which the Conformal curvature tensor K^h_{ijk}, satisfies

$$\nabla_a (F^i_h F^j_t F^k_s K_{ijkm}) = 0, \tag{3.5}$$

and

$$\nabla_a (F^i_h F^j_t F^k_s F^m_n K_{ijkm}) = 0, \tag{3.6}$$

Will be called respectively an almost Kaehlerian Conformal symmetric manifold of the third order and the fourth order.

We, now, have the following:

Theorem (3.1): The condition that an almost Tachibana manifold be a Conformal symmetric manifold of the first order and first kind is

$$\nabla_a K_{rjkm} - F^i_a F^h_r \nabla_h K_{ijkm} = 0. \tag{3.7}$$

Proof: Equation (3.2) is equivalent to

$$(\nabla_a F^i_h)\, K_{ijkm} + F^i_h\, \nabla_a K_{ijkm} = 0. \tag{3.8}$$

Interchanging the indices **a** and **h** in the above equation and adding the result thus obtained in equation (3.8), we get after using (1.8),

$$F^i_h\, \nabla_a K_{ijkm} + F^i_a\, \nabla_h K_{ijkm} = 0, \tag{3.9}$$

Transvecting the above equation with F^h_r and using (1.1), we have the required condition (3.7).

Theorem (3.2): The condition that an almost Tachibana manifold be a Conformal symmetric manifold of the second order and first kind is

$$\nabla_a (F^j_t\, K_{rjkm}) - F^i_a\, F^h_r\, \nabla_h (F^j_t\, K_{ijkm}) = 0 \tag{3.10}$$

Proof: Equation (3.3) is equivalent to

$$(\nabla_a F^i_h)\, F^j_t\, K_{ijkm} + F^i_h\, \nabla_a (F^j_t\, K_{ijkm}) = 0 \tag{3.11}$$

Interchanging the indices **a** and **h** in the above equation and adding the result thus obtained in (3.11), we get after using (1.8),

$$F^i_h\, \nabla_a (F^j_t\, K_{ijkm}) + F^i_a\, \nabla_h (F^j_t\, K_{ijkm}) = 0 \tag{3.12}$$

Transvecting the above equation with F^h_r and using (1.1), we get the required condition (3.10).

Similarly, we can prove the following:

Theorem (3.3): The condition that an almost Tachibana manifold be a Conformal symmetric manifold of the second order and second kind is

$$\nabla_a (F^k_s\, K_{rjkm}) - F^i_a\, F^h_r\, \nabla_h (F^k_s\, K_{ijkm}) = 0. \tag{3.13}$$

Theorem (3.4): The conditions that an almost Tachibana manifold be a Conformal symmetric manifold of the third and fourth orders are:

$$\nabla_a (F^j_t\, F^k_s\, K_{rjkm}) - F^i_a\, F^h_r\, \nabla_h (F^j_t\, F^k_s\, K_{ijkm}) = 0. \tag{3.14}$$

and

$$\nabla_a (F^j_t\, F^k_s\, F^m_n\, K_{rjkm}) - F^i_a\, F^h_r\, \nabla_h (F^j_t\, F^k_s\, F^m_n\, K_{ijkm}) = 0. \tag{3.15}$$

respectively.

Proof: The equation (3.5) is equivalent to

$$(\nabla_a F^i_h)\, F^j_t\, F^k_s\, K_{ijkm} + F^i_h\, \nabla_a (F^j_t\, F^k_s\, K_{ijkm}) = 0. \tag{3.16}$$

Interchanging the indices **a** and **h** in the above equation and adding the result thus obtained in (3.16), we get after using (1.8):

$$F^i_h\, \nabla_a (F^j_t\, F^k_s\, K_{ijkm}) + F^i_a\, \nabla_h (F^j_t\, F^k_s\, K_{ijkm}) = 0. \tag{3.17}$$

Transvecting the above equation with F^h_r and using (1.1), we get the required condition (3.14).

Similarly, we can prove the condition (3.15).

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